ACTUARIAL FUNDING OF DISMISSAL AND RESIGNATION RISKS

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Abstract

Besides the usual pension benefits the pension plan of a firm may be forced by law in some countries to offer wage based lump sum payments by death, retirement and dismissal by the employer, but no payment by the employer is made when due to resignation by the employee. An actuarial risk model for funding severance payment liabilities is formulated and studied. The yearly aggregate lump sum payments are supposed to follow a classical collective model of risk theory with compound distributions. The final wealth at an arbitrary time is described explicitly including formulas for the mean and variance. Annual initial level premiums required for "dismissal funding" are determined and useful gamma approximations for confidence intervals of the wealth are proposed. A specific numerical example illustrates the non-negligible probability of a bankruptcy in case the employee structure of a "dismissal plan" is not well balanced.

Key words

ALM, actuarial funding, dismissal risk, resignation risk, compound distributions

1. **Introduction**.

In some countries, for example Austria, modern social legislation stipulates besides usual pension benefits fixed wage based lump sum payments by death and retirement as well as through dismissal by the employer of a firm, so-called severance payments (see e.g. Holzmann et al.(2003), Koman et al.(2005), Grund(2006), "Abfertigung neu"(2002), Abfertigung neu und alt"(2005), "Abfindung im Arbeitsrecht"(2007)). However, if the contract terminates due to resignation by the employee, no lump sum payment is made by the employer. In this situation, there are four causes of decrement, which have a random effect on the actuarial funding of the additional liabilities in the pension plan, called here "dismissal plan".

We are interested in actuarial risk models, which are able to describe all random lump sum payments until retirement for the dismissal plan of a firm. The aggregate lump sum payments in each year are supposed to follow a classical collective model of risk theory with compound distributions. The evaluation of the mean and standard deviation of these yearly payments requires a separate analysis of the four causes of decrement. Details are found in Section 2.

Actuarial funding with dismissal payments is based on the dynamic stochastic evolution of the random wealth of the dismissal fund at a specific time. The final wealth at the end of a time horizon can be described explicitly, and formulas for the mean and variance are obtained. In particular, given the initial capital of the dismissal fund as well as the funding capital, which should be available at the end of a time horizon to cover all expected future random lump sum payments until retirement of all employees, it is possible to determine the required annual initial level premium necessary for dismissal funding. This is described in Section 3.

Section 4 considers the dynamic stochastic evolution of the random wealth at an arbitrary time and proposes a useful gamma approximation for confidence intervals of the wealth.

The final Section 5 is devoted to the analysis of a specific numerical example, which illustrates the non-negligible probability of a bankruptcy of a dismissal fund in case the employee structure is not well balanced.

2. **The dismissal and resignation causes of decrement**.

Consider the "dismissal plan" of a firm, which offers wage based lump sum payments by death and retirement as well as through dismissal by the employer. However, if the contract terminates due to resignation by the employee, no lump sum payment is made by the employer. In this situation, there are four causes of decrement, which have a random effect on the dismissal funding. They are described as follows:

- dismissal by the employer with a probability PD_x at age x
- resignation by the employee with a probability PR_x at age x
- death of the employee with a probability PT_x at age x
- survival to the deterministic retirement age s with a probability PS_x^s at age x

Survival to retirement age s of an employee aged x happens if the employee does not die and there is neither dismissal by the employer nor resignation by the employee. The probability of this event depends on the probabilities that an employee aged x survives to age $x+k$, namely

$$_{k}PS_{x}=\prod_{j=0}^{k-1}\left(1-PD_{x+j}-PR_{x+j}-PT_{x+j}\right),\quad {}_{0}PS_{x}=1, \tag{2.1}$$

and equals $PS_{x}^{s}={}_{s-x}PS_{x}$. Note that if an employee attains the common retirement age s, then retirement payment due to survival takes place and neither dismissal, resignation and death is possible. Therefore it can be assumed that $PD_{x}=PR_{x}=PT_{x}=0$ for all $x\geq s$.

We consider an actuarial risk model, which describes all random lump sum payments until retirement for the dismissal plan of a firm with M employees at the initial time of valuation $t=0$. For a longer *time horizon* H, say 25 or 30 years, and for an *initial capital* K_{0}, let P be the *annual initial level premium* of the dismissal fund required to reach at fixed *interest rate* i the *funding capital* K_{H} at time H. The latter quantity is supposed to cover at time H all expected future random payments until retirement of all employees (see formula (3.9)). The introduced (overall) funding premium should not be confused with the individual contributions of the employees for their benefits, which may vary between employees. Since lump sum payments are proportional to the wages of the employees, it is assumed that the annual premium increases proportionally to the wages. With an *annual wage increase* of $100\cdot g\%$ the annual premium at time t reads

$$P_{t}=P\cdot(1+g)^{t-1},\quad t=1,\dots,H. \tag{2.2}$$

Let X_{t} be a time dependent random variable, which represents the *aggregate lump sum payments* in year t due to the above four causes of decrement. We assume that this random variable can be described by a random sum of the type

$$X_{t}=\sum_{j=1}^{N_{t}}Y_{t,j},\quad t=1,\dots,H, \tag{2.3}$$

where N_{t} counts the *number of employee withdrawals* due to any cause of decrement, and $Y_{t,j}$ is the *individual random lump sum payment* given the *j*-th withdrawal occurs. Under the assumption of a collective model of risk theory the $Y_{t,j}$ are independent and identically distributed like a random variable Y_{t}, and they are independent from N_{t}. Assuming that N_{t} has mean $\lambda_{t}=E[N_{t}]$ and standard deviation $\sigma_{N_{t}}$, the mean $\mu_{X_{t}}$ and the standard deviation $\sigma_{X_{t}}$ of X_{t} are given by (e.g. Beard et al.(1984), Chapter 3, Bowers et al.(1986), Chapter 11, Panjer and Willmot(1992), Chapter 6, Kaas et al.(2001), Chapter 3, etc.)

