Mean-variance inefficiency of CRRA and CARA utility functions for portfolio selection in defined contribution pension schemes*

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EXTENDED ABSTRACT

1 The problem

The crisis of international Pay As You Go public pension systems is forcing governments of most countries to drastically cut pension benefits of future generations and to encourage the development of fully funded pension schemes. It is well-known that the reforms undertaken in most industrialized countries give a preference towards defined contribution (DC) plans rather than defined benefit (DB) plans. Thus, defined contribution pension schemes will play a crucial role in the social pension systems and financial advisors of DC plans will be needing flexible decision making tools to be appropriately tailored to a member's needs, to help her making optimal and conscious choices. Given that the member of a defined contribution pension scheme has some freedom in choosing the investment allocation of her fund in the accumulation phase, she has to solve a portfolio selection problem. Traditionally, the usual way to deal with it has been maximization of expected utility of final wealth. In this paper, we show that the optimal portfolios derived with the popular CARA and CRRA utility functions are not mean-variance efficient. In fact, we prove that the variance of the optimal portfolios is not the minimal variance. As a byproduct, we propose a natural measure of inefficiency, based on the difference between optimal portfolio variance and minimal variance. We show that inefficiency increases with time horizon and Sharpe ratio of the risky asset and decreases with risk aversion. We also prove that the amount of wealth invested in the risky asset at any time is strictly positive. This means that short-selling of the risky asset is prevented by the adoption of mean-variance efficient strategies in continuous-time. This result, that is a desirable feature for pension funds management, is standard in the single-period framework and is proven to hold true also in the continuous-time setting.

1.1 Review of the literature

The literature on the accumulation phase of defined contribution pension schemes is full of examples of optimal investment strategies resulting from expected utility maximization. See for instance Battocchio and Menoncin (2004), Boulier, Huang and Taillard (2001), Cairns, Blake and Dowd (2006),

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Deelstra, Grasselli and Koehl (2003), Devolder, Bosch Princep and Dominguez Fabian (2003), Di Giacinto, Gozzi and Federico (2009), Gao (2008), Haberman and Vigna (2002), Xiao, Zhai and Qin (2007). Consistently with the economics and financial literature, the most widely used utility function exhibits constant relative risk aversion (CRRA); i.e., the power or logarithmic utility function, see, e.g., Boulier et al. (2001), Cairns et al. (2006), Deelstra et al. (2003), Devolder et al. (2003), Gao (2008), Xiao et al. (2007). Some papers use the utility function that exhibits constant absolute risk aversion (CARA); i.e., the exponential utility function, see, e.g., Battocchio and Menoncin (2004), Devolder et al. (2003). Finally, Di Giacinto et al. (2009) use a general form of utility function that includes as a special case a modified version of the power utility function, and Haberman and Vigna (2002) minimize expected loss using a quadratic loss function, a common approach in pension schemes optimization.

In the context of DC pension funds the problem of finding the optimal investment strategy that is mean-variance efficient, i.e. minimizes the variance of the final fund given a certain level of expected value of the fund, or, equivalently, maximizes the expected value of the final fund given a certain level of variance, has not been reported in published articles. This is not surprising and is mainly due to the fact that the exact and rigourous multi-period and continuous-time versions of the mean-variance problem have been produced only quite recently. The main reason of this delay in solving such a relevant problem, since Markowitz (1952) and Markowitz (1959), lies in the difficulty inherent in the extension from single-period to multi-period or continuous-time framework. A real breakthrough in this area was introduced by Li and Ng (2000) in a discrete-time multi-period framework and Zhou and Li (2000) in a continuous-time model. They show how to transform the difficult problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can then be solved through standard methods. These seminal papers have been followed by a number of extensions; see, for instance, Bielecky, Jin, Pliska and Zhou (2005) and references therein.

On the other hand, expected utility maximization, the leading alternative to the mean-variance approach in the last few decades, was pioneered by Samuelson (1969) in discrete time and by Merton (1969) and Merton (1971) in continuous time. Since their seminal papers, there has been a proliferation of papers adopting dynamic programming to solve expected utility problems in all financial sectors, included defined contribution pension schemes.

Mean-variance and expected utility are two different approaches for dealing with portfolio selection. It is well known that in the single-period framework the mean-variance approach and expected utility optimization coincide if either the utility function is quadratic or the asset returns are normal. Furthermore, in the continuous-time framework when prices are log-normal there is consistency between optimal choices and mean-variance efficiency at instantaneous level (see Merton (1971) and also Campbell and Viceira (2002)). However, this does not imply that an optimal policy should *remain* efficient also after two consecutive instants or, more in general, on a time interval greater than the instantaneous one. In fact, in general it does not. This paper clearly shows it, by proving inefficiency on a given time horizon $t > 0$ when CRRA and CARA utility functions are maximized in the presence of lognormal prices. This drawback becomes particularly relevant in real world applications, given that investors should care more about behaving efficiently on the entire time horizon rather than in each single instant.

In previous financial literature, the lack of efficiency of optimal policies in continuous-time was noted for instance by some empirical works prior to the papers by Li and Ng (2000) and Zhou and Li (2000). These empirical papers that compare mean-variance efficient portfolios with expected utility optimal portfolios find that there are indeed differences between those portfolios. Among

them, Hakansson (1971), Grauer (1981) and Grauer and Hakansson (1993). Related work can be found in Zhou (2003). We believe that now that the connection between the mean-variance approach and standard LQ control problems has been rigourously established, the quite rich stochastic control arsenal should be exploited to investigate further the problem of comparing the two leading methodologies for portfolio selection problems. This paper is a first step in this direction, stating and proving some basic results in a simple model for defined contribution pension schemes.

