Stochastic Modeling
Theory and Reality from an Actuarial Perspective
Introduction

As recently as the mid-1990s, most models used in financial analysis of insurance were deterministic. Based on sets of static parameters and assumptions, these models largely ignored random fluctuations that were likely to occur. Sensitivity analyses were performed but were generally limited to a fixed number of defined scenarios. This deterministic approach is rapidly being replaced by stochastic modeling that can better inform insurers on pricing, financial planning, and capital assessment strategies. Huge advancements in computing power have made it possible for actuaries and financial planners to better understand the increasingly complex risk profiles of insurers’ evolving product design.

Rather than focus on a specific use of stochastic modeling, as many others have done, this book is intended to provide actuaries with a comprehensive resource that details current stochastic methods, provides background on the stochastic technique as well as their advantages and disadvantages.

Many of these techniques transcend international borders and will be relevant in any country. For this reason, we have not focused on any specific country or accounting framework. Parameters and assumptions may be illustrated for a specific region or country, in which case possible modifications have been noted. The goal of this book, however, is not to address all possible risk scenarios, nor generate specific factors for reserve loads, but rather to explain how an approach is derived. Readers can then develop their own parameters and assumptions based on current and historical experience for use in a specific and appropriate methodology.

Furthermore, because insurance products have and will continue to evolve differently around the world, certain products may be more or less advanced depending on the reader’s location.

This book is presented by the International Actuarial Association (IAA) in collaboration with Milliman. It is intended for actuaries, actuarial students and other readers involved in actuarial modeling in both the life and non-life sectors.

Certain concepts, such as mean, standard deviation, and percentile distribution, will be illustrated but not defined. In many cases, technical formulas used in this document were obtained from other publications noted in the bibliography. Additional information on formulas, including their development, will not be discussed here and can be obtained from the source document. In addition, we are not promoting one method or technique over any other. Our goal is to illustrate
commonly used methods employed in risk assessment. To this end, the book covers the following five major sections:

- General methodology, including a discussion of “risk-neutral versus real-world” scenarios, techniques used, distributions and fitting, random number generation, and risk measures.
- Current applications, including economic scenarios, life, health, and non-life models, and country- or region-specific issues.
- Evaluation of results, including calibration, validation, audit, peer review, and communication.
- Seven case studies that show various applications of scenario development in real situations, including pricing, economic capital analysis, and embedded value analysis.
- References and additional resources available to the reader.

This book is intended to be an on-going resource; however, stochastic modeling is expected to continue evolving following this publication. Still, we hope to remove much of the “black-box” mystique that surrounds these models by illustrating the methods that are currently known.

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A book of this breadth and depth cannot be undertaken without broad international participation. The IAA would like to acknowledge Milliman who has assembled an international group of actuaries and economists to gather information and share their knowledge. We would like to thank the contributing authors to the book and the assistance of the IAA Ad Hoc Project Oversight Group, whose guidance and support were the impetus for completion of this document.

The opinions, viewpoints, and conclusions presented in this book are those of the contributing authors and are not necessarily those of the IAA, the supporting organisations, or Milliman, Inc. They should not be interpreted as prescribing actuarial standards of practice in any respect.
I. General Methodology

The often-quoted definition of actuarial science as “the quantification, analysis, and management of future contingent risk and its financial consequences”\(^{16}\) was probably not made specifically in reference to stochastic modeling, but in many ways it could have been. We have only to consider the simplest of examples, such as the need to determine the present value at a fixed interest rate of $1 to be paid at the death of an individual at some future time, or the amount of payment an insurer might be obliged to make for the negligent action of one of its policyholders, to appreciate how this definition applies. In these and nearly all applications in actuarial science, we are faced with a range of possible future outcomes, some more likely than others, and are called upon to make assertions about quantities whose actual values are uncertain. This function – the determination of various possible values of unknown quantities – provides a broad definition of stochastic modeling.