$$\mu_{X_{t}}=\lambda_{t}\cdot\mu_{Y_{t}},\quad \sigma_{X_{t}}=\sqrt{\lambda_{t}\cdot\sigma_{Y_{t}}{}^{2}+\sigma_{N_{t}}{}^{2}\cdot\mu_{Y_{t}}{}^{2}}, \tag{2.4}$$

where μ_{Y_t} and σ_{Y_t} denote the mean and standard deviation of Y_t. The evaluation of these quantities requires a separate analysis for each of the four causes of decrement.

Dismissal by the employer

Let N_t^D be the random number of dismissals in year t, and let $Y_{t,j}^D \sim Y_t^D$ be the independent and identically distributed individual random lump sum payments in year t given the *j*-th dismissal by the employer occurs. If A_k is the age of the employee number k at the initial time of valuation, then the expected number of dismissals in year t equals

$$\lambda_t^D = E\left[N_t^D\right] = \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot PD_{A_k+t-1} \,, \tag{2.5}$$

Consider the probability of dismissal of an employer in year t given a population of M employees at initial time defined by the ratio

$$p_t^D = \frac{\lambda_t^D}{M} \,. \tag{2.6}$$

Since decrement by the cause of dismissal follows a binomial distribution with parameter p_t^D, the variance of the number of dismissals is given by

$$\sigma_{N_t^D}^2 = M \cdot p_t^D \cdot \left(1 - p_t^D\right) = \frac{\lambda_t^D \cdot \left(M - \lambda_t^D\right)}{M} \,. \tag{2.7}$$

Furthermore, suppose that at the initial time of valuation, it is known that by dismissal the k-th employee will receive the lump sum payment $B_{0,k}$. Since the lump sum payment are wage based and the wages increase at the rate $100 \cdot g\%$, the effective lump sum payment in year t equals $B_{0,k}\left(1+g\right)^{t-1}$. Under the assumption of a compound distributed model for the aggregate lump sum payments due to dismissal by the employer, that is $X_t^D = \sum_{j=1}^{N_t^D} Y_{t,j}^D, \quad t = 1,...,H$, it follows that the mean and variance of Y_t^D are given by

$$\begin{aligned} \mu_t^D &= E\left[Y_t^D\right] = \frac{1}{\lambda_t^D} \cdot E\left[X_t^D\right], \\ [\sigma_t^D]^2 &= Var\left[Y_t^D\right] = \frac{1}{\lambda_t^D} \cdot \left(Var\left[X_t^D\right] - \sigma_{N_t^D}^2 \cdot [\mu_t^D]^2\right), \end{aligned} \tag{2.8}$$

where the mean and variance of the aggregate lump sum payments are obtained from

$$\begin{aligned} E\left[X_t^D\right] &= \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot PD_{A_k+t-1} \cdot B_{0,k} \cdot (1+g)^{t-1}, \\ Var\left[X_t^D\right] &= \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot PD_{A_k+t-1} \cdot \left[B_{0,k} \cdot (1+g)^{t-1}\right]^2 - E\left[X_t^D\right]^2. \end{aligned} \tag{2.9}$$

Resignation by the employee

The evaluation is similar to the situation of dismissal by the employer, with the difference that the foreseen lump sum payment is released to the remaining beneficiaries of the dismissal fund. Let N_t^R be the random number of resignations in year t, and let $Y_{t,j}^R \sim Y_t^R$ be the independent and identically distributed individual random lump sum payments in year t given the j-th resignation by the employee occurs. Again we assume a compound distributed model for the aggregate lump sum payments due to resignation by the employee, that is $X_t^R = \sum_{j=1}^{N_t^R} Y_{t,j}^R, \quad t = 1,\dots,H$. The expected number of resignations in year t equals

$$\lambda_t^R = \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot PR_{A_k+t-1}\,. \tag{2.10}$$

The probability of resignation of an employer in year t given a population of M employees at initial time is defined by the ratio

$$p_t^R = \frac{\lambda_t^R}{M}\,. \tag{2.11}$$

Since decrement by the cause of resignation follows a binomial distribution with parameter p_t^R, the variance of the number of resignations is given by

$$\sigma_{N_t^R}^2 = M \cdot p_t^R \cdot \left(1 - p_t^R\right) = \frac{\lambda_t^R \cdot \left(M - \lambda_t^R\right)}{M}\,. \tag{2.12}$$

