Mortality and Longevity: a Risk Management Perspective *

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Abstract

Advantages provided by "large" portfolio sizes in respect of random fluctuations risk justify, to some extent, the traditional deterministic approach to mortality in life insurance calculations. However, the presence of other mortality risk components should be recognized. In particular, risks due to uncertainty in level as well as in trend of future mortality may heavily affect portfolio results. Special attention should be placed when addressing long-term insurance products, for example life annuities.

Enterprise Risk Management can provide sound guidelines when dealing with mortality and longevity risks. Various steps constitute the risk management process, ranging from risk identification and risk assessment to portfolio strategies, such as product design, appropriate pricing, natural hedging, risk transfers and capital allocation.

Keywords: Stochastic mortality, process risk, uncertainty risk, longevity risk, life annuity, term assurance, solvency.

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1 Introduction

The foundation of life insurance mathematics can be traced back to the second half of the 17th century: Jan de Witt and Edmond Halley proposed the first formulae for calculating what we now call the expected present value (or the "actuarial value") of life annuities. Seminal contributions followed in the 18th century, in particular the procedure proposed by James Dodson for calculating annual level premiums for insurance policies providing death benefits. See, for example, Haberman and Sibbett [14], Hald [15] and Pitacco [28], and references therein.

The calculation procedures proposed by these Authors rely, in a modern perspective, on "deterministic" actuarial models, as only expected values are addressed. Progression towards a "stochastic" approach to life insurance mathematics started at the end of the 18th century. In 1786 Johannes Tetens first addressed the analysis of mortality risk inherent in an insurance portfolio. The evidence of the role of $\sqrt{n}$ in determining the riskiness of a portfolio, where n denotes the number of policies in the portfolio itself, can be traced back to Tetens' contribution. In particular, as pointed out by Haberman [13], Tetens showed that the risk in absolute terms increases as the portfolio size n increases, whereas the risk in respect of each insured decreases in proportion to $\sqrt{n}$. In a modern perspective, Tetens' ideas constitute a pioneering contribution to individual risk theory.

The stochastic approach to life insurance problems was progressed further, thanks to seminal contributions throughout the following centuries. Important examples were provided, in the second half of the 19th century, by Carl Bremiker and Karl Hattendorff (see Haberman [13]). Both Bremiker and Hattendorff also focussed in particular on the problem of facing adverse fluctuations in mortality. The need for an appropriate fund and, respectively, for a convenient safety loading of premiums emerged in their contributions.

Early contributions to stochastic modeling in life insurance did not allow for sources of risk other than mortality. In particular, the idea of a random financial result will be achieved after the seminal contribution of Louis Bachelier in 1900, concerning the stochastic modeling of investment problems. It is worth noting that stochastic finance enters much later the life insurance actuarial context, in particular thanks to the work of F.M. Redington, dated 1952, addressing the principles of life office valuation.

Despite the way towards stochastic modeling paved by a number of significant contributions, a deterministic approach to mortality is still frequently adopted in actuarial practice, in particular for calculating premiums according to the well known equivalence principle. It is worthwhile to stress that adopting a deterministic approach to actuarial calculations is, to some extent, underpinned by the nature of the insurance process, which consists in "transforming" individual risks through aggregation, so lowering the relevant impact, as proved by Tetens.

However, this justification can be accepted under the assumption that only the risk of random fluctuations in the mortality of insured lives is allowed for. In this paper we will focus on mortality and longevity issues; so, other source of randomness, such as investment yields, are not addressed. In such a context, the existence of risk components other than random fluctuations must be recognized, and a special attention should be devoted to the risk of systematic deviations arising from the uncertainty in representing future mortality patterns.

The need for a sound assessment of the insurer's risk profile (as, in particular, emerges from new solvency standards) suggests a comprehensive approach to the formal representation of the life insurance business, more general than that provided by traditional actuarial mathematics. A comprehensive approach should in particular provide a unifying point of view, including risk identification, risk assessment and risk management.

Enterprise Risk Management (ERM) offers a sound framework for dealing with life insurance business, and in particular with the related technical issues. In this framework, risk identification and risk assessment constitute preliminary steps towards the choice of appropriate tools for managing the risk themselves. Besides the "traditional" tools given by reinsurance and capital allocation, a risk management perspective suggests as effective tools, for example, a careful product design, an appropriate hedging involving opposite exposures to mortality/longevity risks, etc.

The paper is organized as follows. Section 2 has still an introductory purpose, even though a specific actuarial problem, namely the portfolio evaluation, is dealt with. Looking at methodologies of portfolio valuation frequently adopted in actuarial practice, the need for a stochastic approach clearly emerges. In Section 3 the risk management process is briefly sketched. Risk identification and risk assessment are then dealt with in Section 4, referring to both death benefits and life annuities. A wide range of problems emerge, ranging from the use of approximations in representing the portfolio payout to the need for mortality projections and the quantification of the impact of longevity risk on portfolio results. Some strategies of "risk mitigation" are addressed in Section 5. Finally, some remarks in Section 6 conclude the paper.

The present paper is mostly based on research work recently performed by the author jointly with Annamaria Olivieri (University of Parma). Important suggestions also arose while preparing material for the Groupe Consultatif Actuariel Européen Summer School on "Modeling Mortality Dynamics for Pensions and Annuity Business" (Trieste, 2005 and Parma, 2006; lecturers: M. Denuit, S. Haberman, A. Olivieri, E. Pitacco), as well as from discussion following lectures.

2 Valuations in life insurance

Various results are commonly referred to for assessing the performance of a life insurance portfolio. The traditional approach to portfolio valuation is based on the so-called "Value of the In-Force business" (VIF), defined as the present value of future distributable earnings calculated with a given Risk Discount Rate (RDR), net of the amount of shareholders' capital currently within portfolio assets. The distributable earning related to a given period, say a year, is defined as the flow from the portfolio assets to the residual assets of the insurance company (or vice versa) such that portfolio assets amount to a given level, viz the technical provision (or mathematical reserve) plus the target capital.

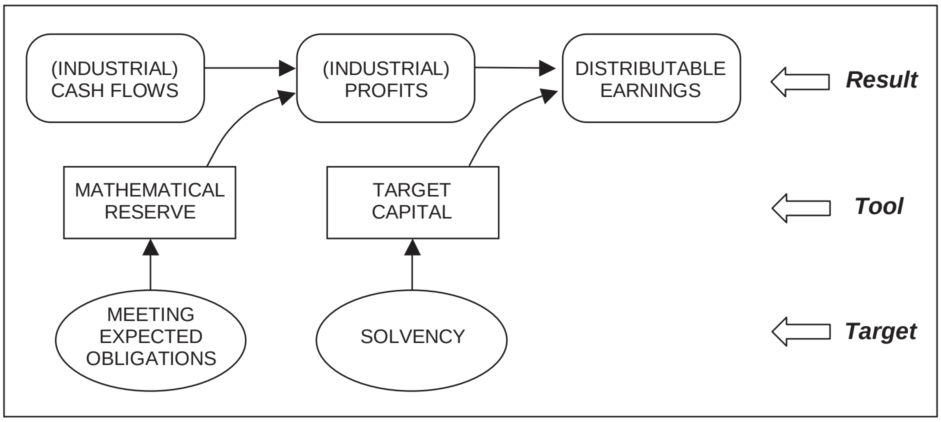

Figure 1: *From cash flows to distributable earnings*

Shareholders' capital within portfolio assets originates, year by year, from undistributable earnings, as well as from returns on pertaining assets. The former may be meant as the share of industrial profit to be maintained within portfolio assets (of course, in case of either an industrial loss or in case of an industrial profit below the amount to be maintained within portfolio assets, it turns out that undistributable earnings are negative, so that some residual assets must be allocated to the portfolio).

In turn, industrial profits originate from industrial cash flows, net of allocation to mathematical reserves which are determined accounting for future expected obligations. Figure 1 illustrates the basic steps of the valuation process which leads from industrial cash flows (premiums, benefits, expenses, etc.) to industrial profits, and finally to distributable earnings.

In order to better understand the nature of the valuation process, as commonly implemented in practice, it is interesting to analyse how riskiness enters the various steps of the process itself. Industrial cash flows include a number of random items: investment yield, mortality, lapses, expenses, etc. These random quantities are usually replaced by estimates (e.g. the estimated yield), in particular by expected values (e.g. the expected outflow for benefits, based on the number of insureds expected to dye/survive according to a given life table). A similar approach underpins the calculation of mathematical reserves and, hence, industrial profits.

The weakness of this procedure clearly emerges as soon as a critical importance is recognized to risks, and to the related impact on portfolio results. Actually, risks are only accounted for via some rules of thumb. For example, mathematical reserves are traditionally calculated adopting safe-side (or "conservative") technical bases. To calculate single-figure indicators. e.g. the VIF, present values of cash flows are prudentially discounted at a RDR, as mentioned above.

When shifting to distributable earnings, a capital allocation policy has to be stated. Trivially, capital allocation can just comply with regulation requirements, and possibly no specific risk assessment is needed. Conversely, if shareholders' capital has to cope with the specific risk profile of the insurer and the regulatory capital is not felt to provide a proper risk measure, a sound assessment of the impact of risks is required.

