IAA PAPER
VALUATION OF RISK ADJUSTED CASH FLOWS AND THE SETTING OF DISCOUNT RATES – THEORY AND PRACTICE

Introduction

This document refers to sub-issue 11G of the IASC Insurance Issues paper and proposes a method to value risk-adjusted cash flows (refer to the IAA paper "INSURANCE LIABILITIES - VALUATION & CAPITAL REQUIREMENTS, GENERAL OVERVIEW OF A POSSIBLE APPROACH", sections 2.2 and 2.3).

The principles and techniques described should work for all cash flows, irrespective of whether they are liability or asset related.

With regard to the cash flows, it is assumed that appropriate market value based adjustment to "insurance-specific" assumptions (like mortality, morbidity, non-interest sensitive lapse, etc) have been made. These risk-corrected cash flows are then comparable to "normal" asset cash flows, so generally accepted asset valuation techniques can be used. Basically, all underlying factors which influence the uncertainty of liability cash flows and are not, or not frequently traded in the markets, will have to be adjusted by means of inclusion of a Market Value Margin (MVM). Uncertainty that arises from factors that are non-traded can be categorized as being non-diversifiable risk.

If these "risk-adjusted" liability cash flows could be matched perfectly to a set of assets generating the same cash flows (cash flow matching), then parallel movements in values of assets and liabilities arising from changes in interest rates would be identical. In that case, there is no profit and loss impact as a result of interest rate movements. This implies that the liabilities would need to be discounted at the rate of return on the assets matching the liabilities.

Note, that since the liability cash flows are risk-adjusted by inclusion of MVMs, the matching assets should be chosen as risk-free as possible, or, alternatively, "*made*" risk-free by including appropriate credit risk corrections.

For **complete** markets, methodologies have been developed to value cash flows (which may depend on changes in e.g., interest rates or equity returns). These methodologies work under the assumption that the underlying factors driving the uncertainty in cash flows are tradable on some financial market, and hence perfect matching is indeed possible. If this is not the case, in so-called **incomplete** markets, it is harder to *estimate* the value of a cash flow (which depends on a non-tradable factor) and miss-estimation of the *true* value could occur.

Note that actual investments of a particular company do not impact the valuation process. Every company is able to invest as it pleases (within statutory limits) without changing the value of the liabilities. A mismatched position of liabilities versus assets will need to be reflected in the amount of solvency capital required.

This paper is organized as follows:

The first section deals with the valuation of the liabilities in complete (1.2) and incomplete (1.3) markets. This distinction is important since for complete markets, techniques have already been developed for the valuation of derivatives. These techniques can also be used for valuation of liabilities, although they are not fully applicable in case of incomplete markets. Section 1.1 explains the difference between complete and incomplete markets.

The second section describes a practical approach to determine the replicating portfolio. In section 3 the use of these portfolios is explained for performance measurement and risk management purposes. The implication and consequences of some others types of business (property/casualty and unit linked) are briefly touched upon in section 4, as most of this paper deals with the valuation of (traditional) life insurance contracts. Finally, in the appendix, a more formal description of the discount rate curve extrapolating technique is given that is used in the examples of this paper.

Lastly, the theory of derivative valuation (and hence also valuation of insurance liabilities) in *incomplete* markets is extremely difficult. Since this paper is more focused on practice than theory, emphasis is on the examples and issues. Also in any method to value cash flows, subjective elements are part of the process. However, the proposed method using the replicating portfolio is transparent and therefore leads to verifiable outcomes, which will hopefully result in an acceptable audit trail.

1. Fair Value Liability Measurement and Replicating Portfolios

This section deals with the theoretical fundaments of replicating portfolios and discounting (liability) cash flows. The main objective is to value liabilities consistent with the way tradable assets are priced in the market.

1.1 Complete versus incomplete markets

In a **complete** market, all underlying factors which drive the uncertainty of liability cash flows are tradable. Therefore, liabilities, or more general contingent claims, can be hedged perfectly in a complete market by a replicating portfolio.

A contingent claim is a cash flow (e.g., insurance liability, pay-off of a derivative security) that depends on other *basic* uncertain factors. Examples of what usually are tradable factors include equities, bonds, and real estate, while examples of usually non-tradable factors include mortality, longevity, and inflation. A fundamental result of arbitrage free pricing theory is that the price of a contingent claim in a complete market is equal to the cost to set up the hedge: hence the value of the contingent liability will equal the value of the (perfectly) replicating portfolio. Furthermore, the return on the hedged contingent claim or on an asset-liability position that contains liabilities as well as the perfectly replicating assets must equal the risk-free rate, as the position itself is risk-free.

Derivative valuation techniques assume complete markets where the main assumption is that all underlying uncertainty drivers are tradable. Most insurance products could be written as a series of complex derivatives of some financial asset (e.g., bonds, equity). Therefore, one would think that valuation principles for derivatives could readily be applied to valuation of insurance liabilities. However, this is usually not the case since insurance liabilities often depend on financial (e.g., interest rates beyond certain available maturities) and non-financial (mortality, morbidity) factors. The latter are not tradable in a sufficiently liquid market.