The mean and variance of Y_t^R are given by

$$\begin{aligned} \mu_t^R &= E\left[Y_t^R\right] = \frac{1}{\lambda_t^R} \cdot E\left[X_t^R\right], \\ [\sigma_t^R]^2 &= Var\left[Y_t^R\right] = \frac{1}{\lambda_t^R} \cdot \left(Var\left[X_t^R\right] - \sigma_{N_t^R}^2 \cdot [\mu_t^R]^2\right), \end{aligned} \tag{2.13}$$

where the mean and variance of the aggregate lump sum payments are obtained from

$$\begin{aligned} E\left[X_t^R\right] &= \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot PR_{A_k+t-1} \cdot B_{0,k} \cdot (1+g)^{t-1}, \\ Var\left[X_t^R\right] &= \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot PR_{A_k+t-1} \cdot \left[B_{0,k} \cdot (1+g)^{t-1}\right]^2 - E\left[X_t^R\right]^2. \end{aligned} \tag{2.14}$$

Death of the employee

Suppose that by death of an employee the portion θ of the dismissal payment is due to its legal survivor ($\theta = \frac{1}{2}$ in our numerical example). Let N_t^T be the random number of deaths in year t, and let $Y_{t,j}^T \sim Y_t^T$ be the independent and identically distributed individual random lump sum payments in year t given the j-th death occurs. We assume a compound distributed model for the aggregate lump sum payments due to death of an employee, that is $X_t^T = \sum_{j=1}^{N_t^T} Y_{t,j}^T, \quad t = 1,...,H$. The expected number of deaths in year t equals

$$\lambda_t^T = \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot PT_{A_k+t-1} . \tag{2.15}$$

The probability of death of an employer in year t given a population of M employees at initial time is defined by the ratio

$$p_t^T = \frac{\lambda_t^T}{M} . \tag{2.16}$$

Since decrement by the cause of death follows a binomial distribution with parameter p_t^T, the variance of the number of deaths is given by

$$\sigma_{N_t^T}^2 = M \cdot p_t^T \cdot \left(1 - p_t^T\right) = \frac{\lambda_t^T \cdot \left(M - \lambda_t^T\right)}{M} . \tag{2.17}$$

The mean and variance of Y_t^T are given by

$$\begin{aligned} \mu_t^T &= E\left[Y_t^T\right] = \frac{1}{\lambda_t^T} \cdot E\left[X_t^T\right], \\ [\sigma_t^T]^2 &= Var\left[Y_t^T\right] = \frac{1}{\lambda_t^T} \cdot \left(Var\left[X_t^T\right] - \sigma_{N_t^T}^2 \cdot [\mu_t^T]^2\right), \end{aligned} \tag{2.18}$$

where the mean and variance of the aggregate lump sum payments are obtained from

$$E\left[X_t^T\right]=\sum_{k=1}^{M}{}_{t-1}PS_{A_k}\cdot PT_{A_k+t-1}\cdot\theta\cdot B_{0,k}\cdot(1+g)^{t-1},$$
$$Var\left[X_t^T\right]=\sum_{k=1}^{M}{}_{t-1}PS_{A_k}\cdot PT_{A_k+t-1}\cdot\left[\theta\cdot B_{0,k}\cdot(1+g)^{t-1}\right]^2-E\left[X_t^T\right]^2. \qquad (2.19)$$

Survival to the retirement age

Let N_t^S be the random number of retirements in year t, and let $Y_{t,j}^S \sim Y_t^S$ be the independent and identically distributed individual random lump sum payments generated upon retirement of the *j*-th employee in year t. Taking into account that an employee numbered k and aged A_k attains retirement in year t such that $A_k+t-1=s$ and using the definition of the retirement probability $PS_x^s={}_{s-x}PS_x$, one obtains for the expected number of retirements in year t

$$\lambda_t^S=\sum_{k=1}^{M}{}_{t-1}PS_{A_k}\cdot I\left(A_k+t-1=s\right), \qquad (2.20)$$

where $I(\cdot)$ is an indicator function such that $I(W)=1$ if the statement W is true and $I(W)=0$ else. The probability of survival to the retirement age of an employer in year t given a population of M employees at initial time is defined by the ratio

$$p_t^S=\frac{\lambda_t^S}{M}. \qquad (2.21)$$

Since decrement by the cause of survival to retirement follows a binomial distribution with parameter p_t^S, the variance of the number of retirements is given by

$$\sigma_{N_t^S}^2=M\cdot p_t^S\cdot\left(1-p_t^S\right)=\frac{\lambda_t^S\cdot\left(M-\lambda_t^S\right)}{M}. \qquad (2.16)$$

Furthermore, suppose that at the initial time of valuation, it is known that at retirement the *k*-th employee will receive the lump sum payment C_k. Due to wages increase the effective sum in year t equals $C_k(1+g)^{t-1}$. Again, assume a compound distributed model for the aggregate lump sum payments due to retirement, that is $X_t^S=\sum_{j=1}^{N_t^S}Y_{t,j}^S,\quad t=1,...,H$. The mean and variance of Y_t^S are given by

$$\begin{aligned} \mu_t^S &= E\left[Y_t^S\right] = \frac{1}{\lambda_t^S} \cdot E\left[X_t^S\right], \\ [\sigma_t^S]^2 &= Var\left[Y_t^S\right] = \frac{1}{\lambda_t^S} \cdot \left(Var\left[X_t^S\right] - \sigma_{N_t^S}^2 \cdot [\mu_t^S]^2\right), \end{aligned} \tag{2.17}$$

where the mean and variance of the aggregate lump sum payments are obtained from

$$\begin{aligned} E\left[X_t^S\right] &= \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot I\left(A_k + t - 1 = s\right) \cdot C_k \cdot (1+g)^{t-1}, \\ Var\left[X_t^S\right] &= \sum_{k=1}^{M} {}_{t-1}PS_{A_k} \cdot I\left(A_k + t - 1 = s\right) \cdot \left[C_k \cdot (1+g)^{t-1}\right]^2 - E\left[X_t^S\right]^2. \end{aligned} \tag{2.18}$$