From these remarks, the need for risk-oriented valuation procedures emerges. It is clear that the traditional valuation approach can be weak in this respect: risk-adjustments are involved in many steps (reserve, target capital, RDR), possibly not consistent one with the other. As an alternative, in recent years market-consistent techniques have been addressed. Typically, a risk-neutral valuation principle is adopted, according to which annual flows must be adjusted with a risk margin assessed consistently with the price of securities suitable to transfer to the market the risk itself; risk-adjusted flows are then discounted with a risk-free rate. It is worth stressing that, according to this setting, only undiversifiable risks (in particular systematic risks common to any agent) are rewarded. In practical terms, the value of risks with a market evidence can be assessed by applying marked-to-market arguments. The value of the portfolio (in particular if one looks for the value to shareholders) is anyhow affected by:

1. systematic risks with poor or no market evidence;
2. inefficiencies in managing the portfolio (for example, diversifiable risks not fully diversified);
3. agency costs.

As regards the assessment of the specific risk profile of an insurer, a deeper analysis is then required, in order to adopt appropriate risk management actions, e.g. reinsurance and capital allocation. The Enterprise Risk Management framework provides the basic ideas which should underpin first risks recognition, then risk assessment (possibly via appropriate "internal models"), finally the choice of strategies aiming at risk mitigation. For a detailed discussion about the Risk Management framework, see for example Tapiero [32] and references therein.

3 The risk management process

As sketched in Figure 2, the Risk Management (RM) process consists of three basic steps, namely the identification of risks, the assessment (or measurement) of the relevant consequences, and the choice of RM techniques. In what follows we obviously refer to the RM process applied to life insurance and annuity portfolios.

The **identification** of risks affecting an insurance company can follow, for example, the guidelines provided by IAA [17], or those provided by the Solvency 2 project (see CEIOPS [9]). Mortality/longevity risks belong to underwriting risks. Components of these risks will be dealt with in Section 4.2. Obviously, the importance of the longevity risk is strictly related to the relative weight of the life annuity portfolio with respect to the overall life business [1].

[1] Terminology problems should not be underestimated when identifying risks. A typical example of possible misunderstanding arises in the field of mortality/longevity risks. We will deal with this aspect in Section 4.2.

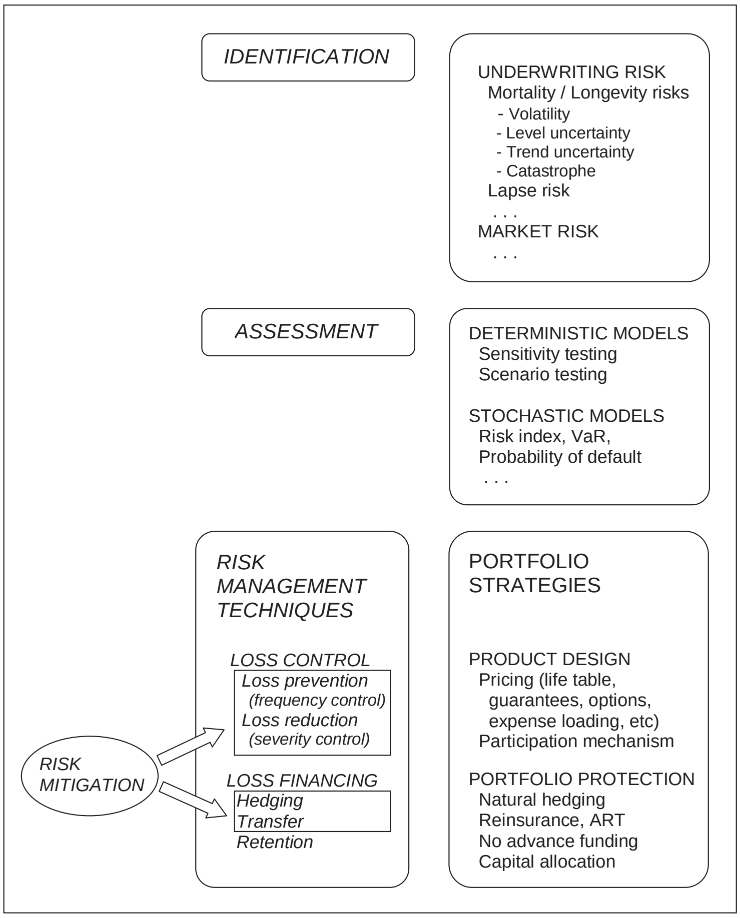

Figure 2: *The Risk Management process*

A rigorous **assessment** of mortality/longevity risks requires the use of stochastic models. Nonetheless, deterministic models are often used in actuarial practice and can provide useful, although rough, insights on the impact of these risks on portfolio results. In particular, as we will see in Section 4.3, deterministic models allow us to calculate the range of values that some results (cash flows, profits, etc.) may assume as one “variable”, viz. the age pattern of mortality, varies (sensitivity testing), or the variables of a given set describing the scenario vary (scenario testing).

Risk management techniques to face mortality/longevity risks include a wide set of tools, which can be interpreted, under an insurance perspective, as portfolio strategies, aiming at risk mitigation. These strategies are dealt with in Section 5.

4 Allowing for mortality/longevity risks

Sections 4 and 5 constitute the core of the paper. First, in Section 4.1 we focus on some topics concerning the representation of the age pattern of mortality. Special attention is placed on the need for mortality projections when long-term products (e.g. life annuities) are concerned. Then, in Section 4.2 "components" of mortality/longevity risks are illustrated. Approaches to the assessment of mortality/longevity risks are discussed in Section 4.3. Finally, two examples are presented in Sections 4.4 and 4.5

4.1 The age pattern of mortality. Projected tables

Usual tools for representing the age pattern of mortality are the life table, i.e. the sequence of expected numbers l_x of survivors at age x ($x = 0, 1, \dots, \omega$) out of a notional cohort of l_0 individuals, and the survival function $S(x)$, defined as the probability for a newborn of being alive at age x ($x > 0$). The life table is commonly used in a time-discrete context, whereas the survival function is adopted in a time-continuous context and is usually represented via mathematical laws.

Both the life table and the survival function are the ultimate result of a statistical process starting from mortality observations and producing, as the first result, the one-year probability of death q_x or the force of mortality μ_x. The life table and the survival function are then derived as follows:

$$l_{x+1} = l_x\,(1 - q_x) \quad \text{for} \quad x = 0, 1, \dots \tag{4.1}$$

$$S(x) = e^{-\int_0^x \mu_t\,dt} \quad \text{for} \quad x > 0 \tag{4.2}$$

Both the probabilities of death q_x and the force of mortality μ_x are usually produced on the basis of "period" observations, i.e. on frequencies of death at the various ages observed throughout a given period, say one year. Hence, calculation of the l_x's and $S(x)$ according to (4.1) and (4.2) relies on the assumption that the mortality pattern does not change in the future.

In many countries, however, statistical evidence shows that human mortality declined over the 20th century, and in particular over its last decades. So, the hypothesis of "static" mortality cannot be assumed in principle, at least when long periods of time are referred to. Figures 3 and 4 illustrate the mortality dynamics in terms of l_x and respectively $d_x = l_x - l_{x+1}$ as it emerges from Italian population tables. Figure 5 illustrates the mortality dynamics in terms of q_x; in particular, the age patterns of mortality corresponding to various period observations and the behavior of q_x, for some fixed ages x, as a function of the observation calendar year (i.e. the so called mortality profiles) are represented.

When we recognize that time affects the age pattern of mortality, functions like $q_x(t)$ must be introduced, the symbol $q_x(t)$ denoting the probability of dying within one year for an individual age x in calendar year t (and thus born in year $t - x$).

Experienced dynamics makes mortality forecasts one of the most important topics in demography and life insurance technique as well. Because of a huge range of problems, methods and controversial issues, mortality forecasting constitutes a stimulating field for research work. For a comprehensive insight on these aspects the reader can refer, for example, to Benjamin and Soliman [1], Delwarde and Denuit [12], Pitacco [27], Tabeau

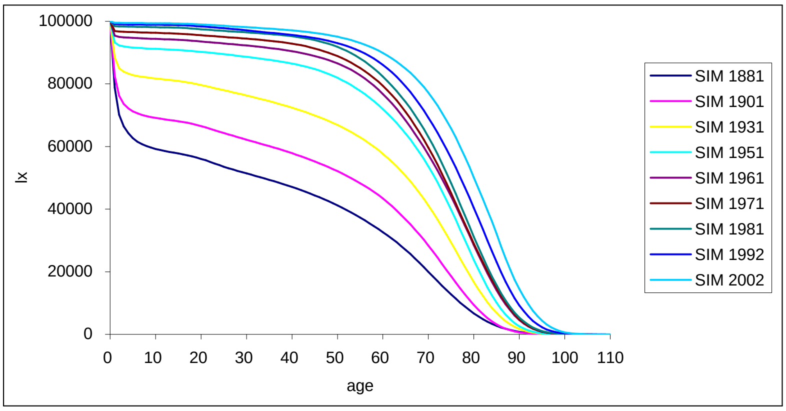

Figure 3: *Survival functions. Source: ISTAT (males)*

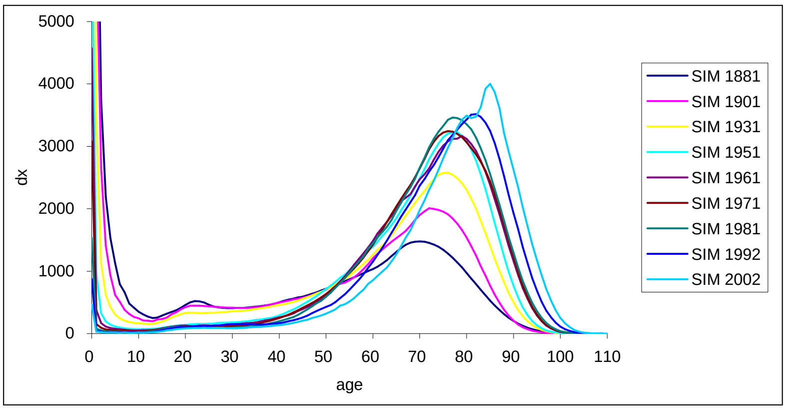

Figure 4: *Curves of deaths. Source: ISTAT (males)*

et al. [31], Wong-Fupuy and Haberman [34], and references therein. In this paper we just address a feature of special interest when dealing with stochastic mortality.