In DC plans, the mean-variance problem has been addressed by Højgaard and Vigna (2007). They show that the target-based approach, based on the minimization of a quadratic loss function, can be formulated as a mean-variance optimization problem. Not surprisingly, also the opposite is true; namely, each point on the efficient frontier corresponds to the solution of a target-based optimization problem. This is expected, because it is a direct consequence of the fact that the quadratic utility (or loss) function is consistent with mean-variance analysis. In this paper, we add new results to that analysis. In fact, we prove that the widely used CARA and CRRA utility functions produce optimal portfolios that are not mean-variance efficient. As a corollary, we show that these results hold also when the contribution rate is null, i.e. for the typical portfolio selection problem. We introduce a natural measure of inefficiency of optimal portfolio derived with those utility functions, and find the intuitive results that inefficiency increases with time horizon and Sharpe ratio of the risky asset and decreases with risk aversion. Finally, exploiting the one-to-one correspondence between efficient frontier and target-based optimization problems, we prove that the amount invested in the risky asset with a mean-variance efficient strategy is strictly positive. In other words, short-selling is prevented a priori with adoption of efficient strategies. This desirable feature, that is straightforward in single-period setting, is here proven to hold also in continuous time. In addition, this result, together with the recent result by Chiu and Zhou (2009) that an efficient portfolio must have a non-zero but not necessarily positive allocation to the riskless asset, sheds further light on the composition of efficient portfolios. We end with a numerical example, aimed at showing, in the context of a defined contribution pension scheme, the extent of inefficiency of optimal portfolios derived with CRRA and CARA utility functions with typical risk aversion coefficients.

References

Battocchio, P. and Menoncin, F. (2004). Optimal pension management in a stochastic framework, *Insurance: Mathematics and Economics* **34**: 79–95.

Bielecky, T., Jin, H., Pliska, S. and Zhou, X. (2005). Continuous-time mean-variance portfolio selection with bankruptcy prohibition, *Mathematical Finance* **15**: 213–244.

Boulier, J.-F., Huang, S. and Taillard, G. (2001). Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund, *Insurance: Mathematics and Economics* **28**: 173–189.

Cairns, A. J. G., Blake, D. and Dowd, K. (2006). Stochastic lifestyling: Optimal dynamic asset allocation for defined-contribution pension plans, *Journal of Economic Dynamic and Control* **30(5)**: 843–877.

Campbell, J. and Viceira, L. (2002). *Strategic Asset Allocation*, Oxford University Press.

Chiu, C. and Zhou, X. (2009). The premium of dynamic trading, *Quantitative Finance, forthcoming.*

Deelstra, G., Grasselli, M. and Koehl, P.-F. (2003). Optimal investment strategies in the presence of a minimum guarantee, *Insurance: Mathematics and Economics* **33**: 189–207.

Devolder, P., Bosch Princep, M. and Dominguez Fabian, I. (2003). Stochastic optimal control of annuity contracts, *Insurance: Mathematics and Economics* **33**: 227–238.

Di Giacinto, M., Gozzi, F. and Federico, S. (2009). Pension funds with a minimum guarantee: a stochastic control approach, *to appear in Finance and Stochastics*.

Gao, J. (2008). Stochastic optimal control of dc pension funds, *Insurance: Mathematics and Economics* **42**: 1159–1164.

Grauer, R. (1981). A comparison of growth optimal portfolio and mean-variance investment policies, *Journal of Financial and Quantitative Analysis* **16**: 1–21.

Grauer, R. and Hakansson, N. (1993). On the use of mean-variance and quadratic approximations in implementing dynamic investment strategies: A comparison of returns and investment policies, *Management Science* **39**: 856–871.

Haberman, S. and Vigna, E. (2002). Optimal investment strategies and risk measures in defined contribution pension schemes, *Insurance: Mathematics and Economics* **31**: 35–69.

Hakansson, N. (1971). Capital growth and the mean-variance approach to portfolio selection, *Journal of Financial and Quantitative Analysis* **6**: 517–557.

Højgaard, B. and Vigna, E. (2007). Mean-variance portfolio selection and efficient frontier for defined contribution pension schemes, *Technical report R-2007-13. Department of Mathematical Sciences, Aalborg University*.

Li, D. and Ng, W.-L. (2000). Optimal dynamic portfolio selection: multiperiod mean-variance formulation, *Mathematical Finance* **10(3)**: 387–406.

Markowitz, H. (1952). Portfolio selection, *Journal of Finance* **7**: 77–91.

Markowitz, H. (1959). *Portfolio selection: efficient diversification of investments*, New York, Wiley.

Merton, R. (1969). Lifetime portfolio selection under uncertainty: the continuous time case, *Review of Economics and Statistics* **51**: 247–257.

Merton, R. (1971). Optimum consumption and portfolio rules in a continuous time model, *Journal of Economic Theory* **3**: 373–413.

Samuelson, P. (1969). Lifetime portfolio selection by dynamic programming, *Review of Economics and Statistic* **51**: 293–246.

Xiao, J., Zhai, H. and Qin, C. (2007). The constant elasticity of variance (cev) model and the legendre transform-dual solution for annuity contracts, *Insurance: Mathematics and Economics* **40**: 302–310.

Zhou, X. (2003). Markowitz's world in continuous time and beyond, *in Stochastic modelling and optimization*, D.D.Yao, H.Zhang and X.Y.Zhou, Springer, New York, pp. 279–310.

Zhou, X. and Li, D. (2000). Continuous-time mean-variance portfolio selection: A stochastic LQ framework, *Applied Mathematics and Optimization* **42**: 19–33.