OPTIMAL REINSURANCE UNDER VaR AND CVaR RISK MEASURES: A SIMPLIFIED APPROACH

BY

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ABSTRACT

In this paper, we study two classes of optimal reinsurance models by minimizing the total risk exposure of an insurer under the criteria of value at risk (VaR) and conditional value at risk (CVaR). We assume that the reinsurance premium is calculated according to the expected value principle. Explicit solutions for the optimal reinsurance policies are derived over ceded loss functions with increasing degrees of generality. More precisely, we establish formally that under the VaR minimization model, (i) the stop-loss reinsurance is optimal among the class of increasing convex ceded loss functions; (ii) when the constraints on both ceded and retained loss functions are relaxed to increasing functions, the stop-loss reinsurance with an upper limit is shown to be optimal; (iii) and finally under the set of general increasing and left-continuous retained loss functions, the truncated stop-loss reinsurance is shown to be optimal. In contrast, under CVaR risk measure, the stop-loss reinsurance is shown to be always optimal. These results suggest that the VaR-based reinsurance models are sensitive with respect to the constraints imposed on both ceded and retained loss functions while the corresponding CVaR-based reinsurance models are quite robust.

KEYWORDS

Conditional value at risk; Value at risk; Stop-loss reinsurance; Limited stop-loss design; Truncated stop-loss reinsurance; Optimal reinsurance.

1. INTRODUCTION

The study of optimal reinsurance is a classical problem in actuarial science. From a practical point of view, an appropriate use of reinsurance can be an effective risk management tool for managing and mitigating an insurer's risk exposure. From a theoretical point of view, the quest for optimal reinsurance is typically formulated as an optimization problem. Both arguments spark a tremendous surge of interest among practising actuaries and researchers in constantly seeking better and more effective reinsurance strategies.

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More explicitly, suppose X denotes the loss initially assumed by an insurer (i.e. in the absence of reinsurance). We assume X is a non-negative random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with cumulative distribution function (c.d.f.) $F_X(x) = \mathbb{P}(X \leq x)$ and $\mathbb{E}[X] < \infty$. The problem of optimal reinsurance is concerned with the optimal partitioning of X into $f(X)$ and $R_f(X)$ where $X = f(X) + R_f(X)$. Here $f(X)$, satisfying $0 \leq f(X) \leq X$, captures the portion of loss that is ceded to a reinsurer while $R_f(X)$ is the residual loss retained by the insurer (cedent). Consequently, $f(x)$ is known as the ceded loss function while $R_f(x)$ is denoted as the retained loss function.

The first formal analysis of optimal reinsurance is attributed to Borch (1960) who shows that the *stop-loss reinsurance* of the form

$$f(x) = (x - d)_+,$$

where $d \geq 0$ is the retention level and $(x)_+ = \max\{x, 0\}$, is optimal in the sense that it minimizes the variance of the retained loss of an insurer if reinsurance premium is calculated by the expected value principle. Under the maximization of the expected utility of the terminal wealth of a risk-averse insurer, Arrow (1963) obtains a similar result in favor of the stop-loss contract. These classical results have been generalized in a number of important directions by using more sophisticated optimality criteria and/or more realistic premium principles. Just to name a few, Kaluszka (2001) extends Borch (1960)'s result by considering mean-variance premium principles including the standard deviation principle and the variance principle. Young (1999) maximizes the expected utility of the terminal wealth of an insurer under the Wang's premium principle. Motivated by the prevalent use of risk measures such as the value at risk (VaR) and the conditional value at risk (CVaR) among banks, insurance companies and other financial institutions for quantifying risk, Cai and Tan (2007), Cai et al. (2008) and Tan et al. (2011) propose a series of risk measure based optimal reinsurance models. More specifically, by minimizing VaR or CVaR of the insurer's total risk exposure and under the assumption of the expected value premium principle, Cai and Tan (2007) derive analytically the optimal retention for a stop-loss reinsurance. Cai et al. (2008) generalize these results by exploring the optimal reinsurance designs among the class of increasing convex reinsurance treaties[1]. Cheung (2010) revisits these optimal reinsurance models with the help of a geometric approach, and generalizes the results in Cai et al. (2008) by showing that under the VaR-minimization problem, the stop-loss reinsurance is also optimal when the reinsurance premium is calculated by Wang's principle. By incorporating a constraint which reflects either the profitability guarantee or the reinsurance premium budget of the

[1] Throughout this paper, the terms "increasing" and "decreasing" mean "non-decreasing" and "non-increasing", respectively.

insurer, Tan et al. (2011) consider a general CVaR-based optimal reinsurance model and verify that the stop-loss reinsurance is optimal. See also Balbás et al. (2009) who investigate optimal reinsurance with general risk measures.

While many studies have supported that the standard stop-loss reinsurance is often optimal, there are some recent findings which deviate from this result. In fact, the optimal reinsurance treaties can be more elaborate and more complex. For example, Cummins and Mahul (2004) demonstrate that the *limited stop-loss* treaty with the following structure

$$f(x) = \min\{(x-a)_+, b\}$$

for some $a \geq 0$, $b > 0$, is optimal. Note that the limited stop-loss treaty is similar to the standard stop-loss reinsurance except that it imposes an upper limit on the loss for that a reinsurer is responsible. Under the criterion of maximizing either the expected utility or the stability of the cedent, Kaluszka and Okolewski (2008) similarly establish that the limited stop-loss treaty is optimal for a fixed reinsurance premium calculated according to the maximal possible claims principle. Gajek and Zagrodny (2004a) consider more general symmetric and even asymmetric risk measures and show that the limited stop-loss treaty is optimal. In contrast, some other recent studies (see e.g. Gajek and Zagrodny (2004b), Kaluszka (2005), Kaluszka and Okolewski (2008), and Bernard and Tian (2009)) have identified that the following reinsurance treaty

$$f(x) = (x-d)_+ \, \mathbb{I}(x \leq m) \tag{1.1}$$

where $0 \leq d \leq m$ and $\mathbb{I}(\mathcal{D})$ denotes the indicator function of an event $\mathcal{D}$, can be optimal. The above ceded loss function, that is left-continuous (l.c.), is commonly known as the *truncated stop-loss*, and it has the peculiar property that once the loss amount exceeds m, the reinsurer will have zero obligation to the insurer. In other words, there is no reimbursement from the reinsurer to the insurer for any loss exceeding m.