A number of projection methods used in actuarial practice simply consists in interpolation of past mortality trends (as these result from period observations) and then extrapolation of the trends themselves. Clearly these methods rely on the assumption that the experienced trend will continue in the future. Moreover, it should be stressed that these methods do not allow for the stochastic nature of mortality, as they are simply based on observed numbers.

A more rigorous approach to mortality forecasts should take into account stochastic features of mortality. In particular, the following points should underpin a stochastic projection model:

– observed mortality rates are outcomes of random variables representing past mortality;

– forecasted mortality rates are estimates of random variables representing future mortality.

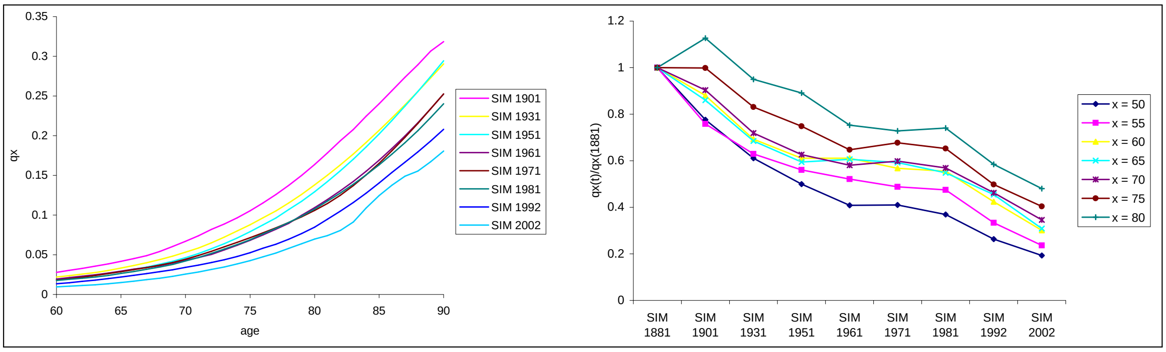

Figure 5: *Mortality rates. Source: ISTAT (males)*

Hence, stochastic assumptions about mortality are required, as well as a statistical structure linking forecasts to observations (see Figure 6).

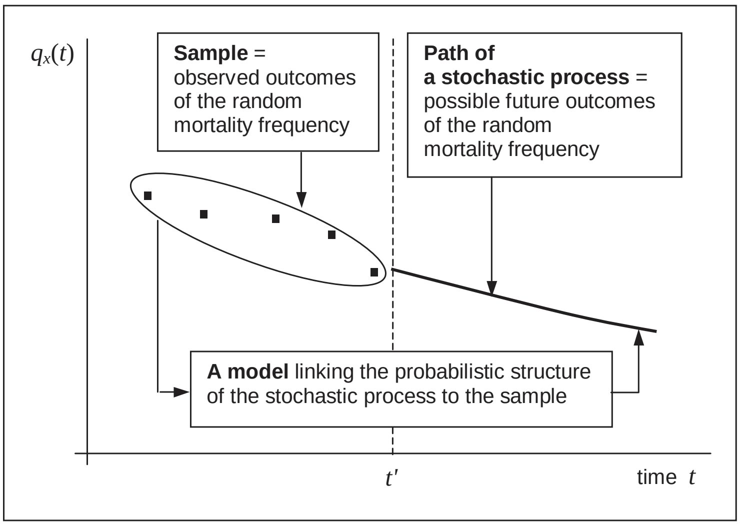

Figure 6: *From past to future: a statistical approach*

In a stochastic framework, results of projection procedures consist in both point estimates and interval estimates of future mortality rates (see Figure 7) and other life table functions. Clearly, traditional interpolation-extrapolation procedures, not explicitly allowing for randomness in mortality, produce just one numerical value for each future mortality rate (or some other age-specific quantity). Moreover, such values can be hardly interpreted as point estimates, because of the lack of an appropriate statistical structure.

The stochastic nature of mortality and the related role in mortality projections can be expressed in several ways. The method proposed by R.D. Lee and L.R. Carter (see Lee and Carter [19], Lee [18] and references therein) constitutes a milestone in stochastic projection methods. The Lee-Carter model has been improved and generalized in many papers, in particular aiming at removing some simplifying hypotheses which are not satisfactory in actuarial applications; see for example Renshaw and Haberman [29], [30]. Brouhns *et al.* [5], [6] investigate possible improvements of the Lee-Carter method, describing the number of deaths as Poisson-distributed random variables. For a deep discussion of stochastic projection methods the reader can refer for example to Delwarde and Denuit [12]. For applications to experience data, see Cairns *et al.* [8].

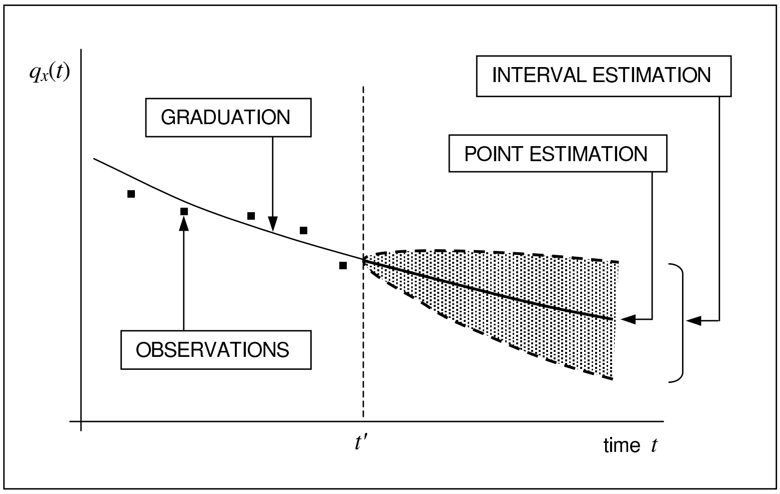

Figure 7: *Mortality forecasts: point estimation vs interval estimation*

4.2 Risk components

Figures 8(a), 8(b) and 8(c) show projected mortality rates at a given age x (the solid line) and three sets of possible future mortality experience (the dots).

Deviations from the projected mortality rates in Figure 8(a) can be reasonably explained in terms of random fluctuations of the outcomes (the observed mortality rates) around the relevant expected values (the projected mortality rates).

Random fluctuations constitute a well-known component of risk in the insurance business, in both the life and the non-life area, often named "process risk". Early fundamental results in risk theory (see Section 1) state that the severity of the process risk decreases, in relative terms, as the portfolio size increases.

The experienced profile depicted in Figure 8(b) can hardly be attributed to random fluctuations only. Much more likely, this profile can be explained as the result of an actual mortality trend other than the forecasted one. So, systematic deviations arise. The risk of systematic deviations can be thought of as a "model risk" or a "parameter risk" referring to the model used for projecting mortality and the relevant parameters (or even a "table risk", clearly referring to the projected life table adopted). The expression "uncertainty risk" is often used to refer to model and parameter (and table) risk jointly, meaning uncertainty in the representation of a phenomenon (viz. the future mortality).

The risk of systematic deviations cannot be hedged by increasing the portfolio size.

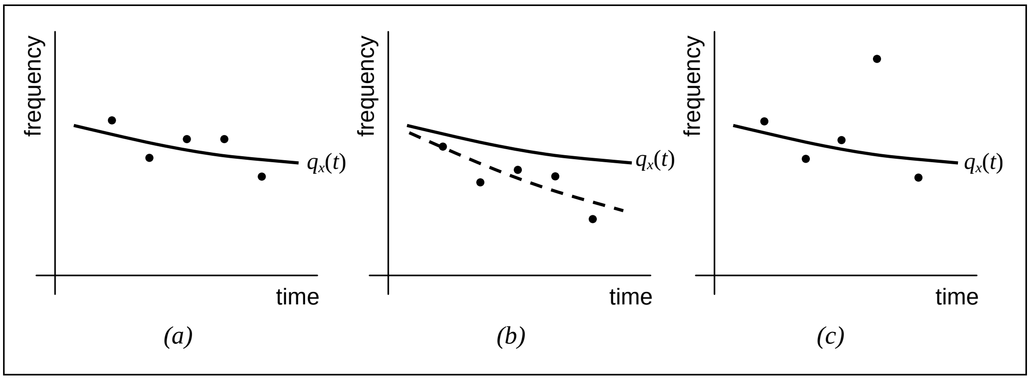

Figure 8: *Effects of mortality/longevity risks*

Actually, in relative terms its financial impact does not reduce as the portfolio size increases, since deviations concern all the insureds or annuitants in the same direction.

The experienced mortality profile depicted in Figure 8(c) likely represents the effect of the "catastrophe risk", namely the risk of a sudden and short-term rise in the mortality frequency, because, for example, of an epidemic or a natural disaster.