I.A Stochastic Models vs. Non-stochastic Models

A stochastic model is a mathematical simplification of a process – financial or otherwise – involving random variables. The primary purpose of a deterministic method is to provide a projection based on a single set of assumptions selected by the user. Depending on the purpose of the model, the single set of assumptions may represent an average expected outcome, a “stress” scenario, or some other single outcome the user is interested in understanding. Both of these are quite useful in understanding more about the processes they are modeling, but there are several aspects of modeling we can examine to determine when stochastic models should be used.

When should stochastic models be used?

As computing power has become cheaper and more widely available, the use of stochastic modeling has grown more and more popular. But to what extent has this growth been driven by necessity or the general perception that a more complex, computation-intensive model is better?

The situations for which stochastic modeling is required generally fall into the following categories:

- **When required by regulation and/or standards of professionalism**
  Increasingly, government regulations, accounting standards, and/or professional guidelines dictate the use of stochastic modeling. In the United States, for example, the principles-based approach to calculation of statutory liabilities for life insurers requires stochastic modeling on many blocks of business. Stochastic techniques are also sometimes indicated for the calculation of risk-based capital.

- **When analyzing extreme outcomes or “tail risks” that are not well understood**
  Scenarios intended to represent one-in-100-year events or one-in-1,000-year events are typically used to stress-test asset or liability portfolios. In some cases, such as a “Black Tuesday” stock market event, the modeler may even believe that he or she has a handle on the likelihood of an extreme event. But recent experience has proved differently in many cases. How many would have included the “perfect storm” scenario of low interest rates and low equity returns of the early 2000s or an event that could cause the catastrophic losses of the World Trade Center terrorist attack? How many would have contemplated the 2008/2009 credit crisis that has affected the United States and European markets? Stochastic modeling would likely have revealed these scenarios, even if indicating they were extremely unlikely.

Yet financial risks are among the best understood risks with which actuaries deal. Mortality and morbidity risks are often much more complex, especially when considering the scope of assessing pandemic risk such as the 1918 Spanish flu event. Such an event could represent a 1-in-1,000 event, or it might not. Insufficient frequency data, rapidly evolving healthcare technology, and increasing globalization could alter the likelihood of a contemporary pandemic and call into question the credibility of using a few select stress scenarios to understand the true exposure. Another approach would be to develop a stochastic model that relies on a few well-understood parameters and is appropriately calibrated for the risk manager’s intended modeling purpose.

- **When using certain risk measures, such as Value at Risk (VaR) or Conditional Tail Expectation (CTE)**
  These risk measures, which quantify a company’s exposure to tail risks, provide an understanding of how an observed phenomenon – interest rates, mortality, etc. – behaves in the tail. Stochastic modeling is required in order to completely describe the probability distribution of the tail.

- **When certain percentiles are required**
  Assigning a percentile to a particular outcome is an excellent way of gaining insight into how extreme the occurrence of an event is. For example, would we rather know that the “Black Tuesday” stock market scenario represents an extreme event, or that the probability of stock market returns being worse than the “Black Tuesday” scenario is 0.005%? Stochastic modeling is the approach to use if probabilities are required.

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Note that here and throughout the term “recent” will refer to the period of approximately 2007-2009, when this book was written.
When one wants to understand where stress tests fall in the broader spectrum of possible outcomes

In many cases, stress tests are prescribed by regulators or other professional organizations. A prime example is the use of seven interest rate scenarios for cash flow testing of life insurance liabilities in the United States. These scenarios satisfy regulatory requirements, but provide little insight into a company’s true risk exposure, especially if the stress scenarios fall between the 25th and 75th percentiles. Thorough risk management, however, requires an understanding of where the stress scenarios fall in the broader range of possible outcomes.

In other cases, knowing the range of reasonably possible outcomes is desirable, but not strictly required. When optional, stochastic modeling should be used at the discretion of the actuary, who must weigh the benefits of a sophisticated stochastic analysis against the increased costs.

When should use of stochastic models be questioned?

In still other cases, stochastic modeling may not be of sufficient value to justify the effort. Examples include:

- **When it is difficult or impossible to determine the appropriate probability distribution**

  The normal or lognormal distributions are commonly used for many stochastic projections because of their relative ease of use. However, there are times when these distributions do not properly describe the process under review. In these situations a distribution should not be forced onto the variable. Indeed, there are many other distributions available as described in other parts of this book; the analysis of which distribution best describes the available information is a critical part of the overall modeling exercise. However, there are still times when it is impossible to determine the appropriate distribution.

