FAIR VALUATION OF LIFE INSURANCE CONTRACTS UNDER
A CORRELATED JUMP DIFFUSION MODEL

BY

YINGHUI DONG

ABSTRACT

In this paper, we study the fair valuation of participating life insurance contract, which is one of the most common life insurance products, under the jump diffusion model with the consideration of default risk. The participating life insurance contracts considered here can be expressed as portfolios of options as shown by Grosen and Jørgensen (1997). We use the Laplace transforms methods to price these options.

KEYWORDS

Participating life insurance policies; correlated jump diffusion process; default.

1. INTRODUCTION

Life insurance companies offer complex contracts written with the following many covenants: interest rate guarantees, bonus and surrender options, equity-linked policies, participating policies, and so on. Each particular covenant has a value and is part of the company liabilities. So they should be properly valued and reported separately on the liability side of the balance sheet. But historically this has not been done. Many life-insurance companies increased the difficulties they faced in the 1990s for their neglecting them for a long time. A large number of companies have been unable to meet their obligations and have simply defaulted. Luckily, the research of fair valuation of liabilities has been attracting a lot of attention. Brennan and Schwarz (1976) firstly use the contingent claim analysis in this area. Briys and de Varenne (1994, 1997) value the assets and liabilities of an insurance company and obtain closed-form formulae by assuming default can occur only at maturity. Their framework is of the Merton type. Grosen and Jørgensen (1997, 2002), Bernard et al. (2005) focus on the modelling of early default of the insurance company in an extended Black-Scholes economy. Bacinello (2001, 2003) who focuses in particular on the valuation of the surrender option in a discrete time framework.
by means of the CRR model. Ballotta et al. (2006) analyze the fair design of participating contracts in the case of a very general structure, which includes a minimum guarantee, a scheme for the distribution of the annual profit of the insurance company, a terminal bonus, the so-called default option and shareholders’ contribution, in a simple Black-Scholes economy with constant interest rates. There are many contributions, to name only a few. Many of the recent studies rely on the Black-Scholes model (BSM). The BSM assumes the asset returns are normally distributed and their variances remain constant. But empirical studies invalidate such assumptions by suggesting two observations for asset returns: the asymmetric leptokurtic feature, i.e., the actual return has much heavier tails than normal, and the volatility smile, i.e., the volatility implied from equity option prices is not a constant but presents a curve resembling a “smile”. So using this not realistic hypothesis of normal returns can lead to mispricing life insurance contracts. An alternative approach is to describe financial price movements by geometric jump diffusion process (see Ballotta (2005), Riesner (2006)). The double exponential jump diffusion model (DEM) introduced by Kou (2002), Kou and Wang (2003, 2004), Huang and Huang (2003), is a special example. Le Courtois and Quittard-Pinon (2008) obtain the quasi-closed form formulas for the valuation of participating life insurance contracts under the DEM. Recently, Cai and Kou (2007) extend the DEM to a hyper-exponential jump diffusion model (HEM). In this paper, we follow Le Courtois and Quittard-Pinon (2008) to study the fair valuation of participating life insurance contracts, which is the same as in Grosen and Jørgensen, under the jump-diffusion model. But allowing for different kinds of unexpected information may trigger different kinds of jump sizes, we model the firm-value process by considering a more generalized jump diffusion model. We consider the company has the possibility to go bankrupt, and default can occur only at maturity or at any time during the contract life. We aim at using the Laplace transforms, which have been widely used in valuing financial derivatives (see Kou, et al. (2005), Pelsser (2000)), to derive the fair valuation of the contracts.

The paper is organized as follows. In section 2 we present the contracts, the default mechanism and the model. Section 3 focuses on the valuation of the participating life insurance contracts. Section 4 gives a variety of numerical analysis by inverting the Laplace transforms via the Euler inversion algorithm.

2. The model

Participating policies are by far the most important in terms of market size. We show how to price a participating life insurance contract with a minimum guaranteed rate in presence of default risk of the issuing company. This section provides a more detailed description of the life insurance contracts and the default process.
2.1. The contracts

We consider an insurance company on the time period \([0, T]\). The company has only two types of agents: policyholders and shareholders. Agents are assumed to operate in a continuous time frictionless economy with a perfect financial market. The policyholders possess the same unique contract which will be defined precisely in the following. The considered life-insurance company has no debt and its planning horizon is finite with \(T\) as maturity, being also the expiry date of the contract. Let \(A_0\) be the assets initial value, \(L_0 = \alpha A_0\) with \(\alpha \in [0, 1]\) the initial investment by policyholders, and \(E_0 = (1 - \alpha) A_0\) is the initial equity. The policyholders are guaranteed a fixed interest rate \(r_g\). So, the guaranteed amount at \(T\) is a priori \(L^g_T = L_0 e^{r_g T}\). However, when the firm defaults, this amount will be lowered, on the contrary it will be raised if financial earnings are sufficient. The next step is to express these payments according to the firm’s assets dynamics. We examine two cases: when default occurs only at maturity, and when this event can occur at any time between inception and maturity, called here early default.