To summarize, the aforementioned studies have suggested that the optimal reinsurance designs can be categorized as either stop-loss, limited stop-loss or truncated stop-loss, depending on the chosen reinsurance model. Very few papers, however, have been devoted to studying the connection between these three types of reinsurance designs. It is the objective of this paper to shed some light on this topic. In particular, we consider the VaR and CVaR risk measure based optimal reinsurance models recently proposed by Cai and Tan (2007). Under the additional assumption that the reinsurance premium is calculated by the expected value principle, it is illuminating to analyzing the optimal reinsurance policies over different classes of ceded loss functions with increasing degrees of generality. The contributions of the present paper are threefold:

First, the aggregate indemnity X is usually assumed to have a continuous (strictly) increasing cumulative distribution function on $(0, \infty)$ with a possible jump at 0 as in Cai et al. (2008) and Cheung (2010). This assumption excludes many loss types such as losses with a cap. In this paper, we relax the constraints on indemnity distribution.

Second, our results offer some interesting insights on the optimal design of reinsurance policies. These results highlight the importance of risk measures as well as the constraints on the ceded loss functions. We explicitly derive the optimal reinsurance solutions over ceded loss functions with increasing degrees of generality. Specifically, we consider three feasible classes depending on the constraints imposed on the ceded and retained loss functions, namely (1) the class of increasing convex ceded loss functions; (2) both the ceded and retained loss functions are increasing; and finally (3) retained loss functions are l.c. and increasing. Using VaR criterion, it is revealing to learn that the optimal reinsurance policies under these three classes are stop-loss reinsurance, limited stop-loss reinsurance, and truncated stop-loss reinsurance, respectively. On the other hand, the corresponding optimal reinsurance for the CVaR-based reinsurance model is very robust and the stop-loss reinsurance is always optimal.

It is reassuring that the limited stop-loss reinsurance is the optimal treaty under the VaR criterion with the constraints that both the ceded and the retained loss functions are increasing. One of the reasons is that these conditions ensure that the higher the incurred loss, the greater the loss to both the insurer and reinsurer. Hence it potentially reduces moral hazard. Another reason is that our finding is consistent with practice since the limited stop-loss reinsurance is a very common reinsurance treaty in the marketplace. What is even more striking is that under the VaR criterion, the optimal reinsurance design is the truncated stop-loss reinsurance when the retained loss function is restricted to be l.c. and increasing. This suggests that the insurer is only interested in reinsuring moderate losses but not large losses. This seems counter-intuitive since an insurer should be more concerned with large losses. This phenomenon, however, is consistent with the clinical studies examined by Froot (2001) in the market for catastrophe risk.

Third, we note that some of our results have appeared in the literature. In particular, Cai et al. (2008) and Cheung (2010) study the VaR and CVaR based optimal reinsurance problems under the assumption of increasing convex ceded loss functions by using approximation and convergence arguments and a geometric approach, respectively; Balbás et al. (2009) and Tan et al. (2011) resort to the Lagrangian approach in order to solve the CVaR-based optimal reinsurance problem for a general set of ceded loss functions. In contrast, it should be emphasized that the approach we use to derive these optimal reinsurance solutions is new and relatively straightforward. Furthermore, it is new to study the VaR-based reinsurance models under the constraints that both ceded and retained loss functions are increasing (see Subsection 3.2) and that the retained loss functions are l.c. and increasing (see Subsection 3.3). While Section 4 is devoted to analyzing the optimal solutions under CVaR-based reinsurance model, it should be emphasized that the result is quite general in the sense that it can easily be generalized to other optimality criteria that preserve the convex order (see Remark 4.1). In fact, the theory of stochastic orders we use has been shown to be a powerful tool for analyzing the optimal reinsurance problems (see Van Heerwaarden et al. (1989)).

The rest of this paper is organized as follows. Section 2 introduces the risk measure based optimal reinsurance models. Section 3 studies the VaR-based reinsurance problem and derives the optimal reinsurance policies over the ceded loss functions with different constraints. In Section 4, the optimization problem under the CVaR criterion is solved. Finally, we provide some concluding remarks in Section 5.

2. Risk measure based optimal reinsurance models

Recall that when an insurer cedes part of its risk to a reinsurer under a typical reinsurance arrangement, the insurer incurs an additional cost in the form of reinsurance premium which is payable to the reinsurer. We use $\Pi_f(X)$ to denote the reinsurance premium which explicitly depends on the ceded loss function f. While there exist various premium principles in determining the reinsurance premium, we consider the most common one in this paper, i.e. we adopt the expected value premium principle so that the reinsurance premium is calculated by

$$\Pi_f(X) = (1+\rho)\mathbb{E}[f(X)], \tag{2.1}$$

where $\rho > 0$ is the safety loading factor.