- **When it is difficult or impossible to calibrate the model**

  Calibration requires either credible historical experience or observable conditions that provide insight on model inputs. If neither is available, the model cannot be fully calibrated and the actuary should carefully consider whether its use is appropriate. Sometimes, an actuary is required to develop a model when no credible, observable data exists. Such a situation requires professional actuarial judgment and the lack of data on which to calibrate the model should be carefully considered.

- **When it is difficult or impossible to validate the model**

  Stochastic models are less reliable if the model output cannot be thoroughly reviewed, comprehended, and compared to expectations. If the actuary must use a model from which the output cannot be reviewed and validated, professional actuarial judgment is required and other methods of review and checking should perhaps be considered. The method of review and checking will necessarily involve a careful consideration of the specifics of the situation.
Alternatives to stochastic models

When the use of stochastic models is impractical or not of sufficient value, some possible alternatives include:

- **Stress testing/scenario testing**

  Stress testing (also known as scenario testing) examines the outcome of a projection under alternative sets of assumptions. In many cases, the alternative scenarios are intended to represent extreme scenarios, such as pandemic risk or stock market crashes. In this way, stress testing can provide some insights into the risk of an asset or liability portfolio. In other cases, the alternative scenarios are intended to gauge sensitivity of outcomes to certain assumptions. If an actuary lacks full confidence in a certain assumption, for example, a stress test of that assumption may help to understand its materiality. If the outcome of the projection is insensitive to the assumption, it may not make sense to invest the time and effort required for fine-tuning the assumption and/or stochastically modeling the assumption to understand the risk profile.

- **Static factors (or “load factors”)**

  In many cases, an actuary may rely on previously developed or commonly used "load factors" to account for risk. Under this approach, static factors (usually multiplicative) are applied to deterministic results to account for potential variability. For example, a factor, which accounts for variability in the liability assumptions, could be applied to a liability of $1 million developed via a deterministic or formulaic approach. In the United States, this approach was adopted by the National Association of Insurance Commissioners (NAIC) for the calculation of risk-based capital (RBC). However, there are widely recognized weaknesses to this approach, and many countries – the United States included – are moving away from this methodology. Weaknesses of load factors include, among other things, the inability to tailor the risk load to the specific block of business under consideration, the inability to apply professional judgment, and the possibility that the risk loads are not fully understood (e.g., what risk factors are reflected, and to what extent, in the risk load).

- **Ranges**

  Ranges can be used to account for uncertainty of a "best" estimate or, alternatively, the variability of potential outcomes. A range of reasonable best estimates can be selected using multiple "reasonable" estimates or using a best estimate and assigning a range of reasonable estimates such as 90% to 110% of the point estimate. The factors should be based on informed judgment and/or past experience. A range designed for uncertainty of a "best" estimate is generally as narrow as reasonably possible, is focused on describing the central estimate itself, and is not intended as a description of the range of possible outcomes. On the other hand, a range designed to describe variability of potential outcomes is generally wider (how much depends on the target percentile width) and is focused on describing a particular confidence interval (or equivalent).
Disadvantages of stochastic models

Stochastic modeling has its advantages, many of which are fairly apparent from the above discussion, but it also has some limitations. These include:

- **The “black box” phenomenon**
  Stochastic models are quite complex and often viewed as “black boxes” into which data and assumptions are entered and results are magically produced. This perception can be true, if the model is not properly understood.

- **Improper calibration or validation**
  Users often fail to properly calibrate or validate the model. Improper calibration is often caused by a misunderstanding of the model’s purpose or by the use of inappropriate historical data. Much like the “black box” phenomenon, improper validation is often related to a misunderstanding of the model.

- **Uses of inappropriate distributions or parameters**
  Users of stochastic modeling often assume – blindly or with very limited empirical evidence – that a variable has a particular distribution. While many processes can approximate a normal distribution, more often than not the possible outcomes are skewed. Users of stochastic modeling should be cautioned against the uninformed use of a particular distribution, which can generate scenarios producing completely useless results. Such a misguided assumption can be especially problematic when examining tail risk.