The above preliminaries are used to obtain the characteristics (2.4) as follows. The expected number of employee withdrawals in year t due to all four causes of decrement equals

$$\lambda_t = \lambda_t^D + \lambda_t^R + \lambda_t^T + \lambda_t^S. \tag{2.19}$$

Denote by M_t the number of remaining employees in year t. Starting with an initial number M of employees one has $M_0 = M$ and for year $t > 1$ one has $M_t = M_{t-1} - \lambda_t$, which shows that the expected number of remaining employees decreases over time, as should be. The individual lump sum payment in year t satisfies the following equation

$$\lambda_t \cdot Y_t = \lambda_t^D \cdot Y_t^D - \lambda_t^R \cdot Y_t^R + \lambda_t^T \cdot Y_t^T + \lambda_t^S \cdot Y_t^S. \tag{2.21}$$

Indeed, the aggregate lump sum payments in year t are the sum of the payments due to dismissal by the employee, death and retirement less the payments due to resignation of employees. Under the assumption of independence of the different random variables, one obtains for the mean and variance of Y_t the formulas

$$\begin{aligned} \mu_{Y_t} &= E\left[Y_t\right] = \frac{1}{\lambda_t}\left(\lambda_t^D \cdot \mu_t^D - \lambda_t^R \cdot \mu_t^R + \lambda_t^T \cdot \mu_t^T + \lambda_t^S \cdot \mu_t^S\right) \\ \sigma_{Y_t}^2 &= Var\left[Y_t\right] = (\sigma_t^D)^2 + (\sigma_t^R)^2 + (\sigma_t^T)^2 + (\sigma_t^S)^2 \end{aligned}. \tag{2.22}$$

Moreover the variance of the number of withdrawals in year t due to all four causes of decrement is given by

$$\sigma_{N_t}^2 = Var\left[N_t\right] = \sigma_{N_t^D}^2 + \sigma_{N_t^R}^2 + \sigma_{N_t^T}^2 + \sigma_{N_t^S}^2 \tag{2.23}$$

The characteristics (2.4) follow immediately by inserting the formulas (2.19), (2.22) and (2.23).

3. An asset and liability model for dismissal funding.

Let W_t be the random wealth of the dismissal fund at time t, where $t=0$ is the initial time of valuation. The random rate of return on investment in year t is denoted I_t. The wealth at time t satisfies the recursive equation

$$W_t = (W_{t-1} + P_t - X_t) \cdot (1 + I_t). \tag{3.1}$$

Taking into account (2.1), the final wealth at the time horizon H is given by

$$W_H = W_0 \cdot \prod_{t=1}^{H} (1+I_t) + \sum_{t=1}^{H} \left\{ P(1+g)^{t-1} - X_t \right\} \cdot \prod_{j=t}^{H} (1+I_j). \tag{3.2}$$

It is clear that the initial wealth coincides with the initial capital, that is $W_0 = K_0$. For simplicity, assume that the accumulated rates of return in year t are independent and identically log-normally distributed such that

$$1 + I_t = \exp(Z_t), \tag{3.3}$$

where Z_t is normally distributed with mean μ and standard deviation σ. Consider the products

$$\prod_{j=t}^{H} (1+I_j) = \exp(Z_{t,H}), \quad 1 \le t \le H, \tag{3.4}$$

which represent the accumulated rates of return over the time period $[t-1, H]$, where the sums $Z_{t,H} = \sum_{j=t}^{H} Z_j$ are normally distributed with mean and standard deviation

$$\mu_{t,H} = E[Z_{t,H}] = (H-t+1) \cdot \mu, \quad \sigma_{t,H} = \sqrt{Var[Z_{t,H}]} = \sqrt{H-t+1} \cdot \sigma. \tag{3.5}$$

The mean and variance of the final wealth are given by the following result.

Theorem 3.1. Under the simplifying assumption that the random rates of return $I_1, ..., I_H$ are independent from the aggregate lump sum payments X_t, the mean of the final wealth is given by the expression

$$E[W_H] = K_0 \cdot r^H + P \cdot r \cdot \frac{r^H - (1+g)^H}{r - (1+g)} - \sum_{t=1}^{H} \mu_{X_t} \cdot r^{H-t+1} \tag{3.6}$$

and the variance of the final wealth by the formula

$$\begin{aligned}
Var\left[W_H\right] &= K_0^2 \cdot r^{2H} \cdot \left(e^{H\sigma^2} - 1\right) + K_0 \cdot \sum_{t=1}^{H} r^{2H-t+1} \cdot \left(P(1+g)^{t-1} - \mu_{X_t}\right) \cdot \left(e^{(H-t+1)\sigma^2} - 1\right) \\
&+ \sum_{t=1}^{H} r^{2(H-t+1)} \cdot \left\{\left(P(1+g)^{t-1} - \mu_{X_t}\right)^2 \cdot \left(e^{(H-t+1)\sigma^2} - 1\right) + \sigma_{X_t}^2 \cdot e^{(H-t+1)\sigma^2}\right\} \\
&+ 2 \cdot \sum_{1 \le s < t \le H} r^{2H-t-s+2} \cdot \left(P(1+g)^{s-1} - \mu_{X_s}\right) \cdot \left(P(1+g)^{t-1} - \mu_{X_t}\right) \cdot \left(e^{(H-t+1)\sigma^2} - 1\right)
\end{aligned} \qquad (3.7)$$

where $r = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$ is the one-year *risk-free accumulated rate of return* over the time horizon $[0, H]$.