Under the reinsurance arrangement, the risk exposure of the insurer is no longer captured by X. In fact, the total risk exposure of the insurer is the sum of the retained loss and the incurred reinsurance premium. Using $T_f(X)$ to denote the total risk exposure of the insurer in the presence of reinsurance, we have

$$T_f(X) = R_f(X) + \Pi_f(X). \tag{2.2}$$

Consequently, a reasonable criterion in determining an optimal ceded loss function can be formulated as one that minimizes an appropriately chosen risk measure on $T_f(X)$. This corresponds to the optimal reinsurance models proposed in Cai and Tan (2007). In Cai and Tan (2007), they consider VaR and CVaR risk measures. Their studies are prompted by the popularity of these risk measures among banks and insurance companies for risk management and for setting regulatory capital.

We now formally provide the definitions of VaR and CVaR risk measures:

Definition 2.1. *The VaR of a non-negative random variable X at a confidence level $1-\alpha$ where $0 < \alpha < 1$ is defined as*

$$VaR_\alpha(X) \triangleq \inf\{x \geq 0 : \mathbb{P}(X > x) \leq \alpha\}. \tag{2.3}$$

Based upon the definition of VaR, the CVaR of X at a confidence level $1-\alpha$ *is defined as*

$$CVaR_{\alpha}(X) = \frac{1}{\alpha}\int_0^{\alpha} VaR_s(X)\,\mathrm{d}s. \tag{2.4}$$

Note that CVaR is also known as the "average value at risk", "expected shortfall", or the "conditional tail expectation (CTE)". VaR is more appropriately referred to as the quantile risk measure since $VaR_{\alpha}(X)$ is exactly a $(1-\alpha)$-*quantile* of the random variable X. It follows from the definition of $VaR_{\alpha}(X)$ that

$$VaR_{\alpha}(X) \leq x \Leftrightarrow \bar{F}_X(x) \leq \alpha, \tag{2.5}$$

where $\bar{F}_X(x) = 1 - F_X(x)$. Therefore, $VaR_{\alpha}(X) = 0$ for $\alpha \geq \bar{F}_X(0)$. For this reason, we assume in this paper that the parameter α satisfies $0 < \alpha < \bar{F}_X(0)$ to avoid the discussion of trivial cases. Another important property associated with $VaR_{\alpha}(X)$ is that for any l.c. and increasing function g, we have (see Theorem 1 in Dhaene et al. (2002))

$$VaR_{\alpha}(g(X)) = g(VaR_{\alpha}(X)). \tag{2.6}$$

A key advantage of $CVaR_{\alpha}(X)$ over $VaR_{\alpha}(X)$ is that $CVaR$ is a coherent risk measure while VaR fails to satisfy subadditivity property. More detailed discussions on these properties can be found in Artzer et al. (1999) and Föllmer and Schied (2004).

Based upon these two risk measures, the risk measure based optimal reinsurance models proposed in Cai and Tan (2007) and Cai et al. (2008) can be summarized succinctly as follows:

$$\textbf{VaR-optimization}: \; VaR_{\alpha}(T_{f^*}(X)) = \min_{f\in C} VaR_{\alpha}(T_f(X)) \tag{2.7}$$

and

$$\textbf{CVaR-optimization}: \; CVaR_{\alpha}(T_{f^*}(X)) = \min_{f\in C} CVaR_{\alpha}(T_f(X)), \tag{2.8}$$

where C is the set of admissible ceded loss functions and $f^* \in C$ is the resulting optimal ceded loss function. In Cai and Tan (2007), the admissible set C is confined to the class of stop-loss functions so that the optimal reinsurance models simplify to one-parameter minimization problems of determining the optimal retention level. Cai et al. (2008) generalize these results by extending the admissible ceded loss functions to be a class of increasing convex functions. More recently, Tan et al. (2011) solve the CVaR based optimal reinsurance model for the general ceded loss functions.

In this paper, we provide further analysis on the risk measure based optimal reinsurance models (2.7) and (2.8). More specifically, we are interested in seeking an optimal reinsurance policy over the following three admissible sets of ceded loss functions:

Definition 2.2. *Define the sets of ceded loss functions by*

$$C^1 \triangleq \{0 \le f(x) \le x : f(x) \textit{ is an increasing convex function}\} \tag{2.9}$$

and

$$C^2 \triangleq \left\{0 \le f(x) \le x : \textit{both } R_f(x) \textit{ and } f(x) \textit{ are increasing functions}\right\} \tag{2.10}$$

and

$$C^3 \triangleq \left\{0 \le f(x) \le x : R_f(x) \textit{ is an increasing l.c. function}\right\}. \tag{2.11}$$

We recall that the risk measure based optimal reinsurance models with admissible set C^1 are analyzed extensively in Cai et al. (2008). We revisit these models in this paper by providing simpler derivations of the results.

We conclude the section by stating some additional relationships on the admissible sets $\{C^j;\, j = 1, 2, 3\}$. First, as pointed out earlier, the increasing condition on both ceded and retained loss functions is important since this condition reduces moral hazard. Second, Lemma A.1 in Cai et al. (2008) shows that for any $f \in C^1$, $R_f(x)$ is increasing and concave in x so that $f \in C^2$. Third, for any $f \in C^2$, we have $R_f(x_1) \le R_f(x_2)$ for any $0 \le x_1 \le x_2$. Then together with the increasing property of $f(x)$ results in

$$0 \le f(x_2) - f(x_1) \le x_2 - x_1. \tag{2.12}$$

Consequently, f is Lipschitz continuous and hence $f \in C^3$. Finally, since the truncated ceded loss function defined in (1.1) is included in C^3 but not in C^2, then we have

$$C^1 \subsetneqq C^2 \subsetneqq C^3.$$

3. Optimal reinsurance under VaR risk measure

In this section, we derive the optimal solutions corresponding to the reinsurance model (2.7) under the admissible ceded loss function sets $\{C^j;\, 1 \le j \le 3\}$. We discuss these results in the following three subsections.