2.2. Default at Maturity

At maturity \(T\), if \(A_T < L^g_T\), the company cannot fulfill its commitments and the firm is declared bankrupt. Policyholders receive \(A_T\) and equityholders nothing. If \(A_T \geq L^g_T\), we distinguish between two scenarios: \(A_T > \frac{L^g_T}{\alpha}\), and \(L^g_T \leq A_T \leq \frac{L^g_T}{\alpha}\). In the former case, the policyholders are given a bonus, say \(\beta\), a contractual part of the surplus, known as the participation coefficient, whereas in the latter case, the guaranteed amount \(L^g_T\) is paid out to the policyholders. To sum up, policyholders receive at \(T\), assuming no prior bankruptcy:

\[
\Theta^*_T(T) = \begin{cases} 
A_T & \text{if } A_T < L^g_T \\
L^g_T & \text{if } L^g_T \leq A_T \leq \frac{L^g_T}{\alpha} \\
L^g_T + \beta(\alpha A_T - L^g_T) & \text{if } A_T > \frac{L^g_T}{\alpha}
\end{cases}
\]

Here we assume the life insurance contracts are considered as purely financial assets traded on liquid market among perfectly informed investor and the firm’s assets are earmarked from the beginning, therefore the above formula means the policyholders have a contingent claim for a payoff on the maturity. This claim is very similar to financial derivatives with the firm’s assets as the well-defined underlying asset. Hence, we can using the contingent claim valuation approach. We rewrite this payoff in a more concise form:

\[
\Theta^*_T(T) = L^g_T + \beta(\alpha A_T - L^g_T)^+ - (L^g_T - A_T)^+,
\]
where we have used \((x)^+ = \max\{x, 0\}\). This payoff consists of three parts: the first term is the promised amount, the second term a bonus option, is linked to the participating clause, the third one is a put option associated with the default risk. These two last payoffs share the same features as usual European options. According to our fundamental hypothesis and assuming that the assets dynamics follows a DEM, Kou and Wang (2004) have priced them.

2.3. Early default

We assume that default can occur prior to the maturity \(T\). The default mechanism we choose is of a structural type, so we introduce an activating barrier on the firm’s assets. From now on, bankruptcy can occur at any time \(t\) before \(T\). The contract value depends on the assets price before the expiry of the contract and not only on their price at \(T\). The barrier is chosen exponential and is denoted by \(B_t\).

The firm pursues its activities until \(T\) if:

\[
\forall t \in [0, T], A_t > \gamma L_0 e^{\nu t} \equiv B_t.
\] (2.2)

As soon as condition (2.2) is not satisfied, the company is declared bankrupt. The default time \(\tau\) is defined as the first time \(A\) hits or crosses the barrier \(B\):

\[
\tau = \inf\{t \in [0, T] | A_t \leq B_t\}.
\] (2.3)

As Le Courtois and Quittard-Pinon (2008) interpreted: with \(\gamma\) greater than 1, the managers are conservative and declare default as soon as the assets become inferior or equal to the amount \(L_0 e^{\nu T}\) which is superior to the nominal amount due. On the contrary, \(\gamma < 1\) means that the managers are optimistic: they believe that even though the assets become smaller than \(L_0 e^{\nu T}\), it is not necessary to declare bankruptcy immediately. In the case of \(\gamma < 1\), the firm is totally insolvent in the case of bankruptcy and unable to meet its commitments.

Upon early default, as upon any type of default, the residual value of the assets is redistributed to the liabilityholders. Following Le Courtois and Quittard-Pinon (2008), here we assume \(\gamma < 1\), so that \(A_t\), the amount remaining after the jump causing default and which can be inferior to \(\gamma L_0 e^{\nu T} < L_0 e^{\nu T}\), is completely paid out to the policyholders, that is, policyholders will receive \(A_t\) in case of early default.

3. Fair valuation in a correlated jump diffusion model

Using the standard machinery of arbitrage theory in continuous time, we can price the contracts under the BSM. But under a diffusion process, firms never default by surprise because a sudden drop in the firm value is impossible. Thus, many authors model the firm-value process as a jump diffusion process.
With the jump risk, a firm can default instantaneously because of a sudden jump in its value. But it is hard to give the closed form formulas under the jump diffusion model. Le Courtois and Quittard-Pinon (2008) obtain the quasi-closed form formulas under the DEM. In this section, we use the Laplace transforms, inspired by Kou (2005) to price the liabilities by considering a more generalized jump diffusion model.

Consider an economy defined on a filtered complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\). The insurance and financial markets are embedded in this economy and we assume them pure and perfect. Moreover, we assume that an equivalent martingale measure \(\mathbb{Q}\), such that the discounted prices at the constant interest rate \(r\) are \(\mathbb{Q}\) martingales has been chosen.

In the jump diffusion process, the value of the firm jumps because of the arrival of unexpected market information or special events, such as a bulk order, winning a lawsuit and there are different kinds of information, such as, macroeconomic variables, interest rates, and idiosyncratic risk of a specific industry sector. In this article, we consider a company may have \(n\) dependent classes of business. For example, a company deals mainly in mineral products, but it also engages in real estate. Hence the total returns of the company is determined by the returns of every class of business. Obviously not all information is relevant for every class of business, and therefore we should classify all the unexpected market information or special events. Motivated by Wang and Yuen (2005), we assume there are \(m\) different types of unexpected information or events and the different varieties of unexpected information or events arrive as mutually independent Poisson processes \(N_1(t), N_2(t), \ldots, N_m(t)\) with parameters \(\lambda_1, \lambda_2, \ldots, \lambda_m\). Suppose that each event occurred in the \(k\)th type may cause a jump in the \(j\)th class of business with probability \(p_{kj}\), for \(k = 1, 2, \ldots, m\), \(j = 1, 2, \ldots, n\) and denote by \(N_{kj}(t)\) the number of jumps of the \(j\)th class caused by the events in group \(k\) occurred up to time \(t\). Obiously \(\{N_{kj}(t); t \geq 0\}\) is a Poisson process with parameter \(\lambda_k p_{kj}\), \(k = 1, 2, \ldots, m; j = 1, 2, \ldots, n\). Therefore, the jump-number process of the \(i\)th class can be expressed as \(N_i(t) = N_{i1}(t) + \cdots + N_{im}(t)\), and \(N_i(t), \ldots, N_n(t)\) may be dependent Poisson processes. Let \(X_{ij}\) be the amount of the \(i\)th jump in the \(j\)th class. It is assumed that \(\{X_{ij}; i = 1, 2, \ldots\}\) is a sequence of i.i.d. random variables with common distribution \(F_j\) for each \(j\), and the \(n\) sequences \(\{X_{1j}; i = 1, 2, \ldots\}, \ldots, \{X_{nj}; i = 1, 2, \ldots\}\) are mutually independent and are independent of \(N_1(t), \ldots, N_m(t)\).