Process risk, uncertainty risk and catastrophe risk constitute the three risk "components". The same terminology is usually adopted in relation to other risk causes, e.g. the market risk (and in particular the interest rate risk, the equity risk, etc).

Still referring to mortality/longevity risk, it is interesting to note what follows. Period observations suggest that the general trend consists in a decline in time of mortality rates. However, due to specific events (such as an epidemic, or critical weather conditions), it may happen that in some years the trend is reversed, especially in relation to some ages. Further, going deeper into the analysis of data, it may turn out that some cohorts are experiencing a specific improvement (higher or lower than the average one). From such considerations, the notions of "cohort" effect and "period" effect follow. See for example Willets *et al.* [33]. Hence, the idea is that each cohort has its own mortality trend; nonetheless, some changes (usually, temporary) are common to more than one cohort (possibly, even to the overall population).

This idea can be placed in the framework of risk component classification. From a specific cohort trend, different from the forecasted trend, systematic deviations follow, whence the uncertainty risk is involved. Conversely, a temporary period effect can be interpreted as an outcome of the catastrophe risk, though not necessarily with a huge severity.

Remark It is worthwhile to note that, according to a rather established terminology, the expression "mortality risk" denotes any risk arising from the randomness of individual lifetimes; conversely the expression "longevity risk" only refers to the risk of systematic deviations of experienced mortality from projected mortality (of particular interest in relation to pensions and life annuity products), and hence constitutes a particular mortality risk. On the contrary, the language adopted in Solvency 2 documentation denotes with the expression "longevity risk" the risk of experiencing a mortality lower than expected, whatever the reason may be (i.e. random fluctuations or systematic deviations); conversely, the expression "mortality risk" refers to a mortality higher than expected (because of random fluctuations or systematic deviations).

4.3 Stochastic modeling: an introduction

In order to deal with mortality/longevity risks, we have

(a) to choose an appropriate representation of some quantities directly related with mortality/survivorship, e.g. the number of insureds dying in the various years;

(b) to focus on portfolio results (e.g. cash flows, profits, etc.) which can significantly witness the financial impact of mortality/longevity risks.

While point (b) simply consists in an appropriate choice of one or more results and in expressing the relation between these and the quantities describing mortality, point (a) is a non-trivial issue of stochastic modeling. More precisely, a number of choices are actually available, ranging from a purely deterministic approach to very complex models allowing for uncertainty risk.

Clearly stochastic mortality modeling can be placed in the (more general) framework of stochastic modeling for life insurance. In what follows we refer to this framework.

Assume that a result of interest, Y (e.g. a one-year cashflow), depends on some input variables, say X_1, X_2, X_3 (e.g. number of insureds alive, expenses, etc.)

$$Y = \Phi(X_1, X_2, X_3) \tag{4.3}$$

Figure 9 presents various approaches to investigations about the result Y. Approach 1 is purely deterministic. Assigning specific values, x_1, x_2, x_3, to the three random variables, the corresponding outcome y of the result variable is simply calculated as $y = \Phi(x_1, x_2, x_3)$.

First, it is interesting to note that classical actuarial calculations follow this approach, replacing random variables with their expected values, or anyhow with appropriate estimates. Secondly, in a more modern perspective this approach is adopted for example when performing stress testing (assigning to some variables "extreme" values), or in general scenario testing.

Randomness in input variables is, to some extent, accounted for when approach 2 is adopted. Reasonable ranges for the outcomes of the input variables are chosen, and consequently a range $(y_{\min}, y_{\max})$ for the result Y is derived.

Approach 3 provides a basic example of stochastic modeling, typically adopted for assessing the impact of process risk. A probabilistic structure is assigned to the input variables, in term of the joint probability distribution, or via marginal distributions (see Figure 9) and appropriate assumptions about correlations. The probability distribution of Y can be found using just analytical tools only in very simple (or simplified) circumstances. Numerical methods or stochastic simulation procedures help in most cases.

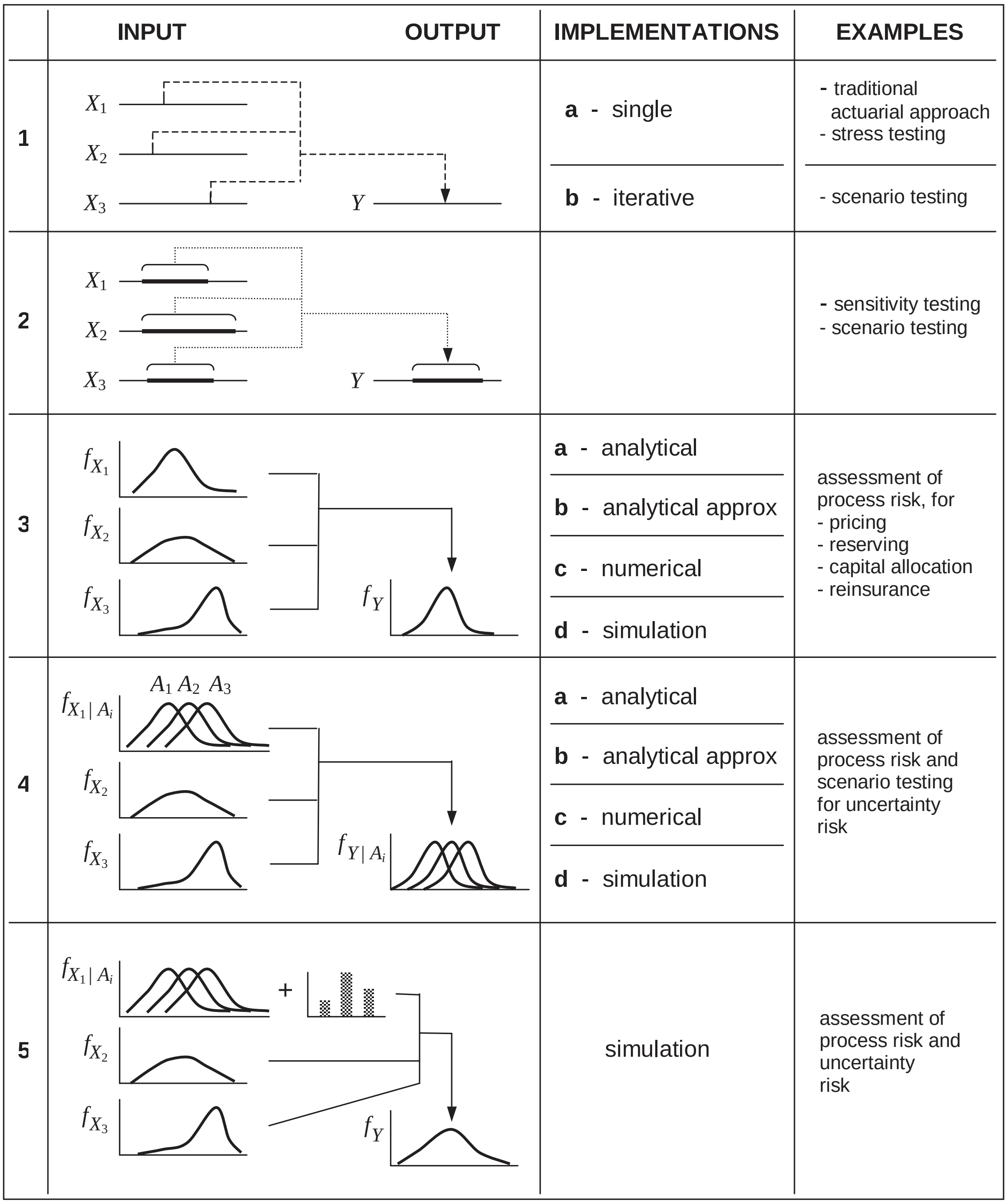

Figure 9: *Modeling approaches*

Dealing with uncertainty risk, in order to assess the impact of systematic deviations, is a crucial issue in particular in life insurance mathematics. Approach 4 simply consists in iterating the procedure implied by approach 3, each iteration corresponding to a specific assumption about the probability distribution of some input variables (the variable X_1 in Figure 9), e.g. a specific set of values for the relevant parameters. Hence, a set of conditional distributions of the result Y is determined.

Finally, approach 5 aims at finding the unconditional probability distribution of the output variable Y, hence allowing for both process risk and uncertainty risk. A more complex probabilistic structure is then required, for example including a probability distribution over the set of assumptions.

Some examples of stochastic models for representing mortality/longevity risks are provided in the following Sections.

4.4 Modeling stochastic mortality: example 1

In this Section we refer to a portfolio of one-year insurance covers only providing a death benefit. In practice, such a portfolio can represent a group insurance, or a one-year section of a more general portfolio consisting of policies with a positive sum at risk due to the presence of some death benefit.

Let n denote the number of insureds, C_j the sum assured for the j-th contract, x_j the insured's age at the beginning of the year, and T_{x_j} her/his remaining lifetime $(j = 1, \dots, n)$. The individual random payout, Y_j, is given by

$$Y_j = \begin{cases} C_j & \text{if } T_{x_j} < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

Hence, the portfolio random payout, Y, is defined as follows:

$$Y = \sum_{j=1}^{n} Y_j \tag{4.5}$$

Assume that the individual lifetimes T_{x_j} are independent, whence the random variables Y_j are also independent. Further, if we assume $C_j = C$ for $j = 1, \dots, n$, and the same probability of dying q for all the insureds, Y has $0, C, \dots, n\,C$ as the possible outcomes, with the binomial probability distribution:

$$\mathbb{P}[Y = h\,C] = \binom{n}{h} q^h (1-q)^{n-h} \tag{4.6}$$

In more general situations the (exact) distribution can be calculated via recursion formulae (see for example Panjer and Willmot [25]). Using the binomial distribution or exact distributions derived via recursion formulae constitute an example of approach 3a (see Figure 9).