Proof. Using the notation (3.4) the expression (3.2) can be rewritten as

$$W_H = W_0 \cdot \exp\left(Z_{1,H}\right) + \sum_{t=1}^{H} \left\{P(1+g)^{t-1} - X_t\right\} \cdot \exp\left(Z_{t,H}\right), \qquad (3.8)$$

from which one gets without difficulty (3.6). To get the expression for the variance, several terms must be calculated. One has

$$Var\left[W_0 \cdot \exp\left(Z_{1,H}\right)\right] = K_0^2 \cdot \left(e^{2H\left(\mu+\sigma^2\right)} - e^{H\left(2\mu+\sigma^2\right)}\right) = K_0^2 \cdot r^{2H} \cdot \left(e^{H\sigma^2} - 1\right).$$

For $1 \le t \le H$ one has

$$\begin{aligned}
& Var\left[\left(P(1+g)^{t-1} - X_t\right) \cdot \exp\left(Z_{t,H}\right)\right] \\
&= Var\left[E\left[\left(P(1+g)^{t-1} - X_t\right) \cdot \exp\left(Z_{t,H}\right) \middle| Z_{t,H}\right]\right] + E\left[Var\left[\left(P(1+g)^{t-1} - X_t\right) \cdot \exp\left(Z_{t,H}\right) \middle| Z_{t,H}\right]\right] \\
&= \left(P(1+g)^{t-1} - \mu_{X_t}\right)^2 \cdot Var\left[\exp\left(Z_{t,H}\right)\right] + \sigma_{X_t}^2 \cdot E\left[\exp\left(2 \cdot Z_{t,H}\right)\right] \\
&= \left(P(1+g)^{t-1} - \mu_{X_t}\right)^2 \cdot \left(e^{2(H-t+1)\left(\mu+\sigma^2\right)} - e^{(H-t+1)\left(2\mu+\sigma^2\right)}\right) + \sigma_{X_t}^2 \cdot e^{2(H-t+1)\left(\mu+\sigma^2\right)} \\
&= r^{2(H-t+1)} \cdot \left\{\left(P(1+g)^{t-1} - \mu_{X_t}\right)^2 \cdot \left(e^{(H-t+1)\sigma^2} - 1\right) + \sigma_{X_t}^2 \cdot e^{(H-t+1)\sigma^2}\right\}
\end{aligned}$$

For $1 \le s < t \le H$ one has

$$\begin{aligned}
& Cov\left[\left(P(1+g)^{s-1} - X_s\right) \cdot \exp\left(Z_{s,H}\right), \left(P(1+g)^{t-1} - X_t\right) \cdot \exp\left(Z_{t,H}\right)\right] \\
&= \left(P(1+g)^{s-1} - \mu_{X_s}\right) \cdot \left(P(1+g)^{t-1} - \mu_{X_t}\right) \cdot Cov\left[\exp\left(Z_{s,H}\right), \exp\left(Z_{t,H}\right)\right]
\end{aligned}$$

where for the covariance term one gets

$$\begin{aligned}&Cov\left[\exp\left(Z_{s,H}\right),\exp\left(Z_{t,H}\right)\right]=Cov\left[E\left[\exp\left(Z_{s,H}\right)\middle|Z_{s,H}-Z_{t,H}\right],E\left[\exp\left(Z_{t,H}\right)\middle|Z_{s,H}-Z_{t,H}\right]\right]\\&+E\left[Cov\left[\exp\left(Z_{s,H}\right),\exp\left(Z_{t,H}\right)\middle|Z_{s,H}-Z_{t,H}\right]\right]=E\left[\exp\left(Z_{s,H}-Z_{t,H}\right)\right]\cdot Var\left[\exp\left(Z_{t,H}\right)\right]\\&=e^{(t-s)\left(\mu+\frac{1}{2}\sigma^2\right)}\cdot\left(e^{2(H-t+1)\left(\mu+\sigma^2\right)}-e^{(H-t+1)\left(2\mu+\sigma^2\right)}\right)=r^{2H-t-s+2}\cdot\left(e^{(H-t+1)\sigma^2}-1\right)\end{aligned}$$

In a similar way, for $1\le t\le H$ one has

$$\begin{aligned}&Cov\left[W_0\cdot\exp\left(Z_{1,H}\right),\left(P(1+g)^{t-1}-X_t\right)\cdot\exp\left(Z_{t,H}\right)\right]\\&=K_0\cdot\left(P(1+g)^{t-1}-\mu_{X_t}\right)\cdot Cov\left[\exp\left(Z_{1,H}\right),\exp\left(Z_{t,H}\right)\right]\\&=K_0\cdot\left(P(1+g)^{t-1}-\mu_{X_t}\right)\cdot r^{2H-t+1}\cdot\left(e^{(H-t+1)\sigma^2}-1\right)\end{aligned}$$

Gathering all terms together and summing appropriately one obtains finally (3.7). ◊

Note that the risk-free rate of return r must be realized in order to guarantee with certainty the expected final wealth. We are now interested in the determination of the required premium for dismissal funding with dismissal payments. Let K_0 be the initial capital of the dismissal fund. Suppose that at time H the funding capital K_H should be available in order to cover all expected future random lump sum payments until retirement of all employees. If $H_{\max}$ denotes the maximum time horizon at which all employees from the initial population of M employees have been retired with certainty, then the required funding capital is given by

$$K_H=\sum_{t=H}^{H_{\max}}r^{H_{\max}-t}\cdot\mu_{X_t}\ , \tag{3.9}$$

where r is a fixed one-year guaranteed accumulated rate of return. Setting this quantity equal to the expected final wealth, that is $E\left[W_H\right]=K_H$, one sees that by fixed r and with the formula (3.6) this equation can be solved for the required annual initial level premium P.