**Remark.** A typical example: For \(n = 2, m = 3, p_{11} = p_{22} = p_{31} = p_{32} = 1, p_{12} = p_{21} = 0\). Suppose the company has two classes of business, so the value of the company is determined by two classes of business. \(N_1(t)(N_2(t))\) counts the number of special events that only causes a jump in the first(second) class with probability \(1\), \(N_3(t)\) counts the number of special events that cause a common jump in two classes with probability \(1\), such as macroeconomic policies. This is the so-called “common shock structure” discussed in Cossette and Marceau (2000).

For simplicity, in this paper we only consider the case of \(n = 2\). Now we model the assets value process under the risk-neutral measure \(\mathbb{Q}\) as
\[ A_t = A_0 e^{\mu t + \sigma W(t) + \sum_{i=1}^{N^1(t)} X_i^{(1)} + \sum_{j=1}^{N^2(t)} X_j^{(2)}} \]  

for \( t \geq 0 \), where \( A_0 > 0 \) is the initial assets value; \( \mu \) is the linear growth rate of \( A_t \) under \( Q \) satisfying \( E^Q[e^{-\tau A_t}] = A_0 \), \( \sigma > 0 \) is the volatility of the diffusion component; \( \{W(t)\} \) is a standard Brownian motion. It is assumed that \( \{W(t)\} \), and \( \{X_i^{(j)}\}, j = 1, 2 \) are mutually independent and are independent of \( N_1(t), N_2(t) \). Let

\[ X(t) = \mu t + \sigma W(t) + S(t), \]  

where \( S(t) = S_1(t) + S_2(t) = \sum_{i=1}^{N^1(t)} X_i^{(1)} + \sum_{j=1}^{N^2(t)} X_j^{(2)} \) is the sum of two compound Poisson processes \( S_1(t), S_2(t) \). In Wang and Yuen (2005), the process \( S(t) \) is called thinning-dependence structure.

To make the analysis of \( S(t) \) mathematically tractable, we assume (see Wang and Yuen (2005)):

(A1) For \( k \neq k' \), the two vectors of claim-number processes, \((N^k(t), N_1^k(t), N_2^k(t))\) and \((N^{k'}(t), N_1^{k'}(t), N_2^{k'}(t))\), are independent.

(A2) For each \( k (k = 1, \ldots, m) \), \( N_1^k(t), N_2^k(t) \) are conditionally independent given \( N^k(t) \).

From Wang and Yuen (2005), the covariance of \( N_1(t) \) and \( N_1(t) \) is

\[ \text{Cov}(N_1(t), N_2(t)) = \sum_{k=1}^{m} \lambda_k p_{k1} p_{k2} t. \]

The covariance of \( S_1(t) \) and \( S_1(t) \) is given by

\[ \text{Cov}(S_1(t), S_2(t)) = p_1^{(1)} p_1^{(2)} \sum_{k=1}^{m} \lambda_k p_{k1} p_{k2} t, \]

and the variance of \( S(t) \) is

\[ \text{Var}[S(t)] = \sum_{k=1}^{m} \lambda_k p_{k1} p_1^{(2)} t + \sum_{k=1}^{m} \lambda_k p_{k2} p_2^{(2)} t + 2p_1^{(1)} p_1^{(2)} \sum_{k=1}^{m} \lambda_k p_{k1} p_{k2} t, \]

where \( p_i^{(j)} \) denotes the \( i \)th moments of \( X_i^{(j)} \).

Therefore, we have if there exists \( k \), such that \( p_{k1} p_{k2} \neq 0 \), \( k = 1, 2, \ldots, m \), then \( N_1(t), N_2(t) \) are dependent and \( S(t) \) is the sum of two dependent compound Poisson processes \( S_1(t), S_2(t) \).

The following Lemma shows \( S(t) \) is also a compound Poisson process.

**Lemma 1.** Under the assumptions (A1), (A2), we have the process of \( \{S(t); t \geq 0\} \) is a compound Poisson process with intensity \( \lambda \) and distribution \( F \) given by

\[ \lambda = \lambda_1(1 - (1 - p_{11})(1 - p_{12})) + \cdots + \lambda_m(1 - (1 - p_{m1})(1 - p_{m2})), \]

\[ F(x) = b_1 F_1(x) + b_2 F_2(x) + b_3 F_1 \ast F_2(x), \]
where \( F_j \) denote the common distributions of \( X_i^{(j)} \) for \( j = 1, 2 \), \( F_1 \ast F_2 \) represents the convolution of \( F_1 \) and \( F_2 \),

\[
\begin{align*}
    b_1 &= \frac{\sum_{i=1}^m \hat{\lambda}_i p_{1i}(1-p_{12})}{\lambda}, \\
    b_2 &= \frac{\sum_{i=1}^m \hat{\lambda}_i p_{2i}(1-p_{11})}{\lambda}, \\
    b_3 &= \frac{\sum_{i=1}^m \hat{\lambda}_i p_{1i}p_{12}}{\lambda}.
\end{align*}
\] (3.5)