In actuarial practice, various approximations to the exact distribution of the random payout are frequently used (thus, adopting approach 3b). In particular (see Panjer and Willmot [25]):

- if $C_j = C$ for $j = 1, \dots, n$, the use of the Poisson distribution relies on the Poisson assumption for the annual number of deaths;
- for more general portfolios, the compound Poisson model is adopted;
- in general, the normal approximation is frequently used.

Whatever the approximating distribution may be, the goodness of the approximation must be carefully assessed, especially with regard to the right tail of the distribution itself, as this tail quantifies the probability of large losses.

Assume the following data:

- sum assured $C_j = 1$ for $j = 1, \dots, n$;
- probability of death $q = 0.005$;
- portfolio sizes: $n = 100$, $n = 500$, $n = 5\,000$.

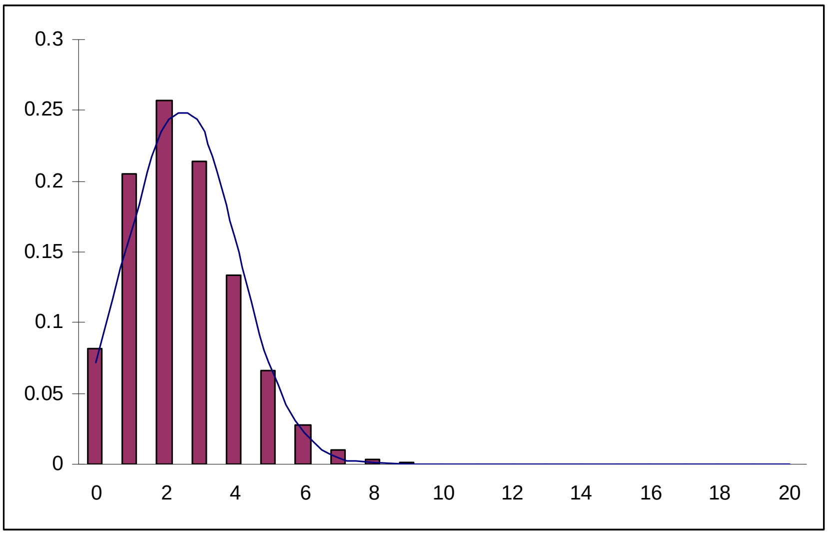

Figure 10: *Probability distribution of the random payout ($n = 500$). Binomial (exact) distribution and Normal approximation*

The (exact) binomial distribution and the normal approximation have been adopted for $n = 500$ and $n = 5\,000$; the (exact) binomial distribution and the Poisson approximation have been used for $n = 100$. Tables 1 to 3 and Figures 10 and 11 show numerical results.

y	$n = 500$ $\mathbb{P}[Y > y]$ Binomial	Normal	y	$n = 5\,000$ $\mathbb{P}[Y > y]$ Binomial	Normal
5	0.04160282	0.056471062	30	0.136121887	0.158048811
6	0.013944069	0.013238288	35	0.022173757	0.022480517
7	0.004135437	0.002164124	40	0.001983179	0.001316908
8	0.001097966	0.000244022	45	0.000101743	3.03545E-05
9	0.000263551	1.88389E-05	50	3.13201E-06	2.68571E-07
10	5.76731E-05	9.90663E-07	55	6.02879E-08	8.9912E-10
...	...	...	...	...	...

Table 1: *Right tails of Binomial (exact) distribution and Normal approximation*

y	$\mathbb{P}[Y = y]$ Binomial	Poisson
0	0.605770436	0.60653066
1	0.304407255	0.30326533
2	0.075719392	0.075816332
3	0.012429649	0.012636055
4	0.001514668	0.001579507
5	0.000146139	0.000157951
6	1.16275E-05	1.31626E-05
7	7.84624E-07	9.40183E-07
8	4.58355E-08	5.87614E-08
9	2.35447E-09	3.26452E-09
10	1.07667E-10	1.63226E-10
...	...	...

Table 2: *Binomial (exact) distribution and Poisson approximation ($n = 100$)*

y	$\mathbb{P}[Y > y]$ Binomial	Poisson
3	0.001673268	0.001752
4	0.000158599	0.000172
5	1.24604E-05	1.42E-05
6	8.32926E-07	1E-06
7	4.83022E-08	6.22E-08
...	...	...

Table 3: *Right tails of Binomial (exact) distribution and Poisson approximation ($n = 100$)*

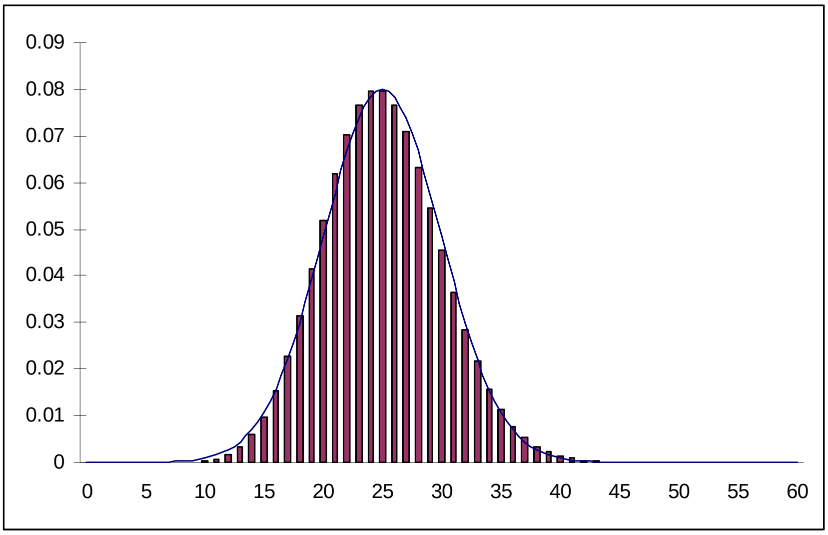

Figure 11: *Probability distribution of the random payout ($n = 5000$). Binomial (exact) distribution and Normal approximation*

The following aspects should be stressed. In relation to portfolio sizes $n = 500$ and $n = 5\,000$, the normal approximation tends to underestimate the right tail of the payout distribution (see Table 1). Conversely, the Poisson distribution provides a good approximation to the exact distribution, also for $n = 100$ (see Tables 2 and 3); unlike the normal approximation, the Poisson model tends to overestimate the right tail, whence a safe-side assessment of liabilities follows.

4.5 Modeling stochastic mortality: example 2

Because of uncertainty in future mortality trend, the stochastic model used for representing mortality, in particular when dealing with life annuities (or other long-term products within the area of insurances of the person; see for example Pitacco [26] and references therein), should allow for the assessment of longevity risk. This can be obtained in various ways. Several proposals focus on the extension of credit risk and interest rate models; see, among the others, Biffis [2], Biffis and Millossovich [3], Cairns *et al.* [7]. A more naive approach consists in designing a finite set of alternative mortality scenarios. This suggests simple and practicable procedures which could be useful for stress tests or for solvency investigations; see CMI [10], [11], and Olivieri and Pitacco [22]. Within the Solvency 2 project, a scenario-based approach should be also addressed for the capital requirement; see CEIOPS [9].

In what follows, we adopt a naive approach, basically consisting of two steps (for details see Olivieri [20], Olivieri and Pitacco [22]).

(1) Choose a set of projected mortality tables (or survival functions, or forces of mor-

tality, etc.) in order to express several alternative hypotheses about future mortality evolution. So, it is possible to perform a scenario testing, assessing the range of variation of quantities such as cash flows, profits, portfolio reserves, etc. This way, the sensitivities of these quantities to future mortality trend is investigated An example of approach 4 (see Figure 9) is thus provided.

(2) Assign a non-negative weight to each mortality hypothesis; the set of weights can be meant as a probability distribution on the space of hypotheses. Hence unconditional (i.e. non conditional on a particular hypothesis) variances, percentiles, etc., of the value of future cash flows, profits, etc. can be calculated; see approach 5.

To assess the impact of longevity risk in a portfolio of life annuities, various "metrics" can be adopted, namely we can focus on several types of results (number of annuitants alive at various times, annual cash flows, discounted cash flows, level of the portfolio fund, etc.), and a number of single-figure indexes. Here we will focus on the random level of the portfolio fund and the shareholders' capital allocated to the portfolio, within a solvency framework. For further information on these issues, the reader can refer to Olivieri [20], and Olivieri and Pitacco [22], [23].

As regards annuitants' mortality, we assume

$$\frac{q_x}{1-q_x} = G\,H^x \tag{4.7}$$

The right-hand side of (4.7) is the third term in the well-known Heligman-Pollard law, i.e. the term describing the old-age pattern of mortality (see Heligman and Pollard [16]). The parameter G expresses the level of senescent mortality and H the rate of increase of senescent mortality itself. The related survival function $S(x)$ can be easily derived. A logistic shape of mortality rates q_x plotted against age x follows.

Note that in a dynamic context probabilities $q_x(t)$ should be addressed. However, in what follows we will address one cohort only, whence the variable t can actually be omitted. Clearly, parameters G and H should be cohort specific.