4. Dynamic stochastic evolution of the dismissal fund random wealth.

The dynamic stochastic evolution of the random wealth at time t is determined by the recursive equation (3.1). Similarly to (3.2) one obtains the explicit expression

$$W_t=W_0\cdot\prod_{j=1}^{t}\left(1+I_j\right)+\sum_{j=1}^{t}\left\{P(1+g)^{j-1}-X_t\right\}\cdot\prod_{k=j}^{t}\left(1+I_k\right). \tag{4.1}$$

Applying the same approach as in the proof of Theorem 3.1, one sees that the mean and variance of the random wealth at time t are given by

$$E[W_t] = W_0 \cdot r^t + P \cdot r \cdot \frac{r^t - (1+g)^t}{r - (1+g)} - \sum_{t=1}^{H} \mu_{X_t} \cdot r^{H-t+1} \tag{4.2}$$

$$\begin{aligned} Var[W_t] = {} & K_0^2 \cdot r^{2t} \cdot \left(e^{t\sigma^2} - 1\right) + K_0 \cdot \sum_{j=1}^{t} r^{2t-j+1} \cdot \left(P(1+g)^{j-1} - \mu_{X_j}\right) \cdot \left(e^{(t-j+1)\sigma^2} - 1\right) \\ & + \sum_{j=1}^{t} r^{2(t-j+1)} \cdot \left\{ \left(P(1+g)^{j-1} - \mu_{X_j}\right)^2 \cdot \left(e^{(t-j+1)\sigma^2} - 1\right) + \sigma_{X_j}^2 \cdot e^{(t-j+1)\sigma^2} \right\} \\ & + 2 \cdot \sum_{1 \le i < j \le t} r^{2t-i-j+2} \cdot \left(P(1+g)^{i-1} - \mu_{X_i}\right) \cdot \left(P(1+g)^{j-1} - \mu_{X_j}\right) \cdot \left(e^{(t-j+1)\sigma^2} - 1\right) \end{aligned} \tag{4.3}$$

Let $k[W_t]$ be the coefficient of variation of the wealth at time t. To estimate a quantile of the random wealth at time t, we suppose that the wealth is approximately gamma distributed. This is a practical approximation under the reasonable assumptions of gamma distributed aggregate lump sum payments and independent identically log-normally distributed accumulated rates of return. Then, the α-quantile of the wealth at time t, $Q_{W_t}^{-1}(\alpha) := \inf\{x : P(W_t \ge x) \ge \alpha\}$, is

$$Q_{W_t}^{-1}(\alpha) = \Gamma_\alpha^{-1}\left(\frac{1}{k[W_t]^2}\right) \cdot k[W_t]^2 \cdot E[W_t], \tag{4.5}$$

where $\Gamma_\alpha^{-1}(\beta)$ is the α-quantile of a standard gamma distribution $\Gamma(\beta,1)$. The α-confidence interval of the wealth at time t contains all possible realisations of the wealth in the interval $\left[Q_{W_t}^{-1}(1-\alpha), Q_{W_t}^{-1}(\alpha)\right]$.

5. The probability of bankruptcy: a numerical example.

The features of the present approach will be illustrated at a concrete example, which is not based on a real life firm. The considered situation is chosen to exemplify what could happen in case a dismissal fund is not well balanced in its employee structure.

Suppose that the employee structure of the dismissal fund by age, term of service and wage is given as in Table 5.1. Each age class is assumed to be represented by 50 employees, of which half is male and half female. Therefore the dismissal fund has a total of $M = 1'000$ employees. The total wages is equal to 61'500'000. The wage based lump sum payment is evaluated using Table 5.2 under the assumption that the wages increase by $100 \cdot g = 3\%$ per year. The used probabilities of withdrawal for the four causes of decrement are summarized in Table 5.3, where linear interpolation is applied for ages between two values. These probabilities are only rough values, but correspond qualitatively to real life data. A distinction is made between male and female probabilities of death.

Following the formulas of Section 2, it is now possible to calculate the expected aggregate lump sum payments for an arbitrary year t (formulas (2.3), (2.19) and (2.22)) as well as the expected aggregate future lump sum payments (formula (3.19) for an arbitrary time horizon $H \le H_{\max} = 45$). The assumed guaranteed rate of return is set at 4%. The obtained results are summarized in Table 5.4. One notes that until every employee attains with certainty the retirement age of 65 years, there is expected a total of 75 dismissals by the employer, 170 resignations by the employee, 121 deaths, and the remaining 634 employees are expected to attain the retirement age.