**Proof.** We refer to the Example 2.1 of Wang and Yuen (2005).

In this paper, we assume \( X_i^{(j)} \) follows a double exponential distribution with density function

\[
f_j(x) = \frac{1}{2} \alpha_j e^{-\alpha_j|x|} 1\{x < 0\} + \frac{1}{2} \alpha_j e^{\alpha_j|x|} 1\{x > 0\}, \quad j = 1, 2,
\] (3.6)

where \( 0 < \alpha_1 < \alpha_2 \). Therefore, the distribution \( F \) has the density

\[
f(x) = (p_1 \alpha_1 e^{-\alpha_1|x|} + p_2 \alpha_2 e^{-\alpha_2|x|}) 1\{x > 0\} + (q_1 \alpha_1 e^{\alpha_1|x|} + q_2 \alpha_2 e^{\alpha_2|x|}) 1\{x < 0\},
\] (3.7)

where

\[
p_1 = q_1 = \frac{b_1}{2} + \frac{b_3 \alpha_2^2}{2(\alpha_2^2 - \alpha_1^2)}, \quad p_2 = q_2 = \frac{b_2}{2} - \frac{b_3 \alpha_1^2}{2(\alpha_2^2 - \alpha_1^2)}.
\]

From the definition (2.3), the default time \( \tau \) can be rewritten as

\[
\tau = \inf \{ t \in [0, T] | u + X_t - r_g t < 0 \},
\] (3.8)

where \( u = \ln \frac{A_0}{L_0} > 0 \). Let \( Y_t = X_t - r_g t \), then the Laplace exponent of \( \{ Y_t \} \) is given by

\[
g(x) = \frac{1}{T} \log E[e^{xY_t}]
= \frac{1}{2} \sigma^2 x^2 + (\mu - r_g) x - \lambda + \lambda \sum_{i=1}^2 \left( \frac{q_i \alpha_i}{x + \alpha_i} - \frac{p_i \alpha_i}{x - \alpha_i} \right),
\] (3.9)

for \( -\alpha_1 < x < \alpha_1 \).

### 3.1. Preliminary results for the pricing

In order to derive the Laplace transforms of the contract fair value, firstly we will present the following Lemma without giving proofs. The proof is referred to Zhang, Yang and Li (2010).

**Lemma 2.** For \( \delta > 0 \), the equation \( g(x) = \delta \) has exactly three roots \( r_1, r_2, r_3 \) on the left-half complex plane and three roots \( p_1, p_2, p_3 \) on the right-half complex plane.
Remark. It is obviously $p_1 = q_1 > 0$ because of $\alpha_1 < \alpha_2$. If $0 < p_2 = q_2 < 1$, then $f(x)$ is the hyper exponential jump-diffusion model in Cai (2009). And the equation $g(x) = \delta$ has three negative real roots and three positive real roots. Here $p_2, q_2$ could be negative, so the roots of the equation $g(x) = \delta$ are not necessarily real.

Define the joint Laplace transform of $\tau$ and $Y(\tau)$ as

$$
\Phi(u, \delta, \eta) = E_Q\left[ e^{-\delta \tau + \eta (u + Y(\tau))} \mid u + Y(0) = u \right],
$$

(3.10)

for $\delta \geq 0$, and $\eta$ satisfies $E_Q[e^{\eta Y(t)} \mid u + Y(0) = u] < \infty$. To simplify the notation, we drop $\delta, \eta$ in the parameters.

Lemma 3. Assume that the jump size distribution $F$ is absolutely continuous. Then, for $u > 0$, the joint Laplace transform $\Phi(u)$ satisfies the integro-differential equation

$$
\frac{1}{2} \sigma^2 \Phi''(u) + \mu - r_g \Phi'(u) - (\lambda + \delta) \Phi(u) = -\lambda \left[ \int_{-\infty}^{-u} e^{\eta(u+x)} dF(x) + \int_{-\infty}^{\infty} \Phi(u + x) dF(x) \right],
$$

(3.11)

with the boundary condition:

$$
\Phi(u) = e^{ru}, \quad u \leq 0.
$$

(3.12)

The proof of Lemma 3 is presented in Appendix.

Remark. Here we assume $\Phi(u)$ is twice continuously differentiable in $u$ over $(0, \infty)$. The sufficient conditions for the continuous differentiability can be found in Chi (2010).

When $F$ has density $f(x)$ given by (3.7), we can obtain the explicit expression for $\Phi(u)$. Let $I$ denote the identity operator and $D$ denote the differential operator. Define differential operator polynomial

$$
h_2(D) = \frac{1}{2} \sigma^2 D^2 + \mu - r_g D - (\lambda + \delta) I,
$$

(3.13)

where, by convention, $D^2 \Phi(u) = D(D \Phi(u))$. Inserting (3.7) into (3.11) and using (3.13) yields

$$
h_2(D) \Phi(u) = -\lambda \left[ \frac{q_1 \alpha_1}{\alpha_1 + \eta} e^{\alpha_1 u} + \frac{q_2 \alpha_2}{\alpha_2 + \eta} e^{\alpha_2 u} \right.
$$

$$
+ \int_{0}^{u} \Phi(s) (q_1 \alpha_1 e^{\alpha_1(s-u)} + q_2 \alpha_2 e^{\alpha_2(s-u)}) ds
$$

$$
+ \int_{u}^{\infty} \Phi(s) (p_1 \alpha_1 e^{-\alpha_1(s-u)} + p_2 \alpha_2 e^{-\alpha_2(s-u)}) ds \right]
$$

(3.14)
Similar to Gerber and Shiu (2005), applying the differential operator polynomial \((D + \alpha_1 I)(D + \alpha_2 I)(D - \alpha_1 I)(D - \alpha_2 I)\) on the both sides of (3.14), we get the differential equation

\[
(D + \alpha_1 I)(D + \alpha_2 I)(D - \alpha_1 I)(D - \alpha_2 I)h_2(D)\Phi(u) + \lambda(D + \alpha_1 I)(D - \alpha_1 I)q_1 \alpha_1 \Phi(u) + \lambda(D + \alpha_1 I)(D - \alpha_1 I)(D - \alpha_2 I)q_2 \alpha_2 \Phi(u)
\]

\[-\lambda(D + \alpha_1 I)(D + \alpha_2 I)(D - \alpha_1 I)p_1 \alpha_1 \Phi(u)
\]

\[-\lambda(D + \alpha_1 I)(D + \alpha_2 I)(D - \alpha_1 I)p_2 \alpha_2 \Phi(u) = 0.
\]  