For the non-financial risk drivers, corrections have been made to the cash flows by including MVMs. The MVMs should capture the uncertainty related to assumption concerning probability distribution of these drivers. In other words MVM attempts to model a market price for this uncertainty. Therefore, the question whether or not the markets are complete relates here to the tradability and availability of products to hedge "financial risks" inherent to the insurance products (e.g., interest sensitivity of the cash flows).

In an **incomplete** market, the uncertain liability cannot be hedged or perfectly replicated with regard to the "financial risks". Note that even markets that can be considered complete, like the European or the U.S. capital markets, can show temporary "market gapping", where in case of a crisis, certain maturities or types of investment can become unavailable.

Examples of complete markets

- A company sells a contract that pays a certain fixed amount at the end of 30 years, while the market trades 30-year discount bonds.
- Term insurance with a maturity shorter than the maximum asset maturity available (although there is no liquid market in "mortality securities", corrections for the market-incompleteness with regard to this factor is taken care of by adjusting the expected cash flows with MVMs).

Examples of incomplete markets

- A company sells a contract that pays a certain fixed amount at the end of 30 years, while the market trades *only* 20-year discount bonds.
- Whole life insurance when the maximum asset-maturity is, for example, *only* 20 years.
- Endowments with inflation options (sold in Eastern Europe): the policyholder has the right to increase future premiums in line with inflation at that moment (at the then prevailing tariffs); this can lead to unpredictable cash flows. In these countries index linked bonds are usually not available and hence inflation is a non-tradable factor.

Most large financially well developed countries have capital markets that can be regarded as complete for the purpose of fair valuing insurance liabilities, as enough long duration assets are available to use the above mentioned derivative valuation techniques. In these cases, markets are assumed to be complete by, for example, treating a fictive 50-year bond the same as a 30-year bond.

Other possibilities include investments in other countries in either the currency one needs (for example, the Bank of Austria sells assets denominated in Czech korunas with a duration longer than available in the Czech Republic itself) or through a combination of investments in other currencies and forward currency contracts. If the market can then still not be considered as sufficiently complete, other methods than those that already exist for "complete" markets are required to obtain the fair value of the liabilities.

To highlight the differences between valuing a cash flow in a complete versus an incomplete market, the following example: suppose two identical companies were selling identical cash flows, but would not have access to the same capital markets to hedge their positions for one reason or another. Furthermore, suppose that these capital markets are very different: the first offered only 2-year bonds, while the other everything that can be imagined (*A* and *B*).

Both companies then decide upon the same investment strategy, i.e. to invest everything in 2-year bonds. Although this results in an identical investment risk that both companies run, one big difference exists: the second company (*B*) voluntarily runs this risk, while the first one (*A*) is forced to (incomplete capital markets).

This voluntary versus involuntary investment risk should influence the way the overall risk is reflected in the balance sheet: the market value of the liabilities will be depend on the hedging opportunities available and thus on the "completeness" of the particular market. In the above example, the value of the liabilities in the case *A* (incomplete market) will be higher than the value in case *B* (complete markets). At the same time, the capital of company *A* required to cover investment risk, will be lower than that of *B* as more of the investment risk is already reflected in the liabilities.

Actual investment opportunities available will therefore determine what *correction* will need to be made to end up with the *true* fair value.

Following first is a description of the valuation in complete markets, then in incomplete markets.

1.2 Valuation of cash flows in *complete* financial markets

The fair value of fixed cash flows is relatively easy to determine. The cash flows can simply be discounted at the existing term structure for interest rates:

Example 1
An insurer sells a contract that pays € 10,000 to a policyholder at the end of 30 years. The bond market trades 30-year discount bonds. Assume that 30-year discount rate is 6%. The market is complete since the "liability underlying" (30 year cash flow) are tradable. Fair value of the liability is € 1,741 (€ 10,000 discounted at 6%).

The perfectly replicating portfolio was easy to construct in this example (a simple zero coupon bond). And although it might become a lot more difficult in case of the more complicated insurance products that include embedded options (like guaranteed surrender values, guaranteed minimum interest rates, etc.), it is, at least in principle, possible to find this perfect replicating portfolio in a complete market.

Given that the perfect replicating portfolio can be found in a complete market, already existing valuation models that are based on the assumption that these portfolios exist can therefore be used to value the interest dependent cash flows.

In general, derivatives like caps, floors and stock-options, are valued using “risk-neutral valuation”. Traditionally, the actuarial profession has valued risky cash flows by including a risk premium in the discount rate. This risk premium was either positive or negative depending on whether the cash flow was a net asset or liability.

The risk-neutral valuation adjusts the cash flows and discounts at the risk-free return. The adjustment is included by *correctly* choosing the interest rate or stock return that drive the cash flow. The cash flow adjustment exactly offsets the change in value caused by discounting at the risk-free rate.