Guidance on stochastic model implementation

Before we begin a discussion of the technical components of stochastic modeling, it is useful to outline the implementation steps. In general, the modeler needs to:

1. Describe the goals and all of the intended uses of the model.
2. Decide if stochastic modeling is necessary or if an alternative approach will yield equally useful results.
3. Determine which of the projection techniques (Monte Carlo, lattice methods, regime-switching models, etc.) should be used, if a stochastic model is deemed appropriate.
4. Decide on the risk metrics (VaR, CTE, etc.) to use.
5. Establish which risk factors need to be modeled stochastically.
6. Determine the approach for modeling these risk factors in terms of which distributions or models should be used, and how to parameterize or “fit” the distributions.
7. Determine the number of scenarios necessary to reach the point at which additional iterations provide no additional information about the shape of distribution. This point is often figured by performing several different runs – one with 100 scenarios, one with 1,000 scenarios, one with 2,000 scenarios, etc.
8. Calibrate the model.
9. Run the model.
10. Validate the model and review output.
11. Conduct a peer review.
12. Communicate results.

We have already discussed the need to first determine whether the use of a stochastic model is appropriate. We now turn to a detailed discussion of various stochastic projection techniques.

I.B Risk-neutral vs. Real-world

Stochastic models have a wide range of applications in insurance and finance. In many of these models, stochastically simulated variables include stock prices, security values, interest rates, or other parameters describing market prices of instruments, and trials (or paths or scenarios) of market prices or rates generated over time. The methodology for determining the stochastic evolution of such prices is referred to as an economic scenario generator (ESG).

The processes underlying an ESG may simply isolate a single-market variable that directly drives the value of the instruments in question, such as a single equity index. Or the ESG may involve the joint simulation of many variable combinations, such as multiple equity indices, several points on the yield curve, currency exchange rates, inflation rates, yield curves in multiple countries, and historical and implied volatilities.

We broadly distinguish between two classes of ESG: real-world and risk-neutral. It is important to note that real-world and risk-neutral scenarios should produce the same expected present value of cash flows.

Real-world scenarios use expected cash flows and a discount rate that reflects the risk believed to be associated with those cash flows. Risk-neutral scenarios use risk-adjusted cash flows and a risk-free discount rate. The risk-adjusted cash flows are adjusted in such a way as to ensure that the expected present value of cash flows between the risk-neutral and real-world scenarios are consistent.

Real-world scenarios can be connected with risk-neutral scenarios via the use of deflators, which establish an equivalence between these two methodologies. The concept of deflators is explored more thoroughly later in the document in Section I.B.5.

This overview of risk-neutral and real-world ESGs is meant to provide a basic context for the more technical discussion to follow. Extensive academic research and reference material also exists, which might help the reader better understand the subject.

I.B.1 Risk-neutral Scenarios

Background

Risk-neutral scenarios are used to value cash flows for the primary purpose of reproducing prices observed in the market as of a specific date. One way to think of the risk-neutral measure is to consider it a representation of the probability distribution of underlying security prices if all investors...
were “risk-neutral” or indifferent to risk. Under this scenario, investors would not demand a premium for holding riskier assets, and all assets would be expected to earn the risk-free rate available over a given time period. Accordingly, in risk-neutral scenarios, the expected return of every asset is the risk-free rate associated with the given term. It should be noted that in construction of risk-neutral scenarios, it does not matter that risk premiums are observed or expected in the real-world; rather, the “risk-neutral world” is an alternate world where such premiums do not exist.

A well-established result of financial economics is the principle of risk-neutral valuation, which states that the value of any financial derivative is the mathematical expectation of the path-wise present value of cash flows under a risk-neutral measure (or probability distribution). For some derivatives, there may be analytical solutions to the mathematical expectation. In many cases, however, there are not, and the solution can be estimated using Monte Carlo methods that simulate the underlying state variables and security prices, and then calculate the sample expectation over a number of trials. As the number of trials used becomes arbitrarily large, the Monte Carlo estimate of the price converges to the theoretical value.