3.1. VaR minimization model with C^1 constraint

We begin this subsection by presenting the following lemma. This lemma gives a new and simplified representation of $f \in C^1$ and this in turn facilitates the derivation of the main result of this subsection.

Lemma 3.1. *We have*

$$\mathcal{C}^1 = \{c\int_{0-}^{\infty}(x-t)_+ v(\mathrm{d}t): \\ 0 \le c \le 1 \text{ and } v \text{ is a probability measure defined on } [0,\infty]\}.$$

Proof. Denote the right derivative of f by f'_+ for any $f \in \mathcal{C}^1$, then f'_+ is increasing and right-continuous. Note that $f(0) = 0$ and $0 \le f'_+(t) \le 1$ according to (2.12), then Corollary 24.2.1 of Rockafellar (1970) implies

$$\begin{aligned} f(x) &= \int_0^x f'_+(s)\mathrm{d}s = \int_0^x \left(\int_0^s \mathrm{d}f'_+(t) + f'_+(0)\right)\mathrm{d}s \\ &= f'_+(0)x + \int_0^x (x-t)_+ \mathrm{d}f'_+(t) = c\int_{0-}^{\infty}(x-t)_+ v(\mathrm{d}t), \end{aligned}$$

where $c = f'_+(\infty) \le 1$ and v is a probability measure generated by $\{f'_+(t)/c;\ t \ge 0\}$.

Conversely, it is easy to verify that $c\int_{0-}^{\infty}(x-t)_+ v(\mathrm{d}t)$ is an increasing convex function and less than x according to the constraints of c and v. Hence the proof is complete. □

To proceed, it is useful to introduce the following two parameters d^* and β:

$$d^* \triangleq VaR_{\frac{1}{1+\rho}}(X); \tag{3.1}$$

$$\beta \triangleq d^* + (1+\rho)\,\mathbb{E}\left[(X-d^*)_+\right]. \tag{3.2}$$

Then by virtue of Lemma 3.1, we obtain the following theorem, which is the main result of this subsection.

Theorem 3.1. *The optimal f^{*1} that solves the VaR-based reinsurance model* (2.7) *over the class of ceded loss functions $\mathcal{C}^1$ is given by*

$$f^{*1}(x) \triangleq \begin{cases} (x-d^*)_+, & VaR_\alpha(X) > \beta; \\ c(x-d^*)_+,\ \forall c \in [0,1], & VaR_\alpha(X) = \beta; \\ 0, & otherwise. \end{cases} \tag{3.3}$$

Moreover,

$$VaR_\alpha(T_{f^{*1}}(X)) = \min_{f\in\mathcal{C}^1} VaR_\alpha(T_f(X)) = \beta \wedge VaR_\alpha(X), \tag{3.4}$$

where $x \wedge y = \min\{x,y\}$.

Proof. For any $f \in C^1$, recall that $R_f(x)$ is increasing and Lipschitz-continuous, then it follows from (2.1), (2.2), (2.6) and Lemma 3.1 that

$$\begin{aligned} VaR_\alpha(T_f(X)) &= VaR_\alpha(R_f(X)) + \Pi_f(X) \\ &= R_f(VaR_\alpha(X)) + (1+\rho)\,\mathbb{E}[f(X)] \\ &= VaR_\alpha(X) - f(VaR_\alpha(X)) + (1+\rho)\,\mathbb{E}[f(X)] \qquad (3.5) \\ &= VaR_\alpha(X) + c\int_{0-}^{\infty} g(t)\,v(\mathrm{d}t) \qquad (3.6) \end{aligned}$$

for some $0 \le c \le 1$ and probability measure v, where

$$g(t) \triangleq (1+\rho)\,\mathbb{E}[(X-t)_+] - (VaR_\alpha(X)-t)_+.$$

It follows from the above results that to analyze the minimization of $VaR_\alpha(T_f(X))$, it is sufficient to focus on the minimum value of g. For any $t \ge VaR_\alpha(X)$, we have

$$g(t) = (1+\rho)\,\mathbb{E}[(X-t)_+] \ge g(\infty) = 0, \qquad (3.7)$$

so the minimum value of $g(t)$ on $[VaR_\alpha(X), \infty]$ is 0. On the other hand, when $0 \le t \le VaR_\alpha(X)$, we have

$$g(t) = t + (1+\rho)\,\mathbb{E}[(X-t)_+] - VaR_\alpha(X) = t + (1+\rho)\int_t^{\infty} \bar{F}_X(s)\,\mathrm{d}s - VaR_\alpha(X).$$

This leads to $g'(t) = 1 - (1+\rho)\bar{F}_X(t)$, and hence $g(t)$ is convex on $[0, VaR_\alpha(X)]$.

If $d^* > VaR_\alpha(X)$, then (2.5) implies $\bar{F}_X(t) > \frac{1}{1+\rho}$ for any $0 \le t \le VaR_\alpha(X)$. Furthermore, $g'(t) \le 0$ for $t \ge 0$ so that g attains its minimum at ∞. In this case, we set $v(\{\infty\}) = 1$, then the corresponding optimal ceded loss function becomes $f^{*1} = 0$ and (3.6) leads to

$$VaR_\alpha(T_{f^{*1}}(X)) = \min_{f \in C^1} VaR_\alpha(T_f(X)) = VaR_\alpha(X). \qquad (3.8)$$

On the other hand, if $d^* \le VaR_\alpha(X)$, then (2.5) implies that the minimum value of g on $[0, VaR_\alpha(X)]$ is attained at d^*. Combining (3.7) yields

$$\min_{x \in \mathbb{R}_+} g(x) = \min\{g(d^*), g(\infty)\} = \min\{\beta - VaR_\alpha(X), 0\}.$$