The reason this technique is called risk-neutral valuation is that it determines the value of a cash flow as if we would live in a risk-neutral world where investors do not require any reward for running risk - the required return on investments as well as the discount rate is equal to the risk-free rate. Although the projected cash flows would not mean anything in the “real world” in absolute terms (since investors would not be risk-neutral and hence demand risk premiums having an impact on the cash flows), when discounted at the risk-free rate, the outcome is also valid in the “real world”. The reason for this is that when moving from a risk-neutral world to a risk averse world (the real world), two things happen – both the expected return and the discount rate change. These two always offset each other exactly.

The value of a derivative equals the weighted average of multiple discounted cash flows, projected under stochastically generated risk-free returns. In other words, it is the expected pay-off of the derivative in the risk-neutral world discounted at the risk-free rate.

Now, why does this work? The derivative value can be written in the form of a differential equation in which no variables are included that are affected by the risk preference of the investors. If risk preferences don’t play a role in the equation, then they cannot affect its solution. Since any set of risk preferences can therefore be used, one can also value the derivative assuming that all investors are risk-neutral. Which means:

1. Assume the **expected return** of the underlying asset is **risk-free**
2. Generate **multiple** interest rates **scenarios**
3. Calculate the **cash flows**
4. **Discount** the cash flows **at the risk-free rate from step 2**
5. Determine the **weighted average**

Note: If we would not have made the risk-neutral assumption, then the return in step (1) would have reflected the risk-aversity of the investor (for instance the historic return would have been used). The discount rate in step (4) would in that case not have been the risk-free rate: a risk premium would be added. Only by coincidence would the *correct* risk premium be chosen that would lead to the same value as in the risk-neutral world.

Therefore, for the interest sensitive liability cash flows the above means that in order to determine the value first, multiple interest rate scenarios need to be generated, then the cash flows have to be determined under each of the scenarios, their present value calculated and finally, the weighted average value. If the interest rate is assumed to have a particular distribution and the pay-off of the derivative is relatively straightforward, this can be simplified and written into a closed formula.

Equity derivatives can be valued in a similar fashion using the same risk-neutral argument. If these are assumed to have a specific distribution (lognormal), that interest rates are fixed (no correlation between stock and bond returns is assumed), and that the derivative pay-out is that of a European option, the closed formula boils down to the famous Black Scholes formula.

The most critical parameters in this valuation process are the volatilities with respect to forward rates and equity returns as these drive the value of the derivatives. Since the embedded options written by the insurance companies have a considerable longer duration than that of the traditional derivatives, the parameters to be used can/should be a topic of discussion.

Finally, the way in which the stochastic interest rate scenarios are generated also has a large impact on the outcome. Frequently these are randomly distributed around the implied forward curve; another possible way is mean reversion to that implied forward curve.

1.3 Valuation of cash flows in *incomplete* financial market

In the introduction it was explained that the value of a set of liabilities determined under the assumption that markets are complete most likely provides a miss-estimation of the value of the liabilities in the case of incomplete markets. In order to arrive at the *true* market value, a correction has to be made that reflects the *completeness* of the market.

Two issues arise with regard to this procedure:

- *How to determine the value assuming that markets are complete?*

  The main problem in incomplete markets, is that for example the term structure of interest rates may only be available for relatively short durations; valuation of cash flows beyond the highest duration available will require assumptions with regard to the level of rates at these durations and their volatilities. For instance one could assume that all yields and volatilities after the maximum maturity are equal to yield and volatility corresponding to that of the maximum maturity (flat forward rates equal to that of the maximum maturity). The resulting value determined under the assumptions that markets are complete will depend on the assumptions made.

- *How to derive an appropriate correction to reflect the incompleteness of the market?*

  In order to reflect the *completeness* of the market, the actual assets available in the market will play an important role. By linking the value of the liabilities in an incomplete market to the actual assets available, a correction to the above-described value "as if the market were complete" can be determined.

Below, in 1.3.1 more detail on the valuation on **interest-insensitive** cash flows in an incomplete market, then, in 1.3.2 on the valuation of **interest-sensitive** cash flows.

1.3.1 Valuation of interest-*insensitive* cash flows in *incomplete* financial market

First, we go back to Example 1 in the previous section. The fact that a 30-year discount rate is available in the market makes life very easy. Consider Example 1 again but now assume that the liability cash flow has a maturity longer than the maximum maturity available in the (local) bond market (the market is incomplete):

Example 2
An insurer pays a policy holder € 10,000 at the end of 30 years. Suppose 20 years is maximum maturity of discount bonds available in the market and the 20-year discount rate is 6%.

In order to value this cash flow, an assumption has to be made with regard to the reinvestment rate at year 20 for a 10-year bond. An assumption could be that the 30-year discount rate is equal to the 20-year bond and hence is equal to 6% (this assumes a 10-year reinvestment rate in year 20 of 6%). Present value of € 10,000 using this discount rate is € 1,741 and equal to the outcome of Example 1 in Section 1.2.
(The insurer could also first buy a 10-year and then a 20-year bond; the optimal strategy will depend on the assumption with regard to the future rates. For more detail please refer to the appendix.)

However, if the 30-year liability cash flow is hedged with the 20-year discount bond then the insurer still has a non-hedgeable mismatch risk. A potential buyer taking over the liability of the insurer possibly would like to receive more than € 1,741 and will add a risk premium for the risk he runs. The impact of different reinvestment assumptions on the fair value of the liabilities would be the starting point of the correction.