(3.15)

Using partial fraction, the characteristic equation of (3.15) is

\[
\frac{\sigma^2}{2}x^2 + \mu - r_s x - \lambda - \delta + \lambda \left( \frac{p_1 \alpha_1}{\alpha_1 - x} + \frac{p_2 \alpha_2}{\alpha_2 - x} + \frac{q_1 \alpha_1}{\alpha_1 + x} + \frac{q_2 \alpha_2}{\alpha_2 + x} \right) = 0. \tag{3.16}
\]

From Lemma 2, we can obtain (3.16) has exactly three roots, \(r_1, r_2, r_3\), on the left-half complex plane, and three roots, \(\rho_1, \rho_2, \rho_3\), on the right-half complex plane. So

\[
\Phi(u) = c_1 e^{r_1 u} + c_2 e^{r_2 u} + c_3 e^{r_3 u}, \tag{3.17}
\]

where \(c_1, c_2, c_3\) are constants

Substituting \(\Phi(u) = \sum_{i=1}^{3} c_i e^{r_i u}\) into the equation (3.11) and equating the coefficients of \(e^{-\alpha_1 u}, e^{-\alpha_2 u}\), with the boundary condition give that

\[
\begin{aligned}
\frac{c_1 \alpha_1}{\alpha_1 + r_1} + \frac{c_2 \alpha_1}{\alpha_1 + r_2} + \frac{c_3 \alpha_1}{\alpha_1 + r_3} &= \frac{\alpha_1}{\alpha_1 + \eta}, \\
\frac{c_1 \alpha_2}{\alpha_2 + r_1} + \frac{c_2 \alpha_2}{\alpha_2 + r_2} + \frac{c_3 \alpha_2}{\alpha_2 + r_3} &= \frac{\alpha_2}{\alpha_2 + \eta}.
\end{aligned} \tag{3.18}
\]

From (3.18), we obtain

\[
\begin{aligned}
c_1 &= \frac{(r_2 - \eta)(r_3 - \eta)(r_1 + \alpha_1)(r_1 + \alpha_2)}{(r_1 - r_2)(r_1 - r_3)(\alpha_1 + \eta)(\alpha_2 + \eta)}, \\
c_2 &= \frac{(r_1 - \eta)(r_3 - \eta)(r_2 + \alpha_1)(r_2 + \alpha_2)}{(r_2 - r_1)(r_2 - r_3)(\alpha_1 + \eta)(\alpha_2 + \eta)}, \\
c_3 &= \frac{(r_1 - \eta)(r_2 - \eta)(r_3 + \alpha_1)(r_3 + \alpha_2)}{(r_3 - r_1)(r_3 - r_2)(\alpha_1 + \eta)(\alpha_2 + \eta)}.
\end{aligned} \tag{3.19}
\]
3.2. Valuation

From the arbitrage pricing theory, the arbitrage free price of the life insurance contract for default at maturity is given by:

\[ V_m(0) = E_Q \left[ e^{-rT} \left( L_T^g + \beta (\alpha A_T - L_T^g) - (L_T^g - A_T) \right) \right] \]

\[ = L_0 e^{(r-r_g)T} + E_Q \left[ \beta e^{-rT} \left( A_T - \frac{L_T^g}{\alpha} \right) \right] - E_Q \left[ e^{-rT} (L_T^g - A_T) \right] \]

\[ = L_0 e^{(r-r_g)T} + \beta e^{-(r-r_g)T} V^1(k) - e^{-(r-r_g)T} V^2(k') \] \hspace{0.5cm} (3.20)

where

\[
\begin{align*}
V^1(k) &= E_Q \left[ (A_T e^{-r_T} - e^{-k}) \right], \quad k = -\log \frac{L_0}{\alpha} \\
V^2(k') &= E_Q \left[ (A_T - e^{-r_T}) \right], \quad k' = \log L_0.
\end{align*}
\] \hspace{0.5cm} (3.21)

Obviously, the above formula does not impose a particular law for the process \( A_T \), the closed form formula can be obtained under BSM and DEM. Here under the HEM, we use Laplace transforms to price it.

**Theorem 1.** For \( \xi > 0 \), the Laplace transforms with respect to \( k \) of \( V^1(k) \) is given by

\[
\hat{L}_1(\xi) = \int_{-\infty}^{\infty} e^{-\xi k} V^1(k) dk = \frac{A_0^{\xi+1}}{\xi(\xi+1)} e^{g(\xi+1)T}. \] \hspace{0.5cm} (3.22)

For \( \xi > 1 \), the Laplace transforms with respect to \( k' \) of \( V^2(k') \) is given by

\[
\hat{L}_2(\xi) = \int_{-\infty}^{\infty} e^{-\xi k'} V^2(k') dk' = \frac{A_0^{-(\xi-1)}}{\xi(\xi-1)} e^{g(1-\xi)T}. \] \hspace{0.5cm} (3.23)

**Proof.** It follows Fubini theorem that

\[
\hat{L}_1(\xi) = \int_{-\infty}^{\infty} e^{-\xi k} V^1(k) dk = E_Q \left[ \int_{-Y_r}^{\infty} e^{-\xi k} (A_0 e^{Y_T} - e^{-k}) dk \right] \]

\[
= E_Q \left[ \int_{-Y_r}^{\infty} e^{-\xi k} (A_0 e^{Y_T} - e^{-k}) dk \right] \]

\[
= \frac{A_0^{\xi+1}}{\xi(\xi+1)} E_Q \left[ e^{(\xi+1)Y_r} \right] = \frac{A_0^{\xi+1}}{\xi(\xi+1)} e^{g(\xi+1)T}, \quad \xi > 0. \] \hspace{0.5cm} (3.24)
The proof of (3.23) is similar.