We now complete the analysis for $\min_{f \in C^1} VaR_\alpha(T_f(X))$ by considering the following three cases:

(i) If $VaR_\alpha(X) > \beta$, then $g(d^*) < 0$. Furthermore, by setting $\nu(\{d^*\}) = 1$ and $c = 1$, the corresponding optimal ceded loss function becomes $f^{*1}(x) = (x - d^*)_+$ and (3.6) implies

$$VaR_\alpha(T_{f^{*1}}(X)) = \min_{f \in C^1} VaR_\alpha(T_f(X)) = \beta.$$

(ii) If $VaR_\alpha(X) = \beta$, then the minimum value of g occurs at both ∞ and d^* with value 0. In this case, we need only to set ν with support on $\{d^*, \infty\}$, then the corresponding optimal ceded loss function becomes $f_1^*(x) = c(x - d^*)_+$ for some $c \in [0, 1]$ and it is easy to verify that f^{*1} satisfies (3.8).

(iii) If $VaR_\alpha(X) < \beta$, then $g(d^*) > 0$. In this case, the minimum of $g(t)$ attains at ∞ with value 0 and hence(3.8) holds for $f^{*1} = 0$.

Collecting all the above results yields the theorem and hence the proof is complete. □

Remark 3.1. *If* $1/(1+\rho) \geq \bar{F}_X(0)$, *then* $d^* = 0$. *In this case,* f^{*1} *simplifies to quota-share ceded loss function, i.e.*

$$f^{*1}(x) \triangleq \begin{cases} x, & VaR_\alpha(X) > (1+\rho)\,\mathbb{E}[X]; \\ cx, \ \forall c \in [0,1], & VaR_\alpha(X) = (1+\rho)\,\mathbb{E}[X]; \\ 0, & otherwise. \end{cases} \tag{3.9}$$

Therefore, when $VaR_\alpha(X) > (1+\rho)\mathbb{E}[X]$, *that is equivalent to saying that the safety loading* ρ *or the parameter* α *is small, the optimal reinsurance strategy for the insurer is to transfer the entire risk to the reinsurer among the choice of increasing convex ceded loss functions.*

Remark 3.2. *Although the results in the above theorem have already been derived by Cai et al. (2008), here we emphasize two important distinctions. First, the results in Cai et al. (2008) are derived under the assumption that X has a strictly increasing continuous distribution function with a possible jump at* 0. *Second, their proof relies on approximation and convergence arguments. In contrast, we relax the constraints on the distribution of X and provide a different but a much simpler proof.*

3.2. VaR minimization model with C^2 constraint

In this subsection, we are interested in the solutions to the following optimal reinsurance model:

$$\min_{f \in C^2} VaR_\alpha(T_f(X)). \tag{3.10}$$

Recall that C^2 denotes the class of ceded loss functions such that both $R_f(x)$ and $f(x)$ are increasing and that $0 \leq f(x) \leq x$. To derive an optimal solution to (3.10), we resort to the following two-step approach. First, we address the solution to the following modified optimal reinsurance model with a fixed reinsurance premium $P = (1+\rho)\mu$:

$$\min_{f \in C_\mu^2} VaR_\alpha(T_f(X)), \tag{3.11}$$

where $C_\mu^2 = \{f \in C^2 : \mathbb{E}[f(X)] = \mu\}$ and $0 < \mu < \mathbb{E}[X]$. Second, we obtain the desired solution by analyzing the effect of μ on the optimal reinsurance.

Proposition 3.1. *The optimal ceded loss function f_μ^* that solves* (3.11) *is given by*

$$f_\mu^*(x) \triangleq \begin{cases} (x \wedge VaR_\alpha(X) - d)_+, & \mathbb{E}[X \wedge VaR_\alpha(X)] \geq \mu; \\ x \wedge VaR_\alpha(X) + \theta(x - VaR_\alpha(X))_+, & \textit{otherwise}, \end{cases} \tag{3.12}$$

where $0 \leq d < VaR_\alpha(X)$ *and* $0 < \theta < 1$ *are determined by* $\mathbb{E}[f_\mu^*(X)] = \mu$. *Moreover,*

$$VaR_\alpha(T_{f_\mu^*}(X)) = \min_{f \in C_\mu^2} VaR_\alpha(T_f(X)) = \begin{cases} d + (1+\rho)\mu, & \mathbb{E}[X \wedge VaR_\alpha(X)] \geq \mu; \\ (1+\rho)\mu, & \textit{otherwise}. \end{cases} \tag{3.13}$$

Proof. Recall that $f \in C^2$ is increasing and Lipschitz continuous, then it follows from (2.1) and (3.5) that

$$VaR_\alpha(T_f(X)) = VaR_\alpha(X) + (1+\rho)\mu - f(VaR_\alpha(X)), \ \forall f \in C_\mu^2. \tag{3.14}$$

The above representation implies if the ceded loss function is a solution to

$$\max_{f \in C_\mu^2} f(VaR_\alpha(X)), \tag{3.15}$$

then it is also a solution to (3.11). We now show that f_μ^* is a solution to the above maximization problem by dividing our analysis into the following two cases:

(i) $\mathbb{E}[X \wedge VaR_\alpha(X)] \geq \mu$ case: If there exists $f \in C_\mu^2$ such that

$$f(VaR_\alpha(X)) > f_\mu^*(VaR_\alpha(X)), \tag{3.16}$$

then for any $d \leq x \leq VaR_\alpha(X)$, we have

$$\begin{aligned} f(x) &\geq f(VaR_\alpha(X)) + x - VaR_\alpha(X) \\ &= f(VaR_\alpha(X)) - f_\mu^*(VaR_\alpha(X)) + f_\mu^*(x) > f_\mu^*(x), \end{aligned} \tag{3.17}$$

where the first inequality is derived by (2.12). Further, the increasing property of $f \in C_\mu^2$, together with (3.16), yields

$$f(x) \geq f(VaR_\alpha(X)) > f_\mu^*(VaR_\alpha(X)) = f_\mu^*(x), \quad \forall x \geq VaR_\alpha(X).$$