Table 5.1: employee structure by age, term of service and wage

age	term of service	wage
20	0	30'000
30	10	50'000
30	5	40'000
30	0	30'000
35	15	60'000
35	10	50'000
35	5	40'000
35	0	30'000
40	15	70'000
40	10	60'000
40	5	50'000
40	0	40'000
50	20	90'000
50	15	80'000
50	10	70'000
50	5	60'000
60	25	100'000
60	20	90'000
65	30	100'000
65	25	90'000

Table 5.2: term of service and lump sum payment

term of service (in years)	number of monthly wage payments
3	2
5	3
10	4
15	6
20	9
25	12

Table 5.3: probabilities of decrement in %

age x	PD_x	PR_x	PT_x^m	PT_x^f
20	2	5	0.2	0.05
35	1	2.5	0.15	0.1
50	0.5	1	0.5	0.2
65	0	0	2	1

Table 5.4: development of lump sum payments

year	expected lump sum payments	expected future lump sum payments	year	expected lump sum payments	expected future lump sum payments
			23	178'102	48'467'033
1	9'429'819	50'964'483	24	225'889	50'220'489
2	-52'479	43'196'050	25	276'384	51'994'383
3	-33'653	44'978'470	26	23'142'266	53'786'720
4	-31'354	46'812'608	27	143'521	31'870'232
5	-9'584	48'717'721	28	178'203	32'995'779
6	10'889'573	50'676'397	29	214'836	34'130'279
7	-105'101	41'378'297	30	253'508	35'272'061
8	-85'453	43'142'734	31	21'921'214	36'419'295
9	-64'532	44'957'314	32	93'753	15'078'004
10	-42'279	46'822'721	33	112'660	15'583'621
11	-41'507	48'739'600	34	132'617	16'089'800
12	-8'093	50'732'351	35	153'670	16'595'471
13	27'364	52'770'062	36	16'890'209	17'099'473
14	64'955	54'852'406	37	10'869	217'634
15	104'777	56'978'949	38	14'926	215'036
16	22'131'057	59'149'139	39	19'217	208'114
17	-42'264	38'498'805	40	23'753	196'453
18	-12'823	40'082'712	41	28'543	179'608
19	18'422	41'699'357	42	33'599	157'108
20	51'553	43'348'172	43	38'933	128'449
21	90'193	45'028'484	44	44'556	93'097
22	132'906	46'735'823	45	50'482	50'482

The required premium to fund the future payments is evaluated for a time horizon between $H = 25$ and 30 years. The initial capital is set at $K_0 = 10$ million to avoid bankruptcy in the first year because expected lump sum payments for this period are 9.43 million according to Table 5.4. The funding capital at time H is set at $K_H = 35.272$ million, which corresponds to the expected future lump sum payments at time $H = 30$. Table 5.5 displays the sensitivity of the required premium depending on the time horizon and the variation of the expected aggregate lump sum payments (mean ± multiple of the standard deviation). In these Tables μ_t, σ_t stand for μ_{X_t}, σ_{X_t}.

Table 5.5: Sensitivity of the required premium

H	$\mu_t - 2\sigma_t$	$\mu_t - \sigma_t$	μ_t	$\mu_t + \sigma_t$	$\mu_t + 2\sigma_t$
25	1'129'202	1'326'777	1'524'352	1'721'926	1'919'501
26	1'374'935	1'600'598	1'826'262	2'051'925	2'277'589
27	1'309'253	1'528'853	1'748'452	1'968'052	2'187'652
28	1'249'018	1'463'125	1'677'231	1'891'338	2'105'444
29	1'193'612	1'402'754	1'611'895	1'821'037	2'030'178
30	1'142'511	1'347'173	1'551'836	1'756'499	1'961'161

Table 5.6: Dynamic stochastic development of the random wealth

time	**mean**	**coefficient of variation**	**95% confidence interval**		**99% confidence interval**	
1	2'206'897	0.530	692'822	4'413'146	399'041	5'785'244
2	4'012'079	0.307	2'223'832	6'227'286	1'710'377	7'414'207
3	5'919'757	0.219	3'961'998	8'198'931	3'327'517	9'341'901
4	7'952'719	0.171	5'851'223	10'319'397	5'128'543	11'462'095
5	10'097'264	0.142	7'859'784	12'566'069	7'063'043	13'729'412
6	1'046'963	1.941	34	4'976'769	0	9'859'078
7	3'125'238	0.684	602'880	7'257'567	264'374	10'073'121
8	5'324'024	0.422	2'239'834	9'474'570	1'503'764	11'883'455
9	7'648'551	0.309	4'220'898	11'900'489	3'239'545	14'181'928
10	10'104'250	0.246	6'397'179	14'501'908	5'237'385	16'753'370
11	12'720'547	0.205	8'755'150	17'292'620	7'447'177	19'560'732
12	15'471'813	0.177	11'255'590	20'239'259	9'815'828	22'551'984
13	18'363'276	0.157	13'896'046	23'343'486	12'332'805	25'719'470
14	21'400'334	0.141	16'677'694	26'608'985	14'994'805	29'062'113
15	24'588'562	0.129	19'603'453	30'040'568	17'802'014	32'582'106
16	5'070'224	0.784	677'515	12'852'005	239'218	18'457'196
17	7'906'838	0.529	2'487'624	15'799'055	1'434'460	20'705'055
18	10'903'994	0.404	4'803'000	19'006'586	3'300'388	23'652'256
19	14'068'569	0.329	7'409'025	22'451'506	5'560'046	27'016'366
20	17'407'696	0.280	10'239'056	26'120'528	8'099'782	30'698'009
21	20'925'103	0.245	13'266'041	30'006'842	10'867'754	34'654'113
22	24'626'232	0.219	16'479'967	34'110'296	13'840'080	38'866'622
23	28'518'472	0.199	19'879'381	38'435'301	17'006'984	43'330'197
24	32'609'476	0.183	23'466'319	42'988'043	20'365'200	48'045'416
25	36'907'161	0.170	27'244'777	47'775'880	23'915'138	53'016'289
26	17'694'659	0.393	7'997'966	30'476'084	5'567'451	37'753'115
27	21'733'726	0.337	11'251'124	34'999'255	8'373'491	42'261'068
28	26'002'703	0.296	14'769'885	39'809'538	11'492'782	47'148'505
29	30'511'889	0.265	18'539'418	44'909'763	14'891'006	52'389'783
30	35'272'000	0.241	22'555'892	50'306'448	18'552'088	57'975'579