We can apply some numerical inversion algorithms to recover the values of the functions for some specific $k$.

Using the standard machinery of arbitrage theory again, the arbitrage free price of the life insurance contract $V_e(0)$ for early default is given by:

$$V_e(0) = E_Q\left[ e^{-rT} \left( L_T^e + \beta (\alpha A_T - L_T^e)^+ - (L_T^e - A_T)^+ \right) 1\{\tau \geq T\} \right] + e^{-r\tau} A_T 1\{\tau < T\} \right]$$

(3.25)

This contract can be split up into four simpler subcontracts:

$$V_e(0) = GF + BO - PO + LR,$$

(3.26)

where $GF$ corresponds to the final guarantee, $BO$ stands for the bonus option, $PO$ for the default put on which policyholders are short, and, at last, $LR$ is the rebate paid to policyholders in case of early default. Individually these four subcontracts can be written as:

$$\begin{align*}
GF &= L_0 e^{-(r-\tau)T} E_Q[1\{\tau \geq T\}] \equiv L_0 e^{-(r-\tau)T} GF'(T), \\
BO &= E_Q\left[ \beta e^{-rT} (\alpha A_T - L_T^e)^+ 1\{\tau \geq T\} \right] \equiv \beta e^{-(r-\tau)T} BO'(T,k), \\
PO &= E_Q\left[ e^{-rT} (L_T^e - A_T)^+ 1\{\tau \geq T\} \right] \equiv e^{-(r-\tau)T} PO'(T,k'), \\
LR &= E_Q\left[ e^{-rT} A_T 1\{\tau \leq T\} \right] \equiv LR(T),
\end{align*}$$

(3.27)

where

$$\begin{align*}
GF'(T) &= E_Q\left[ 1\{\tau \geq T\} \right], \\
BO'(T,k) &= E_Q\left[ (\alpha A_T e^{-rT} - e^{-k})^+ 1\{\tau \geq T\} \right], \quad k = -\log L_0, \\
PO'(T,k') &= E_Q\left[ (e^{k'} - A_T e^{-rT})^+ 1\{\tau \geq T\} \right], \quad k' = \log L_0.
\end{align*}$$

(3.28)

We intend to price the contract by inverting the Laplace transforms numerically, so firstly we give the Laplace transforms of $GF'(T)$, $BO'(T,k)$, and $PO'(T,k')$, $LR(T)$.

**Theorem 2.** For $\delta > 0$, the Laplace transform with respect to $T$ of $GF'(T)$ is:

$$\hat{L}_\delta(\delta) = \int_0^\infty e^{-\delta T} GF'(T) dT = \frac{1 - \sum_{i=1}^{n+1} c_i e^{r,u}}{\delta},$$

(3.29)
where \( c_i' s, r_i' s \) are defined in (3.16) and (3.19).

**Proof.** Applying the Fubini theorem yields that

\[
\hat{L}_3(\delta) = E_Q \left[ \int_0^{\tau} e^{-\delta T} dT \right] = \frac{1 - E_Q \left[ e^{-\delta \tau} \right]}{\delta} = \frac{1 - \sum_{i=1}^{n+1} c_i e^{\eta_i}}{\delta}, \tag{3.30}
\]

the last equality follows from (3.17) by letting \( \eta = 0 \).

**Theorem 3.** For \( \delta > 0 \), the Laplace transform with respect to \( T \) of \( LR(T) \) is:

\[
\hat{L}_4(\delta) = \int_0^\infty e^{-\delta T} LR(T) dT = \frac{A_0 e^{-\eta_1 \delta} \Phi(\delta + r - r_g, 1)}{\delta}, \tag{3.31}
\]

where \( \Phi(\delta + r - r_g, 1) \) is defined in (3.17) with \( \delta + r - r_g \) in place of \( \delta \) and 1 in place of \( \eta \).

**Proof.** Applying the Fubini theorem yields that

\[
\hat{L}_4(\delta) = E_Q \left[ \int_0^{\tau} e^{-\delta T} e^{-r_1 T} dT \right] = \frac{E_Q \left[ A_0 e^{-(\delta + r) T} \right]}{\delta} = \frac{E_Q \left[ A_0 e^{-\delta T + r + Y_1} \right]}{\delta}, \tag{3.32}
\]

so (3.31) can be obtained from (3.17) directly.

**Theorem 4.** For \( \delta > \max\{g(\eta + 1), 0\} \), \( 0 < \eta < \alpha_1 - 1 \), the Laplace transform with respect to \( T \) and \( k \) of \( BO'(T, k) \) is:

\[
\hat{L}_5(\delta, \eta) = \int_0^\infty \int_{-\infty}^{\infty} e^{-\delta T - \eta k} BO'(T, k) dT dk = \frac{\alpha A_0^{\eta+1}}{\eta(\eta+1)(\delta - g(\eta+1))} \left( 1 - e^{-(\eta+1) \alpha T} \Phi(\delta, \eta + 1) \right), \tag{3.33}
\]

where \( \Phi(\delta, \eta + 1) \) is defined in (3.17) with \( \eta + 1 \) in place of \( \eta \).