Risk-neutral scenarios are Monte Carlo draws from the risk-neutral probability distribution of the underlying security prices on which the payouts of a derivative are dependent. A risk-neutral scenario generator is structured so that the stochastic processes for the underlying state variables or security prices have expected growth rates consistent with the risk-neutral measure in each and every time period.

We often use the term “arbitrage-free” to refer to risk-neutral scenarios, as the principle of no arbitrage is the foundation for risk neutrality in the correct pricing of derivatives. In this book, the terms “market-consistent” and “risk-neutral” are used somewhat interchangeably, because one of the purposes of risk-neutral scenarios is to reproduce observable market prices as well as to price exotic or complex derivatives consistently with observed market prices.

Uses

Risk-neutral scenarios are typically used by practitioners to answer the following types of questions:

- What is the market-consistent value or fair value of an insurance liability?
- What is the expected hedging cost of an insurance guarantee with an embedded derivative?
- What is the price or fair value of an exotic derivative?
- How much would a market participant demand to assume liability cash flows?

All of these questions effectively ask: What is a set of cash flows worth in the market? As a result, they cover a much narrower scope than questions addressed by real-world scenarios of possible outcomes. Typically, only the mean present value of cash flows from a set of risk-neutral scenarios has significance; other features of the distribution are not relevant.

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For the purposes of this non-technical discussion, we consider measure and probability distribution to be equivalent.
Use of risk-neutral scenarios for pricing purposes involves discounting cash flows to obtain an expected present value. Typically, these scenarios are generated by assuming cash flows will be discounted under risk neutrality; that is, the scenarios use discounted factors appropriate for valuing bonds with no cash flow uncertainty.

**Calibration and parameterization**

A risk-neutral ESG is typically parameterized to observed market prices of relevant instruments. For instance, a simple lognormal equity model with two constant parameters (equity drift rate and return volatility) can be calibrated to be consistent with prices of two instruments – a single zero-coupon bond (or swap) and a single equity option of a given maturity and strike price. A lognormal equity model with time-dependent drifts and volatility can be calibrated to replicate market prices of a zero-coupon bond and an equity option of every maturity within the projection horizon.

The calibration process becomes increasingly complicated with the complexity of a model. With stochastic equity volatility or stochastic interest rate models, the goal of the calibration might be to fit a large sample of market prices of options or swaptions. When there are more instruments to fit than there are parameters in the model, optimization may be used to find the combination of parameters that constitutes the best fit to the overall set of market prices.

When calibrating a model of stochastic interest rates and equity returns to observed market prices of equity options, the modeler should be careful to define the equity return volatilities being used as excess-return volatilities, rather than total-return volatilities, and then adjust the market inputs accordingly. Excess return refers to the additional return that equities earn above the risk-free rate of return.

Market prices of options and swaptions may themselves be quite volatile from day to day, resulting in parameter instability. When the risk-neutral ESG is used for the management of an ongoing hedging program, practitioners may choose to adjust their parameterization process to sacrifice fit to observed market prices in favour of parameter stability.

Risk-neutral ESGs should be tested by using the generator to calculate prices of instruments (calculating the expected present value of the cash flows) and to confirm they are consistent (within Monte Carlo error) with the observed market prices to which the model has been calibrated. In addition, the user may also gain insight by pricing derivatives outside of the calibration set to test for reasonableness.

Technical discussion on calibration of specific models is covered in other sections of this book.

**Other considerations**

Sometimes risk-neutral ESGs will have a limited number of observable market inputs for calibration. One example is a situation in which the longest available traded option has a 10-year maturity, but the ESG is required to produce scenarios that extend 30 years. In this case, judgment regarding assumptions in the non-traded space is needed; the solution is often a calibration that blends market prices with historical data. In other situations, market prices may be available but unreliable because of illiquidity or for other reasons. The practitioner will have to apply judgment on which prices to use and which ones to bypass.
I.B.2 Real-world Scenarios

Background
In contrast to risk-neutral scenarios, which use an unrealistic assumption about risk premiums for purposes of calculating derivative prices under the no-arbitrage assumption, real-world scenarios take into account investor risk preferences. The compensation required by investors to take on riskier portfolios is either implicitly or explicitly used in the construction of these scenarios. By definition, real-world scenarios seek to avoid the unrealistic assumptions of the risk-free world.