The risk premium will depend on the perception of the buyer. The following table shows the amount of cash needed now (to be invested in 20-year bond at 6 %) such that at the end of 20 year a 10-year bond can be bought to hedge the liability cash flow (assuming that this 10-year bond will generate 10 %, 8%, 6%, 4% and 2% respectively). Call this amount of cash K_0:

10-year bond rate after 20 years	K_0
10 %	1,202
8 %	1,444
6 %	**1,741**
4 %	2,106
2 %	2,558

Explanation of the table:

- The first column gives possible scenarios for the 10-year rate at the end of year 20 (so at the time the insurer can fully hedge the liability);
- The second column gives the amount of cash needed now in order to set up the hedge in each of the 10-year rate scenarios;
- Note that the € 1,741 is equal to the value in the first paragraph of this example since the reinvestment rate (6%) is equal to the current rate.

Note 1: The possible reinvestment scenarios should be stochastically generated, for instance around an extrapolated mean (which of course brings subjectivity into the model as above already mentioned).
Note 2: The risk premium to be included in the market value of the liabilities to reflect this uncertainty with regard to future investment returns could be based on the standard deviation of the curve derived from the stochastically generated scenarios.
Note 3: If the market were perfect, this standard deviation would be zero and no risk premium would be included.

This example assumes that in incomplete markets, the replicating portfolio is always the one with the longest duration available. Although this assumption is probably valid for long duration business, the following algorithm provides a more thorough way of determining this portfolio:

Algorithm for refining fair value calculation in incomplete markets:

- As we will slowly increase the liability value until we have found the *correct* value in incomplete markets, a sufficiently *low* value must be chosen as a starting point for the algorithm. When the value determined under the assumption of complete markets is known, this can be used (€ 1,741 in the above example), as we know that it will be below the value of the liabilities in the incomplete market; otherwise a starting value equal to zero could be chosen. Call this K_0 .
- Define several (dynamic) hedge strategies (initial starting portfolio with a value of K_0 and a reinvestment strategy.
- Check for every strategy whether liability cash flows can be hedged (for instance by stating that the asset and liability cash flows have to stay within certain boundaries of

each other: cash flow matching) starting with initial capital K_0 (for this amount can asset be bought to hedge liabilities).

- If none of the strategies work in bullet point above, test the same hedge requirements with a new $K_0^{new} = K_0^{old} + \varepsilon$ (in the first run $K_0^{old} = K_0$).
- Stop when one of the selected strategies is able to hedge the liability cash flows.

As a last step we need to quantify the cash flow mismatch between these replicating assets and the liabilities in order to reflect this in the value of the liabilities as well. The above described comparison of asset and liability cash flows under different unbiased interest rate scenarios can provide insight into the remaining risk. The distribution of the outcomes under the different scenarios should form the basis for this correction related to this "unhedgeable interest rate risk", depending on the risk appetite of the market. Basically, this correction is the reward the market requires for this unhedgeable interest rate risk.

1.3.2. Valuation of interest-*sensitive* cash flows in *incomplete* financial market

Example 2 was already quite complicated although the liability cash flow itself did not depend on future interest rates. To make things even more complicated a product with embedded options/guarantees is considered. Again, the incomplete market case is assumed (otherwise classical derivatives valuation techniques would give the answer immediately).

Example 3
Consider the following product (for the moment no mortality or other actuarial risk factors are included in the product):

- Maturity 10 years;
- Benefit at maturity: € $1{,}000\prod_{t=1}^{10}(1+R_t)$, with $R_t = \max(\text{1yr rate}_t, 4\%)$. The 1-year rate in the formula is the spot rate at the end of each policy year. This product credits annually the one-year rate with a minimum of 4 %.

About the financial market:

- Maximum available maturity in bond market is 5 year;
  This means that implied 1-year forward rates are available up to 4 years: 4-year and 5-year spot rates are needed to calculate *f*(1,4), the 1 year forward rate with settlement 4 years from now. Note that this example again involves an incomplete market, since forward rates after 4 years are not tradable.
  (Note: the 5-year spot rate is the interest rate on a 5-year zero coupon bond, while the implied forward rate – for instance the 2-year forward rate with settlement 4 years from now, or f(2,4), is the interest rate on a 2-year zero coupon bond bought 4 years from now. These implied forward rates are derived from the spot rates and visa versa.)

A valuation approach that could be pursued for complex products like this (embedded option, path dependent structures) is via Example 2. Like in that example, the discount curve beyond the "tradable maturities" will be extrapolated to be able to calculate the value as if the market was complete.