**Proof.** By Fubini theorem,

\[
\hat{L}_5(\delta, \eta) = E_Q \left[ \int_0^\infty \int_{r_T}^{\infty} e^{-\delta T - \eta k} 1 \{ \tau \geq T \} \left( \alpha A_T e^{-r_1 T} - e^{-k} \right) dk dT \right] = E_Q \left[ \int_0^\infty e^{-\delta T}(\alpha A_0)^{\eta+1} 1 \{ \tau \geq T \} \frac{e^{(\eta+1)Y_1}}{\eta(\eta+1)} dT \right]
\]
The strong Markov property of $Y$ implies that

$$E_Q\left[\int_0^\infty e^{-\delta(t+\tau)+Y_{t+\tau}(\eta+1)} dt \right] = E_Q\left[\int_0^\infty e^{-\delta(t+\tau)+Y_{t+\tau}(\eta+1)} dt \mid \mathcal{F}_t \right] = E_Q\left[e^{-\delta \tau + Y_{\eta}(\eta+1)} \int_0^\infty e^{-\delta t + g(\eta+1)} dt \right].$$

(3.35)

So from (3.34), (3.35) becomes

$$\hat{L}_5(\delta, \eta) = \frac{(\alpha A_0)^{\eta+1}}{\eta(\eta+1)(\delta - g(\eta+1))} \left(1 - E_Q\left[e^{-\delta \tau + (\eta+1)Y_{\eta}}\right]\right).$$

(3.36)

**Theorem 5.** For $\delta > \max\{g(1-\eta), 0\}$, $0 < \eta < \max\{\alpha_1 - 1, 1\}$, the Laplace transform with respect to $T$ and $k$ of $PO'(T, k')$ is:

$$\hat{L}_6(\delta, \eta) = \int_0^\infty \int_0^\infty e^{-\delta T - nk'} PO'(T, k') dTdk'$$

$$= \frac{A_0^{1-\eta}}{\eta(\eta-1)(\delta - g(1-\eta))} \left(1 - e^{(\eta-1)\mu} \Phi(\delta, 1-\eta)\right).$$

(3.37)

where $\Phi(\delta, 1-\eta)$ is defined by (3.17) with $1-\eta$ in place of $\eta$.

**Proof.** By Fubini theorem,

$$\hat{L}_6(\delta, \eta) = E_Q\left[\int_0^\infty \int_0^\infty e^{-\delta T - nk'} 1\{\tau \geq T\} \left(e^{k'} - A_\tau e^{-r_T}\right) dk'dT\right]$$

$$= E_Q\left[\int_0^\infty e^{-\delta T} A_0^{1-\eta} 1\{\tau \geq T\} \frac{e^{-(\eta-1)Y_T}}{\eta(\eta-1)} dt\right]$$

$$= \frac{A_0^{1-\eta}}{\eta(\eta-1)} \left(E_Q\left[\int_0^\infty e^{-\delta T - (\eta-1)Y_T} dt\right] - E_Q\left[\int_0^\infty e^{-\delta T - (\eta-1)Y_T} 1\{\tau \leq T\} dt\right]\right).$$
\[ A_0^{1-\eta} \left( \int_0^\infty e^{-(t+\tau-Y_{\tau+1})} d\tau - E_0 \left[ \int_0^\infty e^{-(t+\tau-Y_{\tau+1})} d\tau \right] \right) \]
\[ = \frac{A_0^{1-\eta}}{\eta(\eta-1)} \left( \int_0^\infty e^{(t+\tau-Y_{\tau+1})} d\tau - E_0 \left[ \int_0^\infty e^{(t+\tau-Y_{\tau+1})} d\tau \right] \right), \quad (3.38) \]

The strong Markov property of \( Y \) implies that
\[ E_0 \left[ \int_0^\infty e^{-\delta(t+\tau-Y_{\tau+1})} d\tau \right] = E_0 \left[ E_0 \left[ \int_0^\infty e^{-\delta(t+\tau-Y_{\tau+1})} d\tau \right] \mid \mathcal{F}_t \right] \]
\[ = E_0 \left[ e^{-\delta t-Y_{\tau+1}} \int_0^\infty e^{-\delta t+g(1-\eta)\tau} d\tau \right]. \quad (3.39) \]

So from (3.38), (3.39) becomes
\[ \hat{L}_0(\delta, \eta) = \frac{A_0^{1-\eta}}{\eta(\eta-1)(\delta-g(1-\eta))} \left( 1 - E_0 \left[ e^{-\delta t+(1-\eta)\tau} \right] \right). \quad (3.40) \]

This completes the proof.

4. Numerical Examples

In this section, we intend to price the participating life insurance contract by inverting the associated Laplace transforms numerically via the Euler inversion algorithm. This algorithm was introduced by Abate and Whitt (1992). Petrella (2004) gives some improvements in inverting a two-sided Laplace transform. His method is faster and more stable numerically than the original Euler inversion when dealing with two-sided transforms, due to the introduction of a scaling factor. For all the computations, the values of certain parameters are held fixed except otherwise indicated: we take \( A_0 = 100, \alpha = 0.85, \gamma = 0.7, r = 0.045, r_\varepsilon = 0.025, \beta = 0.9, T = 5, \alpha_1 = 5, \alpha_2 = 10, \mu = 0.5, \lambda_1 = \lambda_2 = \lambda_3 = 0.1. \)

To investigate the impact of the dependent jumps on the contract value, we consider the following two cases.

CASE 1: We consider the common-shock structure discussed in Section 3. That is to say, \( N_1(t) = N_1^1(t) + N_1^3(t), N_2(t) = N_2^2(t) + N_2^3(t), \) so \( S(t) \) is a compound Poisson process with jump intensity \( \lambda_1 + \lambda_2 + \lambda_3. \)

CASE 2: Let \( N_1^1(t), N_1^2(t) \) and \( S^1(t) \) represent the independent versions of \( N_1(t), N_2(t) \) and \( S(t), \) that is, \( N_1^1(t), N_1^2(t) \) are mutually independent Poisson processes with parameters \( \lambda_1 + \lambda_3 \) and \( \lambda_2 + \lambda_3 \) respectively. Hence, \( S^1(t) \) is a compound Poisson process with jump intensity \( \lambda_1 + \lambda_2 + 2\lambda_3. \)
Note that $S(t)$ has a smaller jump intensity and a larger variance of jump size. By some calculations, we can obtain $\text{Var}[S(t)] = \text{Var}[S'(t)]$, therefore the variations in the contract value and the default probability are relatively mainly caused by the dependence of $N_1(t)$ and $N_2(t)$ rather than by the changes of the total variance of jump processes.