Moreover, $f(x) \geq 0 = f_\mu^*(x)$ for any $0 \leq x \leq d$ according to the definition of $f_\mu^*(x)$ in (3.12). Then combining (3.17) yields $f(x) \geq f_\mu^*(x)$ for any $x \in \mathbb{R}_+$ and the inequality is strict for $x \geq d$. This leads to the contradiction:

$$\mu = \mathbb{E}[f(X)] > \mathbb{E}[f_\mu^*(X)] = \mu.$$

Therefore, $f_\mu^*(x)$ is a solution to (3.15) so that (3.14) leads to (3.13).

(ii) $\mathbb{E}[X \wedge VaR_\alpha(X)] < \mu$ case: In this situation, we have $f_\mu^*(VaR_\alpha(X)) = VaR_\alpha(X) \geq f(VaR_\alpha(X))$ since $0 \leq f(x) \leq x$. Consequently, $f_\mu^*(x)$ is optimal among C_μ^2 so that (3.13) holds and hence this completes the proof.

□

Given a reinsurance premium P satisfying $P \leq (1+\rho)\mathbb{E}[X \wedge VaR_\alpha(X)]$, the above proposition shows that the limited stop-loss reinsurance is optimal under the VaR-based reinsurance model when both the insurer and reinsurer are obligated to pay more for larger loss. Since the optimal ceded loss function $f_\mu^*(x)$ depends on the choice of μ, we now study the effect of μ on f_μ^* and obtain the main result of this subsection.

Theorem 3.2. *The optimal ceded loss function f^{*2} that solves* (3.10) *is given by*

$$f^{*2}(x) \triangleq \begin{cases} \min\left\{(x-d^*)_+, VaR_\alpha(X) - d^*\right\}, & d^* < VaR_\alpha(X); \\ 0, & \textit{otherwise}, \end{cases} \tag{3.18}$$

where d^* is defined in (3.1)*. Moreover,*

$$\begin{aligned} &VaR_\alpha(T_{f^{*2}}(X)) \\ &\quad = \min_{f \in C^2} VaR_\alpha(T_f(X)) \\ &\quad = d^* \wedge VaR_\alpha(X) + (1+\rho)\,\mathbb{E}\Big[\min\left\{(X-d^*)_+, (VaR_\alpha(X)-d^*)_+\right\}\Big]. \end{aligned}$$

Proof. Let $\kappa = \mathbb{E}[X \wedge VaR_\alpha(X)]$. If $\mu > \kappa$, Proposition 3.1 asserts

$$VaR_\alpha(T_{f_\mu^*}(X)) = (1+\rho)\mu > (1+\rho)\kappa = VaR_\alpha(T_{f_\kappa^*}(X)),$$

then the optimal reinsurance appears on $\mu \leq \kappa$.

We now consider the case $\mu \leq \kappa$. Since

$$\mu = \mathbb{E}[f_\mu^*(X)] = \int_0^\infty \mathbb{P}(f_\mu^*(X) > s)\,\mathrm{d}s = \int_d^{VaR_\alpha(X)} \mathbb{P}(X > s)\,\mathrm{d}s,$$

then taking the derivatives of the above equation with respect to d results in

$$\frac{\partial \mu}{\partial d} = -\bar{F}_X(d).$$

The above equation, together with (3.13), implies

$$\frac{\partial\, VaR_\alpha(T_{f_\mu^*}(X))}{\partial d} = 1 + (1+\rho)\,\frac{\partial \mu}{\partial d} = 1 - (1+\rho)\,\bar{F}_X(d).$$

Since $d^* \leq d \Leftrightarrow \bar{F}_X(d) \leq 1/(1+\rho)$ according to (2.5), then we have

$$\frac{\partial\, VaR_\alpha(T_{f_\mu^*}(X))}{\partial d} \gtreqless 0 \quad \text{for} \quad d^* \lesseqgtr d.$$

Recall that $0 \leq d < VaR_\alpha(X)$. If $d^* \geq VaR_\alpha(X)$, then the above equation implies that $VaR_\alpha(T_{f_\mu^*}(X))$ attains its minimum at $d = VaR_\alpha(X)$. In this case, $f_\mu^* = 0$ and hence $\mu = 0$, so that $VaR_\alpha(T_{f_\mu^*}(X)) = VaR_\alpha(X)$ according to (3.13).

On the other hand, if $d^* < VaR_\alpha(X)$, $VaR_\alpha(T_f(X))$ attains its minimum at $d = d^*$, then

$$\min_{\mu \leq \kappa} VaR_\alpha(T_{f_\mu^*}(X)) = d^* + (1+\rho)\,\mathbb{E}\Big[(\min\{X, VaR_\alpha(X)\} - d^*)_+\Big].$$

Collecting all the results above yields the theorem and hence the proof is complete. □