Assume that the spot rates are 3%, 3.25%, 3.5%, 3.75% and 4% for the 1year up to 5-year rates respectively. As in Example 2, interest rate levels beyond 5 years are necessary to construct forward rates all the way up to 10 years. These forward rates will define the spot curve and visa versa. The following assumptions are used in the construction of that "discount curve":

- If a cash flow with maturity beyond 5 year has to be hedged, then first a 5-year bond is purchased; the hedge is completed by purchasing a new bond after 5 years with funds available when original 5-year bond matures. (Note: this assumes that we have already found the replicating portfolio; in practice, if we do not assume that the maximum duration assets are most suitable, we would first need to use the above-described algorithm.)
- We assume that the *actual* future spot rates at time five are distributed around an assumed mean. Below are the seven means we have defined in this example. Each will generate a different MVL given that the particular investment scenario will materialize. The base case is here simply the current curve.

If the market had been complete, we would have known the *true* mean and only 1 scenario, the *true* one, would have been evaluated.

Given that the scenarios on the interest rates after 5 years are defined, the value of the interest sensitive cash flows can be determined for each of the below defined seven scenarios for the spot curve after five years:

Maturity	Scenario 1	Scenario 2	Scenario 3	Base	Scenario 4	Scenario 5	Scenario 6
1	0.00%	1.00%	2.00%	3.00%	4.00%	5.00%	6.00%
2	0.25%	1.25%	2.25%	3.25%	4.25%	5.25%	6.25%
3	0.50%	1.50%	2.50%	3.50%	4.50%	5.50%	6.50%
4	0.75%	1.75%	2.75%	3.75%	4.75%	5.75%	6.75%
5	1.00%	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%

Here is where subjectivity has again come into play (similar to example 2): an assessment for the future spot curve beyond current maximum maturity is made.

In order to determine the interest dependent pay-off, the 1-year spot rates need to be determined for each of the scenarios. These rates are derived from the above-defined scenarios: first the discount yield is determined, then the 1-year forward rates.

Using the above mentioned rates of scenario 2:

- Suppose the K_0 for a 10-year cash flow (that pays € 1) has to be determined. Then first K_0 will be invested in a 5-year bond (yield 4%), then a 5-year bond will be purchased that hedges the original 10-year cash flow (yield according to table above:

2.0%). So $K_0 = (1+z_5)^{-5}(1+\tilde{z}_5)^{-5} = (1.04)^{-5}(1.02)^{-5}$ = € 0.74. (z_5 is the current 5-year rate, $\tilde{z}_5$ is the 5-year rate after 5 years, according to our subjective probability measure). The 10-year extrapolated discount yield (at time 0) is $K_0^{-1/10} - 1 = 3.00\%$.

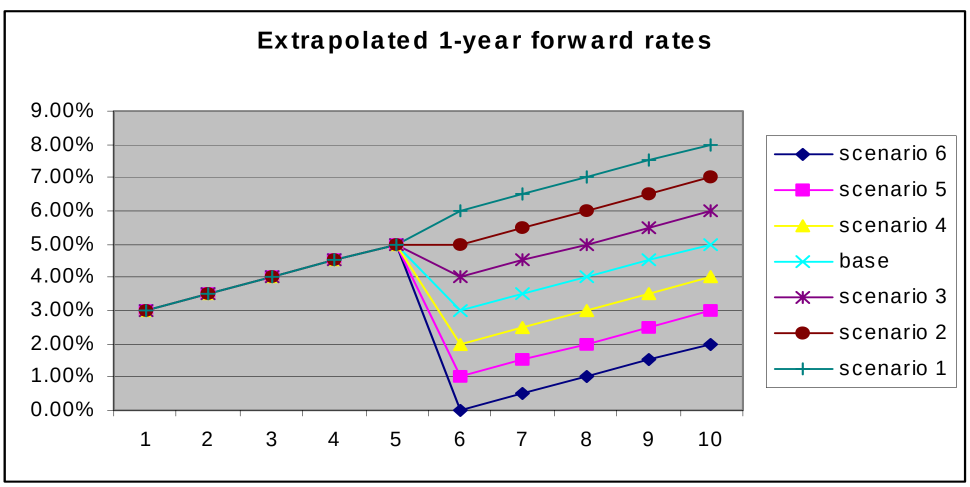
- If the formula in the bullet point above is used for the 6 up to 10-year rate then the following extrapolated discount yield and forward curve is constructed:

	Discount Yield
1	3.00%
2	3.25%
3	3.50%
4	3.75%
5	4.00%
6	3.49%
7	3.21%
8	3.06%
9	2.99%
10	3.00%

	Forwards
(1,0)	3.00%
(1,1)	3.50%
(1,2)	4.00%
(1,3)	4.50%
(1,4)	5.01%
(1,5)	1.00%
(1,6)	1.50%
(1,7)	2.00%
(1,8)	2.50%
(1,9)	3.01%

*Note: for instance the f(1,4), is the 1-year forward rate with settlement 4 years from now and can be derived from the discount curve by: (1 + 5^yr yield)^5 = (1 + 4^yr yield)^4 * (1 + f(1,4)), or f(1.4) = [(1.04^5) / (1.0375^4)] – 1 = 0.0501*

The next figure displays the forward curves (like the one above for a 2 % drop in rates) for several interest rate scenarios: current curve -3 %, -2 %, -1%, 0% (=base), +1%, +2% and +3%. (The higher the drop the lower the extrapolated part of the curve.)