In order to represent mortality trends, we use projected survival functions. More precisely, we define three projected survival functions, denoted by $S^{[\text{min}]}(x)$, $S^{[\text{med}]}(x)$ and $S^{[\text{max}]}(x)$, expressing respectively a little, a medium and a high reduction in mortality with respect to period experience. Probabilities $\rho^{[\text{min}]}$, $\rho^{[\text{med}]}$ and $\rho^{[\text{max}]}$ are respectively assigned to the three survival functions.

We refer to a portfolio consisting in one cohort of immediate single-premium life annuity contracts, issued at time 0. We assume that all annuitants are aged x_0 at time $t = 0$. Their lifetimes are assumed to be independent of each other (conditional on any given survival function), and identically distributed. All annuities have a (constant) annual amount R. Expenses and related expense loadings are disregarded. N_0 denotes the (given) number of annuities at time $t = 0$.

First, consider the random present value at time 0 of the portfolio future payouts, $Y_0^{(\Pi)}$. The riskiness of the payout can be summarized by its variance or its standard

	A_1 [min]	A_2 [med]	A_3 [max]
G	0.000042	0.000002	0.0000001
H	1.09803	1.13451	1.17215

Table 4: *Parameters of Heligman-Pollard law*

deviation. A relative measure of riskiness is provided by the coefficient of variation, defined as the ratio of the standard deviation to the expected value. This relative measure of riskiness is often denoted, in actuarial mathematics, as the risk index.

The risk index can be calculated conditional on a particular survival function S, i.e. an assumption about future mortality scenario expressed by parameters G, H:

$$\mathrm{CV}[Y_0^{(\Pi)}|S] \;=\; \frac{\sqrt{\mathbb{V}[Y_0^{(\Pi)}|S]}}{\mathbb{E}[Y_0^{(\Pi)}|S]} \tag{4.8}$$

in this case, only random fluctuations are accounted for.

Conversely, the risk index can be calculated allowing for uncertainty in future mortality, weighting the scenarios with the relevant probabilities. In this case we have

$$\mathrm{CV}[Y_0^{(\Pi)}] \;=\; \frac{\sqrt{\mathbb{V}[Y_0^{(\Pi)}]}}{\mathbb{E}[Y_0^{(\Pi)}]} \tag{4.9}$$

and both random fluctuations and systematic deviations are allowed for.

Turning to solvency issues, let Z_t denote the random portfolio fund (i.e. assets facing portfolio liabilities) at (future) time t, and $V_t^{(\Pi)}$ the random portfolio reserve set up at time t. The quantity M_t, defined as follows

$$M_t = Z_t - V_t^{(\Pi)} \tag{4.10}$$

represents the shareholders' capital at time t.

Solvency requirements are usually expressed in terms of M_t. For example, given a time horizon of T years, we say that the insurer has a solvency degree $1-\varepsilon$ if and only if

$$\mathbb{P}\left[\bigwedge_{t=1}^{T} M_t \geq 0\right] = 1-\varepsilon \tag{4.11}$$

The capital required at time $t=0$ is the amount $M_0^{(R)}$ such that condition (4.11) is fulfilled.

Choices for the parameters of the three Heligman-Pollard survival functions are shown in Table 4.

Table 5 provides a comparison between the coefficient of variation (or risk index), as a function of the (initial) portfolio size N_0, allowing for random fluctuations only (i.e. the process risk) and, respectively, for both random fluctuations and systematic deviations (process risk and uncertainty risk). Allowing for random fluctuations only, the pooling effect clearly emerges: actually the coefficient of variation tends to 0 as N_0 tends to ∞. Conversely, when accounting for systematic deviations also, the coefficient of variation decreases as N_0 increases, but its limiting value is positive, showing the non-diversifiable part of the risk. It is worth noting that the results above mentioned can be proved in analytical terms; to this purpose the reader can refer to Olivieri [20], Olivieri and Pitacco [23].

We now turn to the investigation of solvency issues. We address a portfolio of identical annuities, paid to annuitants of initial age $x_0 = 65$, with annual amount $R = 100$. As regards mortality assumptions, we adopt the Heligman-Pollard law, with parameters as described in Table 4.

The single premium (to be paid at entry) is calculated, for each policy, according to the survival function $S^{[\text{med}]}(x)$ and with a constant annual interest rate $i = 0.03$. Further, we assume that for each policy in force at time t, $t = 0, 1, \dots$, a reserve must be set up, which is calculated according to such hypotheses.

N_0	$\text{CV}[Y_0^{(\text{II})} \mid A_2\ [\text{med}]]$	$\text{CV}[Y_0^{(\text{II})}]$
1	31.98%	33.01%
10	10.11%	13.22%
100	3.20%	9.13%
1,000	1.01%	8.61%
10,000	0.32%	8.56%
100,000	0.10%	8.56%
...	...	
∞	0.00%	8.56%

Table 5: *The risk index*

Disregarding uncertainty risk and hence allowing for process risk only, the probability distribution of the future lifetime of each insured is stated, the only cause of uncertainty consisting in the time of death. The assessment of the solvency requirement is performed through simulation. In order to obtain results easier to interpret, we disregard profit; the actual life duration of the annuitants is thus simulated with the survival function $S^{[\text{med}]}(x)$. Further, we assume that the yield from investments is equal to $i = 0.03$.

Allowing also for uncertainty risk, the assessment of the solvency requirement is obtained considering explicitly uncertainty in future mortality trends. To this aim, we consider the three survival functions $S^{[\text{min}]}(x)$, $S^{[\text{med}]}(x)$ and $S^{[\text{max}]}(x)$, weighted with the probabilities $\rho^{[\text{min}]}$, $\rho^{[\text{med}]}$ and $\rho^{[\text{max}]}$ representing the "degree of belief" in such functions.

The single premium for each policy and the individual reserve are still calculated with the survival function $S^{[\text{med}]}(x)$ and the interest rate $i = 0.03$. We still assume $\rho^{[\text{min}]} = 0.2$, $\rho^{[\text{med}]} = 0.6$, $\rho^{[\text{max}]} = 0.2$ (reflecting the fact that $S^{[\text{med}]}(x)$, which is used for pricing and reserving, is supposed to provide the most reliable mortality description).

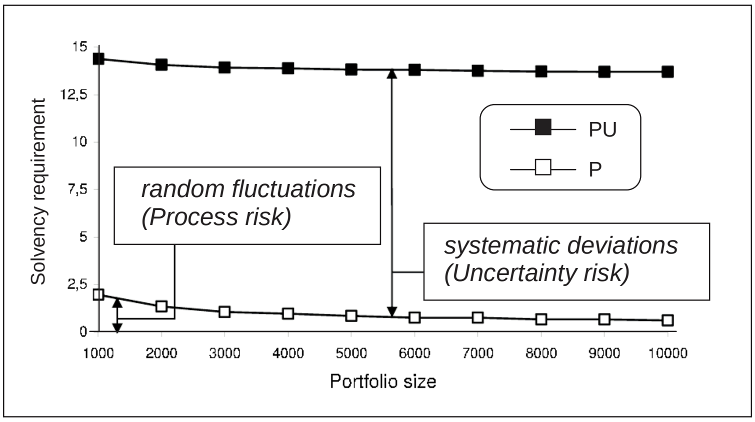

Figure 12: *Solvency requirements for mortality/longevity risks*

Obviously, the investigation is carried out via simulation. We now deal with two causes of uncertainty: the actual distribution of the future lifetimes and the time of death of each insured.

To illustrate the results, we consider the quantity

$$QM^{[\cdot]}(N_0) = \frac{M_0^{(R)}}{V_0^{(\Pi)}} \tag{4.12}$$

In Figure 12 solvency requirements are shown in terms of the ratios (4.12), calculated respectively allowing for process risk only ($QM^{[\mathrm{P}]}(N_0)$), and for both process and uncertainty risk ($QM^{[\mathrm{PU}]}(N_0)$), and plotted against the (initial) portfolio size N_0. A ruin probability $\varepsilon = 0.025$ and a time horizon of $T = 110 - 65 = 45$ years (assuming 110 as the maximum attainable age) have been chosen. When only random fluctuations are accounted for, the solvency requirement tends to 0 as the (initial) portfolio size N_0 diverges, thanks to the pooling effect. Conversely, allowing for systematic deviations also, the solvency requirement keeps high even for large portfolio sizes.