Remark 3.3. *Note that both d^* and $VaR_\alpha(X)$ are fixed for given confidence level $1-\alpha$, safety loading factor ρ and loss random variable X. Consequently, when $d^* < VaR_\alpha(X)$, the optimal ceded loss function f^{*2} in* (3.18) *corresponds to a limited stop-loss treaty with a fixed deductible d^* and a constant cap $VaR_\alpha(X) - d^*$. As pointed out earlier, this type of reinsurance policy is commonly found in the reinsurance marketplace. Moreover, this optimal ceded loss function is equivalently represented as*

$$f^{*2}(x) = (x - d^*)_+ - (x - VaR_\alpha(X))_+, \;\; d^* < VaR_\alpha(X).$$

3.3. VaR minimization model with C^3 constraint

In this subsection, we analyze the VaR-based optimal reinsurance model (2.7) by only requiring that the retained loss function $R_f(x)$ is increasing and l.c. with $0 \leq R_f(x) \leq x$ (i.e. C^3 constraint). As in the last subsection, our approach involves a two-step procedure which entails first seeking the solution to the following constrained optimization problem:

$$\min_{f \in C_\lambda^3} VaR_\alpha(T_f(X)), \tag{3.19}$$

where $C_\lambda^3 = \{f \in C^3 : \mathbb{E}[R_f(X)] = \lambda\}$ and $0 < \lambda < \mathbb{E}[X]$. We then obtain the desired solution by analyzing the effect of λ on the optimal reinsurance.

Proposition 3.2. *Let*

$$R_{f_\lambda^*}(x) = \begin{cases} \frac{\lambda}{l(\alpha)} x\mathbb{I}(x > VaR_\alpha(X)), & l(\alpha) \geq \lambda; \\ x, & l(\alpha) < \lambda \text{ and } x > VaR_\alpha(X); \\ x \wedge d, & \text{otherwise}, \end{cases} \tag{3.20}$$

where $l(\alpha) \triangleq \mathbb{E}[X\mathbb{I}(X > VaR_\alpha(X))]$ *and* $0 \leq d < VaR_\alpha(X)$ *is determined by* $\mathbb{E}[R_{f_\lambda^*}(X)] = \lambda$, then $f_\lambda^*(x) = x - R_{f_\lambda^*}(x)$ *is a solution to* (3.19)*. Moreover,*

$$VaR_\alpha(T_{f_\lambda^*}(X)) = \min_{f \in C_\lambda^3} VaR_\alpha(T_f(X)) = d\mathbb{I}(l(\alpha) < \lambda) + (1+\rho)(\mathbb{E}[X] - \lambda). \tag{3.21}$$

Proof. The proof is similar to that of Proposition 3.1 with a slight modification.

The translation invariance property of VaR risk measure, together with (2.6), implies

$$\begin{aligned} VaR_\alpha(T_f(X)) &= VaR_\alpha(R_f(X)) + (1+\rho)(\mathbb{E}[X] - \lambda) \\ &= R_f(VaR_\alpha(X)) + (1+\rho)(\mathbb{E}[X] - \lambda) \end{aligned} \tag{3.22}$$

for any $f \in C_\lambda^3$. The following analysis is divided into two cases: $l(\alpha) \geq \lambda$ and $l(\alpha) < \lambda$.

If $l(\alpha) \geq \lambda$, $R_{f_\lambda^*}(VaR_\alpha(X)) = 0$, then (3.22) implies that the ceded loss function f_λ^* is optimal and (3.21) holds.

Now, we consider the case $l(\alpha) < \lambda$. If there exists $f \in C_\lambda^3$ such that

$$R_f(VaR_\alpha(X)) < R_{f_\lambda^*}(VaR_\alpha(X)) = d,$$

then we have

$$R_f(x) \leq x \wedge d = R_{f_\lambda^*}(x), \ \forall 0 \leq x \leq VaR_\alpha(X), \tag{3.23}$$

since $R_f(x)$ is increasing with $0 \leq R_f(x) \leq x$. On the other hand, we have

$$R_{f_\lambda^*}(x) = x \geq R_f(x), \ \forall x > VaR_\alpha(X).$$

Combining (3.23) yields the contradiction:

$$\lambda = \mathbb{E}[R_f(X)] < \mathbb{E}[R_{f_\lambda^*}(X)] = \lambda.$$

Therefore, f_λ^* is a solution to (3.19) and that (3.22) leads to (3.21). The proof is complete. □

By virtue of the above proposition, we now study the effect of λ on f_λ^* and obtain the main result of this subsection. To proceed, we introduce two useful parameters ϑ and γ:

$$\vartheta \triangleq \bar{F}_X(VaR_\alpha(X)) + \frac{1}{1+\rho} \quad \text{and} \quad \gamma \triangleq VaR_\vartheta(X). \tag{3.24}$$

Note that $0 \le \gamma \le VaR_\alpha(X)$ according to (2.5).

Theorem 3.3. *The optimal ceded loss function f^{*3} that solves* (2.7) *with constraint C^3 is given by*

$$f^{*3}(x) = (x-\gamma)_+ \, \mathbb{I}(x \le VaR_\alpha(X)). \tag{3.25}$$

Moreover,

$$\begin{aligned} VaR_\alpha(T_{f^{*3}}(X)) &= \min_{f \in C^3} VaR_\alpha(T_f(X)) \\ &= \gamma + (1+\rho)\,\mathbb{E}[(X-\gamma)_+ \, \mathbb{I}(X \le VaR_\alpha(X))]. \end{aligned} \tag{3.26}$$

Proof. For $l(\alpha) \ge \lambda$, Proposition 3.2 shows

$$\begin{aligned} VaR_\alpha(T_{f_\lambda^*}(X)) &= (1+\rho)(\mathbb{E}[X] - \lambda) \ge (1+\rho)(\mathbb{E}[X] - l(\alpha)) \\ &= VaR_\alpha(T_{f_{l(\alpha)}^*}(X)), \end{aligned}$$

then the optimal reinsurance appears on $\lambda \ge l(\alpha)$.