Note: the strange pattern of the 1-year forward curves in the graph above highlights the difficulty in extrapolating beyond the term of available maturities; it is the direct result of the interest rate scenarios we have chosen.

All of these seven scenarios can then be valued by taking the weighted average of the outcome under multiple stochastically generated interest rates scenarios around the particular 1-year forward rates (for the volatilities beyond the fourth year, a subjective assessment is needed).
Alternatively, the "standard" method for valuation in complete markets can be used:

$$FV = K_0 = (1+z_{10}^{extra})^{-8} E\left(1{,}000\prod_{t=1}^{10}(1+\max[Y_t,4\%])\right),$$

where Y_t is assumed to be log normally distributed with parameters $f(1,t)$ and σ_t^2 (σ_t^2 is the estimated volatility of forward rates; again for volatilities beyond the 4 year forwards a subjective assessment is needed).

The distribution of the possible values under each of the scenarios should then be the basis for the calculation of the correction that needs to be applied to the predefined "base case" market value assuming perfect markets.

Like already stated, in practice we would first need to apply the algorithm described in 1.3.1 to determine the replicating portfolio, after which the mismatch can be determined to correct the value of the replicating portfolio for the remaining mismatch risk. The notes mentioned under the algorithm also apply here.

Remark: In complete markets, the perfect replicating portfolio exists (refer to 1.2) and techniques can be used to value the liabilities directly instead of by first determining the actual replicating portfolio itself. This does not mean that the replicating portfolio idea cannot be applied to complete markets; it merely states that we are satisfied with the idea that it exists, so we can use valuation models that use that underlying assumption. In fact, if we would actually determine a perfect replicating portfolio in a complete market and state that the value of the liabilities that it replicates equals the value of these assets, we would end up with the same value as had we used the "direct" valuation from the start.

In Example 4 it is assumed that the financial uncertainty drivers are tradable. This can be done without loss of generality since in the incomplete financial market case the discount rate curve can be extrapolated using the "extrapolation argument" in Example 2.

Example 4
Consider the following liability cash flow at the end of the year: $100\max(y_{10}, 6\%)$ if policyholder is alive at the end of the year and 10 if policyholder dies during the year. Y_{10} is the yield at the end of the year on a 10-year bond. Suppose that q is the probability that the policyholder dies at the end of the year, then the following cash flow profile has to be hedged in order to obtain the fair value: $10q + 100(1-q)\max(y_{10}, 6\%)$. (Note: the q will equal the expected value plus an appropriate MVM, to reflect the uncertainty with regard to the non-tradable mortality risk).

This cash flow above can be replicated with a 10-year bond and a current account. The price to set up the replicating portfolio (which is actually a dynamic strategy here) is exactly the value of the uncertain cash flow.

The easiest way to obtain the fair value (using remark above) is to calculate

$$10q(1+z_1)^{-1} + 100(1-q)(1+z_1)^{-1} E\max(Y, 6\%),$$

where Y is lognormal distributed with parameters f_{10} (the $1x10$ forward rate) and σ^2. In the formula above, z_1 is the 1 year risk-free rate. This essentially is the basic formula for calculating caps and floors on interest rates, which is, of course, logical as the product is in essence a floor: the rate on this product resets annually and has a 4% minimum guarantee. The replicating portfolio would, in a complete market, consist of an asset with a floating rate that also reset once per year and a floor that pays off when rates are below 4%.

Remark: the lognormal distribution used here in the calculation is the risk-neutral probability distribution for this particular valuation problem. Under this probability distribution expected cash flows can be discounted at a risk-free discount rate in order to obtain the value. This approach is equivalent with obtaining the cost of setting up a replicating portfolio that completely replicates the liability cash flow.

2. A Practical Approach to the Replicating Portfolio

Ideally, the algorithm described above leads to a replicating portfolio that is cash flow matched irrespective of movement in interest rates. In practice such cash flow matching is hard to achieve. An alternative approach would be to search for an asset portfolio that replicates the liability *value* change under different interest rate scenarios over a given time horizon. The resulting asset portfolio will need to be rebalanced regularly.

The objective would become to find the asset portfolio that minimizes the net fair value change (defined as change in fair value of liabilities minus change in fair value of assets) over a certain time horizon.

The change in fair value of liabilities would be determined as described above – first extrapolate of the current yield curve (different scenarios), then value the cash flows as if all underlying financial factors are tradable (the fair value of assets needs to be determined on the same basis). The extrapolation would need to be as unbiased as possible.

Note that if the time horizon chosen is one year then this replicating portfolio automatically reduces volatility in the fair value profit and loss.

In practice:
The objective is to choose assets in such a way that the market value of assets minus liabilities is as *riskless* as possible. Risk is quantified through a probability distribution of difference in market value changes at the end of some time interval. Risk-minimization means that the distribution of fair value of assets minus liabilities is as narrow as possible. A degenerate distribution consisting of a single point (which basically means that the value change of assets and liabilities is identical under all scenarios) is the ideal risk-free situation. Although this is usually not possible, we can, for a given fixed time horizon and sample from the probability distribution of spot curves and volatility parameters, set up an optimization problem where the objective is to minimize the volatility of the fair value changes (assets - liabilities).