The dynamic stochastic development of the random wealth is displayed in Table 5.6. The calculation is done with a volatility $\sigma = 2\%$ and a logarithmic rate of return $\mu = \ln(1.04) - \frac{1}{2}\sigma^2 = 3.902\%$. The skew employee structure implies a 5% bankruptcy probability in the 6-th year, and a close to bankruptcy in the first and 16-th year with a probability of 1%. Also, there is a non-negligible probability that the overall goal at time $H = 30$ will not be attained.

Traditionally the life insurance sector has set annual premiums at a constant level. It is interesting to compare this situation with the above one. To do calculations one has to replace the wage increase factor $1+g$ by a factor of one in the relevant formulas. The Tables 5.5 and 5.6 are then being replaced by the Tables 5.5’ and 5.6’.

Table 5.6': Dynamic stochastic development of the random wealth for a level premium

time	mean	coefficient of variation	95% confidence interval		99% confidence interval	
1	2'941'511	0.397	1'318'252	5'086'515	913'730	6'310'620
2	5'462'274	0.225	3'612'278	7'624'833	3'017'245	8'714'272
3	8'064'287	0.160	6'065'441	10'297'587	5'368'605	11'365'858
4	10'767'991	0.126	8'637'159	13'093'124	7'864'281	14'174'014
5	13'557'202	0.105	11'297'847	15'987'332	10'458'998	17'096'979
6	5'122'858	0.395	2'303'596	8'844'611	1'599'411	10'966'572
7	7'785'600	0.273	4'644'543	11'585'772	3'698'604	13'572'662
8	10'534'419	0.212	7'143'772	14'463'928	6'035'767	16'424'579
9	13'371'433	0.176	9'753'547	17'458'432	8'516'055	19'438'873
10	16'298'785	0.152	12'459'791	20'562'819	11'107'927	22'588'170
11	19'342'428	0.134	15'280'195	23'800'275	13'820'489	25'887'037
12	22'473'065	0.121	18'183'285	27'138'816	16'619'029	29'299'116
13	25'692'053	0.112	21'168'118	30'578'994	19'500'096	32'822'585
14	29'000'706	0.104	24'234'430	34'121'858	22'461'838	36'457'189
15	32'400'289	0.098	27'382'281	37'768'730	25'503'221	40'203'525
16	13'028'525	0.304	7'270'643	20'145'833	5'610'061	23'950'965
17	15'942'145	0.261	9'762'520	23'354'574	7'869'927	27'194'971
18	18'941'690	0.231	12'352'045	26'681'485	10'251'288	30'601'159
19	22'028'723	0.209	15'029'295	30'123'576	12'733'256	34'153'059
20	25'204'781	0.192	17'789'267	33'680'084	15'304'421	37'842'703
21	28'467'695	0.179	20'625'684	37'347'836	17'954'575	41'662'938
22	31'816'704	0.169	23'534'829	41'125'765	20'677'303	45'610'332
23	35'252'670	0.160	26'515'276	45'014'849	23'469'223	49'684'428
24	38'776'376	0.153	29'565'925	49'016'294	26'327'767	53'885'625
25	42'388'516	0.147	32'685'827	53'131'449	29'250'836	58'214'874
26	22'364'624	0.310	12'311'133	34'845'671	9'437'490	41'547'960
27	25'458'471	0.286	14'772'358	38'503'688	11'611'309	45'388'811
28	28'640'002	0.267	17'311'308	42'287'094	13'870'655	49'390'007
29	31'910'696	0.252	19'923'914	46'196'695	16'206'918	53'547'485
30	35'272'000	0.240	22'607'494	50'233'935	18'614'135	57'859'685

Table 5.5': Sensitivity of the required level premium

H	$\mu_t - 2\sigma_t$	$\mu_t - \sigma_t$	μ_t	$\mu_t + \sigma_t$	$\mu_t + 2\sigma_t$
25	1'551'100	1'822'493	2'093'887	2'365'280	1'551'100
26	1'910'989	2'224'634	2'538'278	2'851'923	1'910'989
27	1'841'020	2'149'812	2'458'605	2'767'397	1'841'020
28	1'776'695	2'081'256	2'385'817	2'690'378	1'776'695
29	1'717'385	2'018'301	2'319'216	2'620'132	1'717'385
30	1'662'555	1'960'376	2'258'196	2'556'016	1'662'555

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