Figures 1, 2 show the impact of dependent jumps on the default probabilities. It is well known jump risk has great impact on the default probability over a relatively short interval of time at beginning. We can conclude from figures 1,2, the default probability in case 1 is much larger than that in case 2 for a short period and the reverse relation holds for a long period. That is to say, given the volatility of the jump component $\text{Var}[S(t)] = \text{Var}[S'(t)]$, the process with a smaller jump intensity and a larger variance of jump size is more likely to default for a short period.

Figure 3 gives the the contract value as a function of $\lambda_3$ for different $\sigma$ with dependent and independent jumps in the case where default can only occur at
the contract maturity. The general shape of the curves is qualitatively similar to the plots in Le Courtois and Quittard-Pinon (2008). One can first note the existence of optima for both cases: there are levels of jump intensity where the contract value is maximized. And the optima are shifted to the left when $\sigma$ increases. From figure 3, we can conclude the optima for case 1 is on the right of the optima for case 2. It can be understood that when the jump intensity is small the risk of bankruptcy is small, and for the same $\lambda_3$, the jump intensity of $S(t)$ is less than the jump intensity of $S^I(t)$, so a larger jump intensity induces an increase of the contract value. Figure 4 shows the the contract value as a function of $\lambda_3$ for different $\sigma$ with dependent and independent jumps in the case where default can occur before the contract maturity. The general shape of the curves is qualitatively similar to figure 1, but the optimal values of the contract are much higher than those in the default at maturity only case, as explained in Le Courtois and Quittard-Pinon (2008), though early default can mean a loss, because the assets might cross the barrier, but this negative contribution to the contract value is less important than the negative contribution of a massive default at maturity.

From the numerical analysis, we conclude it is very important to correctly specify the dependence of the arrival processes of the market information when we study the fair valuation of participating life insurance contracts.

5. Conclusions

This article considers a jump diffusion process with dependent jumps, providing ways to price the life insurance contracts. The price of the life insurance contracts rely on the Laplace transform of default time and the firm’s expected present market value at default. Under some assumptions, the model with correlated jumps considered in this article can be changed into a double mixed exponential model. The article gives closed form expression for the joint Laplace transform of the default time and the firm’s expected present market value at default when the jumps have double mixed exponential distribution.

The double mixed exponential model is even more flexible and it has heavier tails than normal distributions, so that it can better capture the asymmetric leptokurtic feature of the empirical financial data. Furthermore, since the class of double mixed exponential distributions is rich enough to approximate many other distributions, including some heavy-tail distributions, in the sense of weak convergence, we may use the double mixed exponential jump diffusion process to approximate some ones with jumps being generally two side distributed when evaluating the life insurance contracts for the proposed model.

There are several limitations of our model. Firstly, although the explicit expression for the joint Laplace transform can be obtained similarly when the types of jumps are much more ($n \geq 3$), it is much more complicated. This will increase a little difficulty in calculations. Secondly, because the double mixed exponential distribution has more parameters, estimation and empirical
assessment of this model become more difficult for us. One thing on our future research agenda is to empirically test our model using statistical data, such as daily data for individual stocks and the S&P-500 and the NASDAQ composites. One approach to obtain parameter estimates we can rely on is maximum likelihood method. It is the best method of estimation, because under mild regularity conditions, the estimated parameters are consistent, asymptotically normal and asymptotically efficient.

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**Appendix**

This appendix outlines the proof of Lemma 3.

**Proof of Lemma 3.** We consider an infinitesimal time interval of length $h$. By a heuristic argument, we distinguish according to whether there is or there is not a jump in this interval and when there is a jump in this interval, whether default occurs or not. From the law of total probability it follows that

$$
\Phi(u) = (1-\lambda h) e^{-\lambda h} E[\Phi(u + ch + \sigma W(h))] \\
+ \lambda he^{-\lambda h} \left[ \int_{-\infty}^{-u} \Phi(u + ch + \sigma W(h) + x) dF(x) \right] \\
+ \int_{-\infty}^{-u-ch+\alpha W(h)} e^{-\eta(u + ch + \sigma W(h) + x)} dF(x) + o(h).
$$

(A.1)

where $c = \mu - r_g$.

By Taylor's expansion, we have

$$
\Phi(u + ch + \sigma W(h)) = \Phi(u) + \Phi'(u)(ch + \sigma W(h)) + \frac{1}{2} \Phi''(u)(ch + \sigma W(h))^2 \\
+ \frac{1}{6} \Phi''(u^*)(ch + \sigma W(h))^3,
$$

where $u^*$ is between $u$ and $u + ch + \sigma W(h)$. Noting $E[W(h)] = E[W^3(h)] = 0$, and $E[W^2(h)] = h$, we obtain
\[ E[\Phi(u + ch + \sigma W(h))] = \Phi(u) + c\Phi'(u)h + \frac{\sigma^2}{2} \Phi''(u)h + o(h). \]

Plugging the above formula into (A.1), dividing both sides by \( h \) and letting \( h \to 0 \) gives (3.11).

**REFERENCES**


YINGHUI DONG

*Department of Mathematics and The Center for Financial Engineering
Soochow University
Suzhou, P.R.China*

*Department of Mathematics
Suzhou Science and Technology University
Suzhou, P.R.China*

*Email: dongyinghui1030@163.com*