The asset portfolio that minimizes risk in terms of fair value changes (MVA minus MVL) will be called the *fair value immunizing replicating portfolio*. Consider the following scheme for the construction of such a replicating portfolio:

- Obtain a large number (say K) of independent samples from the probability distribution of discount yield curves and volatility parameters (given a fixed time horizon).
- For each of these samples, calculate the fair value of liabilities and store the K values in a vector L. (the risk-neutral valuation is used here)
- Do the same for the set of tradable assets that can be used for hedging purposes. If there are M assets the resulting asset values can be stored in a (KxM)-matrix S.
- Choose the weights for the assets in the replicating portfolio x so that x minimizes $\sum_{j=1}^{K}\sum_{i=1}^{M}(S_{ji}x_i - L_j)^2$ (volatility squared of net market value change over the given time horizon).

The problem can be modified by imposing restrictions on x. Natural restrictions are the prohibition of short selling certain assets, limits on the amounts invested in a particular asset, or budget constraints. As long as the restrictions are linear in x the optimization problem is relatively easy to solve. The method here is an extension of classical duration/convexity matching. By immunizing duration and convexity (making sure that replicating portfolio has similar duration and convexity to corresponding duration and convexity of the liabilities) the net fair value change of replicating portfolio minus liabilities is very small given a small time horizon and given *non-extreme* changes in the financial markets.

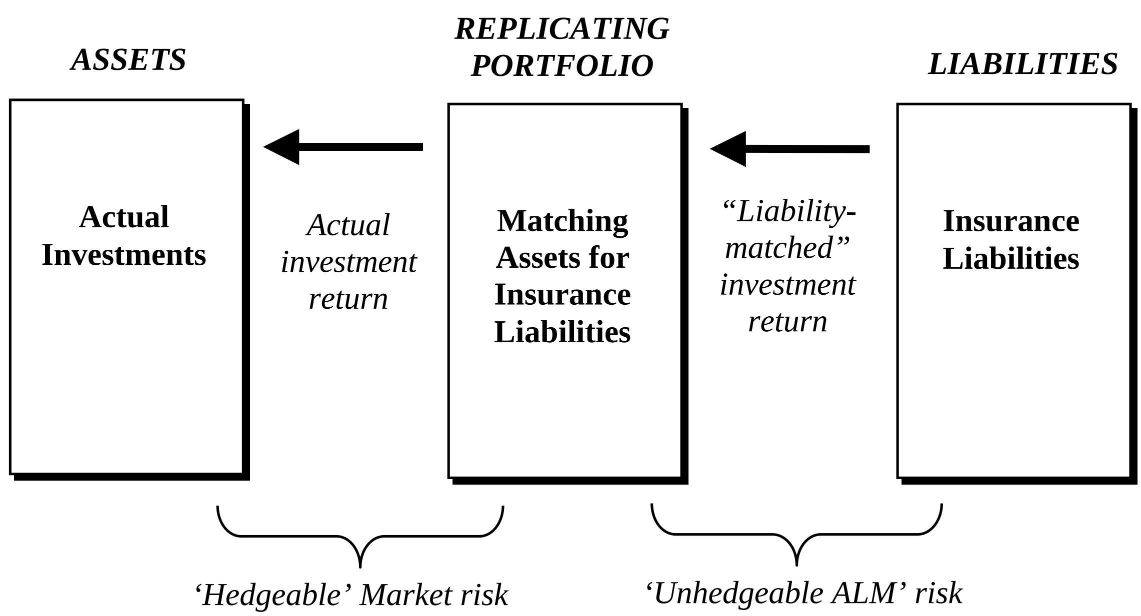
After the weights have been chosen, the distribution of the change in value of assets and liabilities under the K different scenarios can be determined. Again, this distribution should drive the correction that needs to be made to the value of the replicating portfolio to reflect the remaining investment risk.

3. Performance Management and Risk Management

Although the main focus of this paper is the valuation of the risk-adjusted liability cash flows by taking into account the time value of money and the *financial risk*, it is interesting to investigate the relation between the methodology presented here and risk management practice for insurance products.

Replicating portfolios can have another function within an insurance company, namely, for performance measurement and risk management, resulting in specific capital requirements:

- By placing a replicating portfolio between the insurance liabilities and the actual investments, the performance of the insurance and the investment activities can be measured separately.
- An insurance company that desires less volatility in the (fair value) profit and loss accounts can decide to hedge its insurance liabilities with a hedge or replicating portfolio.



Capital requirements with regard to market or ALM risk can be set on the basis of the impact on the fair value profit and loss of different interest rate scenarios. Minimal risk can be taken by investing close to the replicating portfolio. Additional risk can be measured relative to the replicating portfolio together with the additional return it generates.

Other risk-based capital requirements can also be defined as a function of their impact on the fair value profit and loss accounts. For each type of risk (for example, credit risk, life risk, property and casualty risk, etc.) the company can determine a certain level of capital, which, with a certain level of confidence, will prevent it from becoming insolvent (Market Value of Assets < Market Value of Liabilities).