An approach to solvency requirements explicitly allowing only for process risk could be used (and actually is sometime used) taking into account, at least to some extent, uncertainty in future mortality trends (i.e. longevity risk) also. Let $V_0^{(\Pi)[\mathrm{W}]}$ denote the (initial) reserve calculated according to a "worst case" basis, i.e. assuming a very strong mortality improvement, and $V_0^{(\Pi)[\mathrm{B}]}$ the (initial) reserve according to a "bad case" basis, i.e. a strong mortality improvement. Clearly

$$V_0^{(\Pi)} < V_0^{(\Pi)[\mathrm{B}]} < V_0^{(\Pi)[\mathrm{W}]} \tag{4.13}$$

Let $QV^{[\cdot]}$ denote the ratio defined as follows

$$QV^{[\cdot]} = \frac{V_0^{(\Pi)[\cdot]}}{V_0^{(\Pi)}} - 1 \tag{4.14}$$

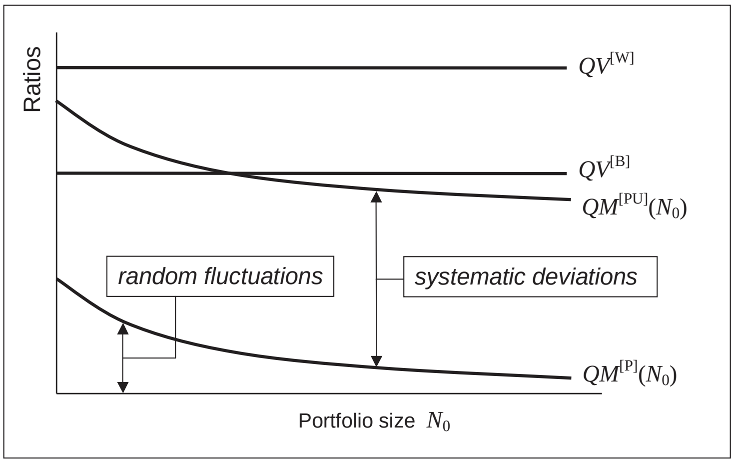

Figure 13: *Reserving and solvency requirements*

Obviously ratios $QV^{[\cdot]}$ are independent of both the portfolio size N_0 and the probability ε. From (4.13) it follows that

$$0 < QV^{[B]} < QV^{[W]} \tag{4.15}$$

Conversely, we find

$$QM^{[P]}(N_0) < QM^{[PU]}(N_0) \tag{4.16}$$

(see, for example, Figure 12). Comparing ratios $QV^{[\cdot]}$ and $QM^{[\cdot]}$ does not lead to general conclusions. However a likely situation is represented by Figure 13. The following aspects should be noted.

(1) Allocating shareholders' capital in the measure suggested by the "worst case" reserve leads to a huge and likely useless capital allocation, whatever the portfolio size N_0 may be; see the value of $QV^{[W]}$ compared to $QM^{[PU]}(N_0)$.

(2) A "bad case" reserve based capital allocation can result in a poor capability of facing the mortality risks when small portfolios are concerned; see the portfolio sizes such that $QV^{[B]} < QM^{[PU]}(N_0)$. Conversely, a too high capital allocation may occur for larger portfolios; see the interval where $QV^{[B]} > QM^{[PU]}(N_0)$.

Thus, setting aside a target capital simply based on the comparison of reserves calculated with different survival functions (as some practice suggests) on the one hand disregards the risk of random fluctuations (which obviously can be considered separately) and on the other disregards a valuation of the probability of ruin, possibly leading to not sound capital allocation.

As regards the process risk, i.e. random fluctuation in mortality, an important aspect of mortality dynamics should be stressed. Looking at mortality trends throughout a long period of time (see for example Figures 3 and 4) the so-called "rectangularization" of the survival curve clearly emerges, meaning an increasing concentration of deaths around

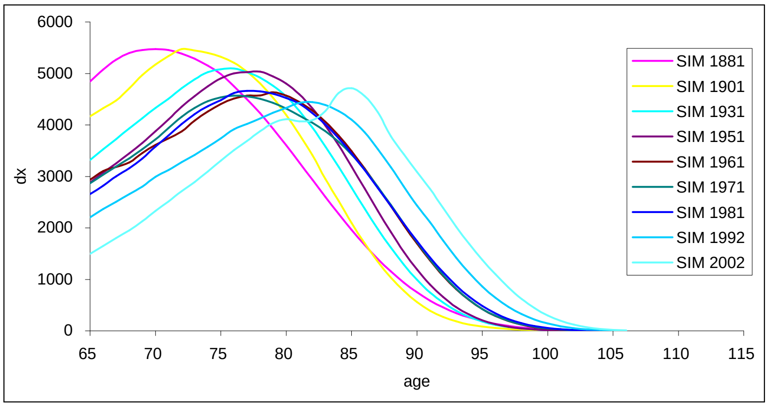

Figure 14: *Curves of deaths referred to people alive at age 65*

the modal age (the Lexis point). Together with the rectangularization, the "expansion" occurs, leading in particular to an increase in the modal value.

	SIM 1881	SIM 1901	SIM 1931	SIM 1951	SIM 1961	SIM 1971	SIM 1981	SIM 1992	SIM 2002
Me[T_{65}]	74.45827	75.09749	76.55215	77.42349	78.21735	77.94686	78.27527	80.23987	82.20066
x_{25}[T_{65}]	69.80944	70.45377	71.45070	72.16008	72.43802	72.32797	72.65518	73.89806	75.73235
x_{75}[T_{65}]	79.95515	80.14873	81.80892	82.63073	83.86049	83.84586	83.96275	86.02055	87.83705
IQR[T_{65}]	10.14570	9.694965	10.35822	10.47065	11.42247	11.51789	11.30757	12.12249	12.10470

Table 6: *Probability distribution of the remaining lifetime at age 65. Some markers*

Clearly, the rectangularization implies a decreasing importance of random fluctuations in mortality, when the whole range of ages is addressed. However, this feature of mortality trends does not impact on riskiness of (immediate) life annuities. Actually, if we consider the probability distribution of the remaining lifetime, say at age 65, i.e. in terms of $\frac{d_x}{l_{65}}$ $(x \geq 65)$, we find a rather stable or even increasing variability in the age of death. This fact clearly emerges from Figure 14, which illustrates the probability distributions of the random lifetime T_{65}, corresponding to various period observations. Table 6 shows the behavior throughout time of the median Me[T_{65}], the $25th$ and $75th$ percentiles, $x_{25}[T_{65}]$ and $x_{75}[T_{65}]$, and the interquartile range $\mathrm{IQR}[T_{65}] = x_{75}[T_{65}] - x_{25}[T_{65}]$.

From these arguments, it follows that the process risk should not be disregarded when portfolios of life annuities are concerned. In particular, small portfolios (and pension funds) still require a sound assessment of the impact of mortality random fluctuations.

5 Risk mitigation

Let now return to RM techniques and, in particular, to portfolio strategies aiming at risk mitigation. Because of the complexity of the problem, we just refer to a portfolio of immediate annuities, consisting in one cohort of annuitants.

A number of portfolio results can be taken as "metrics" to assess the effectiveness of portfolio strategies. In what follows, we focus on cash flows, which anyhow constitute the starting point from which other quantities are derived (e.g. profits), as it clearly appears in Figure 1. As we refer to single-premium annuities, cash flows are actually outflows; disregarding expenses, outflows are originated only by the payment of benefits.

In Figure 15 a sequence of outflows is represented, together with a barrier (the "threshold") which represents a maintainable level of benefit payment. The threshold amount is financed first by (single) premiums via the portfolio mathematical reserve, and by shareholders' capital as the result of the allocation policy (consisting in specific capital allocations as well as accumulation of undistributed profits).

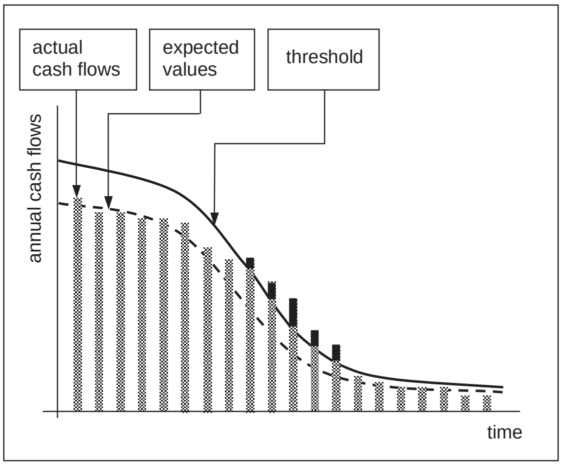

Figure 15: *Annual cash flows in an annuity portfolio (one cohort)*

The situation occurring in Figure 15, namely some annual cash flows above the threshold level, should be clearly avoided. To lower the probability of such critical situations, the insurer can resort to various portfolio strategies, in the framework of the RM process (see Figure 2).

Figure 16 illustrates a wide range of portfolio strategies aiming at risk mitigation, meant as lowering the probability and the severity of events like the situation depicted in Figure 15. In practical terms, a portfolio strategy can have as targets

(a) an uplift of the maintainable annual cash flow, thus a higher threshold level;

(b) lower (and smoother) annual cash flows in the case of unanticipated improvements in portfolio mortality.

Both **loss control** and **loss financing** techniques (according to the RM language) can be adopted to achieve targets (a) and (b). Loss control techniques are mainly performed via product design, i.e. via an appropriate choice of the various items constituting an insurance product. In particular, **loss prevention** is usually meant as the RM technique aiming at mitigating the loss frequency, whereas **loss reduction** aims at lowering the severity of the possible losses.