The goal of real-world scenarios is to produce a realistic pattern of underlying market prices or parameters that will ultimately be used to generate realistic distributions of outcomes. Because the past is often viewed as a good basis for future expectations, real-world scenarios are often developed to maintain consistency with observed historical market experience. But they can also be vulnerable to a considerable amount of subjectivity. Different modelers might have different views about the course of future market movement, the reasonableness of certain scenarios, and the relationship between assumed parameters and results.

Real-world scenarios seek to maintain consistency with stylized facts observed in the actual evolution of market prices. These include the following:

- Risk premiums above the risk-free rate that equities tend to earn over the long term.
- Term premiums that investors in longer-term bonds require over the yields of shorter-term bonds.
- Credit spreads in excess of long-term default costs that compensate the holders of credit risk.
- Implied volatilities reflected in option prices that are in excess of realized return volatility exhibited by equities. Implied volatilities are commonly measured using market-observed prices and a pricing model such as the Black-Scholes equation. For example, the modeler enters values for all variables in the pricing model, including market-observed price, but excluding volatility. The modeler then solves for the volatility in the pricing model that would produce the market-observed price.

Note that none of the above are reflected in the development of risk-neutral scenarios. Specific features of different models are discussed in other sections.

Uses
Real-world scenarios are used to answer what can be broadly classified as “what-if” questions such as:

- What does the distribution of present value earnings look like?
- How much capital does the company need to support worst-case scenarios?
- What level of pricing is required to earn a target return at least X% of the time without assuming excessive downside risk?
- What kind of earnings volatility could a block of business generate?
- How does the distribution of final capital change with changes in investment strategy?
- How much residual risk remains from alternative hedging or risk management strategies?
In a sense, each of these questions asks, “What outcome might we expect if market prices behave as we might realistically expect?” The answers allow managers to see the implications of what might happen given a particular evolution in market prices, rates, or parameters, and thereby guide their understanding of possible outcomes.

### I.B.3 Techniques

In this section, we will discuss the various projection techniques, including Monte Carlo simulations, lattice methods, and regime-switch models, that can be used for financial reporting and capital assessment.

#### I.B.3.a Monte Carlo Simulation

The Monte Carlo (MC) method is one of the most efficient numerical methods in finance, particularly as it applies to risk analysis, stress testing of portfolios, and the valuation of securities. Since this method was first developed, its efficiency has been substantially increased. Its development is, indeed, fortuitous in view of the fact that, as global markets become more efficient, more complex stochastic modeling is needed.

In modern financial theory, the prices of financial instruments are often modeled by continuous-time stochastic processes. Assuming no arbitrage, the price of a derivative security is the expectation of its discounted payoff under the risk-neutral measure. Here, the expectation is truly an integral of the payoff function with respect to the risk-neutral measure. The dimension of the integral is the number of observations. The models should choose an efficient numerical valuation method that can approximate the actual solution as closely as possible in a fairly rapid manner, unless a closed-form solution does not exist.

In order to evaluate the expectation, an MC method draws upon random samples and approximates them with an average of all the outcomes. In general, the modeler will need to:

- Generate sample paths of the underlying asset price or interest rate over a certain time period under the risk-neutral measure.
- Calculate the discounted cash flows of the sample paths.
- Average the discounted cash flows of a security on a sample path.

For example, suppose a stock price follows a lognormal distribution, i.e., the natural logarithm of the stock return is normally distributed, where $S_t$ is a stock price at time $t$. To simulate the stock price at time $t$, the algorithm of MC simulation would be as follows.

**Algorithm I.B-1**

1. Generate $\varepsilon \sim N(0,1)$
2. $S_t = \exp [\mu^* + \sigma^* \varepsilon]$

where $N(0,1)$ denotes a standard normal distribution.