The sum of these capital requirements (corrected for diversification effects) protects against a change in value of the business, such that the likelihood of default or insolvency of the company (on a fair value basis) over a given time period is less than a specified confidence level.

This default rate is set either by the regulators or the market in such a way as to be consistent with the level of comfort (risk-aversion) required by these institutions. Although the levels of prudency required might differ between investors and supervisors, the use of similar methodologies in the evaluation and analysis of the risks of the company would make comparisons possible.

4. Other Products

Unit-Linked/Variable business
These types of insurance products have of course other features than the classical profit sharing guaranteed business. The valuation of these products, however, has similarities with the valuation of the products described in the examples in the previous sections. In fact, usually the valuation is easier since complete market techniques can be used. The hardest part in valuation for unit- linked / variable business is the cash flows interaction from policyholder to the fund (and visa versa), from the fund to the insurance company (and visa versa) and from the policyholder to the insurance company.

In practice, distinctions can be made between fixed cash flows (such as costs charges in nominal terms, if we assume that they are not linked to inflation) and variable cash flows that depend on the value of the fund (e.g. fee income as a percentage of the fund value and lapse rates).

Once the modeling of the cash flow scheme is completed, the valuation of the variable business can be completed by:

- Valuing the fixed cash flows as bonds (discounting against current discount rate curve) and
- Viewing the variable cash flows as derivatives of the underlying fund value (and hence valuing these cash flows using the derivative valuation techniques that are available for the complete markets)

This basically means that replicating portfolio principles are used as these techniques assume that this portfolio can be found.

In the second bullet point it is implicitly assumed that the fund itself is a tradable asset. This may not be true in real life, but it seems this is the most practical solution to determining the value of this type of businesses (thus assuming complete markets exist).

Property & Casualty business
Many property and casualty liabilities are inflation-sensitive. To the extent that interest rates are responsive to inflation there will be an indirect linkage between interest rates and property & casualty insurance liabilities.

At the moment, it is implicitly assumed that the inflationary impact on future claims payments is *reasonably* represented by a zero discount rate (perfect correlation is assumed between inflation and interest rates). Unless this can be demonstrated for the key business segments over a historical period, it would be better to make explicit assumptions for both.

In that case, property & casualty insurance cash flows can be valued using the same techniques as described above; this is especially so for long-tailed businesses.

31 May 2000

Appendix

Discount rates
In Section 1.3.2, a method was proposed to extrapolate existing yield curves using the "subjective" shape of the discount curve after the maximum maturity available. If the maximum maturity was five years, then a cash flow in year 10 would be hedged by first investing in a 5 year and then in a 5-year bond. In this section a formal construction of the extrapolated discount yield curve is stated, given the still subjective future shape of the discount curve.

Notation:
- M maximum maturity tradable bonds.
- $2M$ maturity of the liability cash flow.
- $z_1,\ldots,z_M$ discount yields on the tradable discount bounds.
- $\tilde{z}_{M+1},\ldots,\tilde{z}_{2M}$ extrapolated curve.

Now suppose that $M = 10$ and that we want to hedge a liability cash flow of 1 at time M+1=11. Note that at time 1 we can choose to hedge the liability (with remaining maturity of 10) with the 10-year bond available in the market. We can also wait one additional year and hedge the liability cash flow at time 2 with a 9-year bond. So for an 11-year liability cash flow there are 10 potential hedge moments (if we only allow hedging at the end of each year). Note that in this market there is only one hedge opportunity for a 20-year liability cash flow.

The extrapolated yield curve $\tilde{z}_{M+1},\ldots,\tilde{z}_{2M}$ can only be constructed if we have subjective interest curves for the level of interest rates beyond M years (if we are now at time 0). Fix *s* and *t*, $s = 1,\ldots,M$ and $t = M+1,\ldots,2M$ and assume that $t \le s+M$. Here: *t* is the maturity of the liability cash flow of 1 and *s* is the potential final hedge time after we initially invested in an *s* year discount bond.

Note that if $P(z_{t-s} \ge \Delta z_{t-s}) = 1$(with Δz_{t-s}, being the assumed drop in interest rates following the current level of z_{t-s}), then $(1+z_s)^{-s}(1+z_{t-s}-\Delta z_{t-s})^{-(t-s)}$ is the amount of money needed at time 0 to hedge the liability cash flow at time *t*, assuming that the amount is initially invested in a *s* year bond and funds available at time *s* are invested in an *t-s* year bond with a yield always larger than (according to our subjective probability measure) $z_{t-s} - \Delta z_{t-s}$.

If $p_{s,t}$ is defined as the amount of cash mentioned above and if $t \le s+M$ then the extrapolated curve can be defined as:

$$\tilde{z}_t = \left(\min_{1\le s\le M} p_{s,t}\right)^{-1/t} - 1, t = M+1,\ldots,2M .$$

Now, the curve $z_1,\ldots,z_{M,}\tilde{z}_{M+1},\ldots,\tilde{z}_{2M}$ can be used to value fixed (interest rate insensitive) cash flows or as the input curve for a classical "complete market" derivative valuation model.