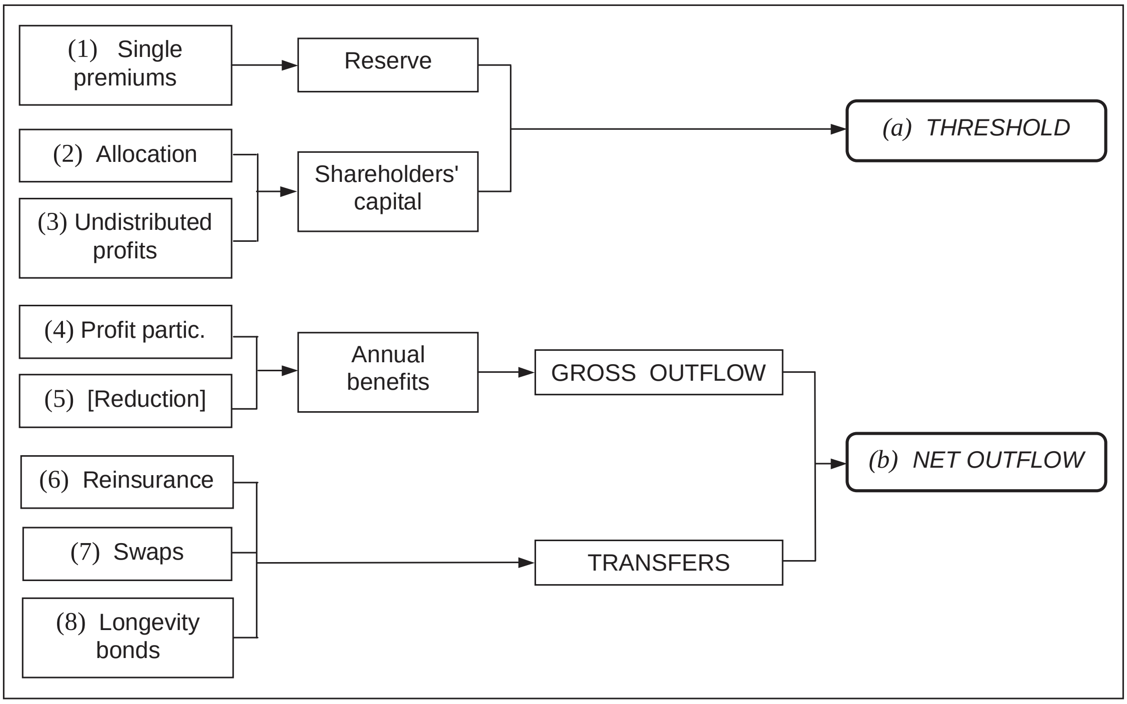

Figure 16: *Portfolio strategies for risk mitigation*

The pricing of insurance products provides a tool for loss prevention. This portfolio strategy is represented by path (1) $\rightarrow$ (a) in Figure 16. Referring to a life annuity product, the following issues in particular should be taken into account.

- Mortality improvements require the use of a projected life table for pricing life annuities, as the hypothesis of "static" mortality should be rejected (see Section 4.1).

- Because of uncertainty in future mortality trend, a premium principle other than the traditional equivalence principle should be adopted. It should be noted that, adopting the equivalence principle, the longevity risk can be accounted for only via a (rough) safety loading, constructed by increasing the survival probabilities resulting from the projected table. Actually, this approach is often adopted in actuarial practice.

- The presence, in an accumulation product such as an endowment, of an option to annuitize at a fixed annuitization rate (the so-called Guaranteed Annuity Option) requires an accurate pricing model accounting for the value of the option itself.

In order to pursue loss reduction, it is necessary to control the amounts of benefits paid. Hence, some flexibility must be added to the annuity product. One action could be the reduction of the annuity amount as a consequence of an unanticipated mortality improvement (path (5) $\rightarrow$ (b) in Figure 16); however, in this case the product would be a non-guaranteed annuity (though with a reasonable minimum amount guaranteed). A more practicable tool, consistent with the features of a guaranteed annuity, consists in reducing the level of investment profit participation while a poor mortality is experienced (path (4) $\rightarrow$ (b)); it is worth stressing that undistributed profits also increase the shareholders' capital within the portfolio, hence uplifting the maintainable threshold (see (3) $\rightarrow$ (a)).

Loss financing techniques require specific strategies involving the whole portfolio, and in some cases even other portfolios of the insurance company. Risk transfer can be realized via (traditional) reinsurance arrangements (see path (6) $\rightarrow$ (b)), swap-like reinsurance ((7) $\rightarrow$ (b)) and securitization, i.e. Alternative Risk Transfers (ART). ART require, when life annuities are concerned, the use of specific financial instruments, e.g. longevity bonds ((8) $\rightarrow$ (b)) whose performance is linked to some measure of longevity in a given population.

It is worth briefly describing the swap-like reinsurance arrangement. Let $R_t^{(\Pi)}$ denote the portfolio annuity payment at time t. Assume that two barriers are stated in relation with the cedant's annual outflow: a lower barrier Λ_t' and un upper barrier Λ_t''. The upper barrier is stated in relation to the threshold level, and in particular can coincide with it. Let

$$B_t^{(R)} = \begin{cases} R_t^{(\Pi)} - \Lambda_t' & \text{if } R_t^{(\Pi)} \leq \Lambda_t' \\ 0 & \text{if } \Lambda_t' < R_t^{(\Pi)} \leq \Lambda_t'' \\ R_t^{(\Pi)} - \Lambda_t'' & \text{if } R_t^{(\Pi)} > \Lambda_t'' \end{cases} \tag{5.1}$$

The swap-like arrangement is defined as follows. If $B_t^{(R)} > 0$ the cedant receives from the reinsurer $B_t^{(R)}$, else, if $B_t^{(R)} < 0$, the cedant pays to the reinsurer $B_t^{(R)}$. Hence, the net outflow of the cedant is given by

$$F_t = R_t^{(\Pi)} - B_t^{(R)} = \begin{cases} \Lambda_t' & \text{if } R_t^{(\Pi)} \leq \Lambda_t' \\ R_t^{(\Pi)} & \text{if } \Lambda_t' < R_t^{(\Pi)} \leq \Lambda_t'' \\ \Lambda_t'' & \text{if } R_t^{(\Pi)} > \Lambda_t'' \end{cases} \tag{5.2}$$

The interest of this reinsurance arrangement is mainly due to the possibility, for the reinsurer, of hedging the risk taken from the cedant via transfer to the capital market, namely via longevity bonds.

Some additional comments on risk transfers are worthwhile. Traditional reinsurance arrangements (e.g. surplus reinsurance, XL reinsurance, etc.) can be applied also to annuity portfolios, at least in principle (for example, see Olivieri [21]). Anyhow, it should be stressed that risk transfer via traditional reinsurance mainly relies on the improved diversification of risks when these are taken by the reinsurer, thanks to a stronger pooling effect.

However, such an improvement can be achieved in relation to process risk (viz. mortality random fluctuations), whilst uncertainty risk (leading to systematic deviations) cannot be diversified "inside" the insurance-reinsurance process. Hence, to gain effectiveness reinsurance transfer must be completed with a further transfer, i.e. a transfer to capital markets. Such a transfer can be realized via bonds, whose yield is linked to some mortality/longevity index, so that the bonds themselves generate flows hedging the payment of benefits. While mortality bonds (hedging the risk of a mortality higher than expected) already exist, longevity bonds (hedging the risk of a mortality lower than expected, and hence of interest in relation to annuity portfolios) constitute a new issue. For detailed information on this topic the reader can refer, for example, to Blake *et al.* [4]. Under a financial perspective, interest in such bonds clearly relies on a likely absence of correlation with other investment yields.

To the extent that mortality/longevity risks are retained by an insurance company, the impact of a poor experience falls on the company itself. To meet an unexpected amount of obligations, an appropriate advance funding may provide a substantial help. To this purpose, shareholders' capital must be allocated to the annuity portfolio (see $(2) \rightarrow (a)$, as well as $(3) \rightarrow (a)$), and the relevant amount should be determined aiming at insurer solvency. Conversely, the expression "no advance funding" (see Figure 2) should be meant as no specific capital allocation facing the risks, whose impact will be (at least partially) met thanks to the capital required by legislation.

Hedging strategies in general consist in assuming a risk offsetting another risk borne by the insurer. In some cases, hedging strategies involve various portfolios or lines of business (LOBs), or even the whole insurance company, whence they cannot be placed in the portfolio framework as depicted by Figure 16.

In particular, a "natural" hedging (see Figure 2) consists in offsetting risks in different LOBs. For example, writing both life insurance providing death benefits and life annuities for similar groups of policyholders may help to provide a hedge against longevity risk. Such a hedge is usually named "across LOBs".

A natural hedge can be realised even inside an annuity portfolio, with the proviso that the product is no longer just a straight life annuity. Assume that the product consists in a life annuity combined with a death benefit with an amount decreasing as the age at death increases. Clearly, in the case of a mortality improvement higher than anticipated, death benefits lower than expected will be paid. Such a hedge is usually named "across time".

Clearly, mortality/longevity risks should be managed by the insurer via an appropriate mix of the tools described above. The choice of the RM tools is also driven by various interrelationships among the tools themselves. For example, the possibility of purchasing profitable reinsurance is strictly related with the features of the insurance products and, in particular, the life tables underlying the pricing as well as with the availability of ART for the reinsurer.

6 Concluding remarks

Among the risks which affect life insurance and annuity portfolios, both investment risks and mortality risks deserve careful analysis and require the adoption of proper management solutions.

Literature on investment risks is very rich. Several tools have been proposed in this respect, and implemented in practice as well. Conversely, the analysis of mortality risks still require investigations, especially as far as the risk of systematic deviations is concerned.

When assessing the risk profile of a life insurer, riskiness arising from the behavior of mortality should be analyzed via appropriate tools. In particular, the traditional deterministic approach to mortality modeling should be rejected and replaced by a stochastic approach allowing for both process risk and uncertainty risk.

In this paper, two examples of stochastic mortality modeling have been presented and discussed, concerning death benefits and life annuities respectively. While the latter focusses on the well known problem of longevity risk (meant as uncertainty risk), the former deals with the risk of random fluctuations in a portfolio of insurance products with a positive sum at risk.

Although in practice the importance of mortality risks is often underestimated when dealing, for example, with endowments or term assurances, and a deterministic approach to mortality is consequently adopted, it should be stressed that an appropriate assessment of the insurer's risk profile should account for all types of risks, not disregarding, in particular, the effectiveness of the pooling effect when small or heterogeneous portfolios are addressed.

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