Now, we consider the case $\lambda \ge l(\alpha)$. In this situation, $0 \le d \le VaR_\alpha(X)$ is determined by $\mathbb{E}[R_{f_\lambda^*}(X)] = \lambda$, i.e.

$$\begin{aligned} \mathbb{E}[X] - \lambda &= \mathbb{E}\big[(X-d)_+ \, \mathbb{I}(X \le VaR_\alpha(X))\big] \\ &= \int_d^{VaR_\alpha(X)} \big(\bar{F}_X(y) - \bar{F}_X(VaR_\alpha(X))\big)\,\mathrm{d}y. \end{aligned} \tag{3.27}$$

Taking derivatives of the above equation with respect to d yields $\frac{\partial \lambda}{\partial d} = \bar{F}_X(d) - \bar{F}_X(VaR_\alpha(X))$, then we have

$$\frac{\partial VaR_\alpha(T_{f_\lambda^*}(X))}{\partial d} = 1 - (1+\rho)\frac{\partial \lambda}{\partial d} = (1+\rho)\big(\vartheta - \bar{F}_X(d)\big)$$

according to (3.21). Thus, $VaR_\alpha(T_{f_\lambda^*}(X))$ attains its minimum at $d = \gamma$ and (3.26) follows from (3.21) and (3.27). This completes the proof. $\square$

In C^3, the retained loss functions are assumed to be l.c. and increasing. However, the l.c. constraint excludes many types of reinsurance policies such as

$$f(x) = (x-d)_+ \mathbb{I}(x < m),\ 0 \le d < m.$$

Under some additional assumption of loss distribution $F_X(x)$, the following theorem generalizes the results of Theorem 3.3 by relaxing such a constraint. Now, the set of admissible ceded loss functions is given by

$$C^{3'} \triangleq \{0 \le f(x) \le x : R_f(x) \text{ is an increasing function}\} \supsetneqq C^3. \quad (3.28)$$

Theorem 3.4. *Provided that $F_X(x)$ is strictly increasing and continuous at a neighborhood of $VaR_\alpha(X)$, the ceded loss function f^{*3} defined in* (3.25) *is also optimal under the reinsurance model* (2.7) *with constraint $C^{3'}$.*

Proof. Since $F_X(x)$ is assumed to be strictly increasing and continuous at a neighborhood of $VaR_\alpha(X)$, then $VaR_p(X)$, that is a function of $p \in [0,1]$, is continuous and strictly decreasing at a neighborhood of α. Following, a two-step procedure is applied to analyze the optimal reinsurance problem (2.7) with constraint $C^{3'}$.

First, we show that f_λ^* defined in Proposition 3.2 is also a solution to the following modified optimal reinsurance problem

$$\min_{f \in C_\lambda^{3'}} VaR_\alpha(T_f(X)), \quad (3.29)$$

where $C_\lambda^{3'} = \{f \in C^{3'} : \mathbb{E}[R_f(X)] = \lambda\}$ with $0 < \lambda < \mathbb{E}[X]$. The proof is similar to that of Proposition 3.2 with a slight modification. Specifically, the translation invariance property of VaR risk measure implies

$$VaR_\alpha(T_f(X)) = VaR_\alpha(R_f(x)) + (1+\rho)(\mathbb{E}[X] - \lambda),\ \forall f \in C_\lambda^{3'}. \quad (3.30)$$

The following analysis is divided into two cases: $l(\alpha) \ge \lambda$ and $l(\alpha) < \lambda$, where $l(\alpha)$ is given in Proposition 3.2.

If $l(\alpha) \ge \lambda$, then we have

$$\mathbb{P}(R_{f_\lambda^*}(X) > 0) = \mathbb{P}(X > VaR_\alpha(X)) = \alpha,$$

where $R_{f_\lambda^*}(x)$ is defined in (3.20). Thus, the definition of VaR in (2.3) implies $VaR_\alpha(R_{f_\lambda^*}(X)) = 0$, then it follows from (3.30) that $f_\lambda^*(x)$ is a solution to (3.29).

Next, we focus on the case $l(\alpha) < \lambda$. Since $R_{f_\lambda^*}(x)$ is increasing and l.c., then (2.6) implies

$$VaR_p(R_{f_\lambda^*}(X)) = R_{f_\lambda^*}(VaR_p(X)) = \begin{cases} VaR_p(X) \wedge d, & \alpha \le p < 1; \\ VaR_p(X), & 0 < p < \alpha, \end{cases}$$

because $VaR_p(X)$ is continuous and strictly decreasing at a neighborhood of α. Further, we have

$$\mathbb{P}(VaR_U(R_{f_\lambda^*}(X)) = d) > 0 \tag{3.31}$$

because of $0 \le d < VaR_\alpha(X)$, where U is uniformly distributed on [0,1].

Compared to $VaR_\alpha(R_{f_\lambda^*}(X))$, if there exists $f \in C_\lambda^{3'}$ such that

$$VaR_\alpha(R_f(X)) < VaR_\alpha(R_{f_\lambda^*}(X)) = d, \tag{3.32}$$

then we have

$$\mathbb{P}(R_f(X) > VaR_p(R_{f_\lambda^*}(X))) \le p, \forall p \in (0,1).$$

The above inequality can be justified by considering the following two cases:

(i) $\alpha \le p < 1$ case: If $VaR_p(X) \le d$, then

$$\begin{aligned} \mathbb{P}(R_f(X) > VaR_p(R_{f_\lambda^*}(X))) &= \mathbb{P}(R_f(X) > VaR_p(X)) \\ &\le \mathbb{P}(X > VaR_p(X)) \le p, \end{aligned}$$

where the last inequality is derived by (2.5); otherwise, if $VaR_p(X) > d$, then

$$\mathbb{P}(R_f(X) > VaR_p(R_{f_\lambda^*}(X))) = \mathbb{P}(R_f(X) > d) = \bar{F}_{R_f(X)}(d) \le \alpha \le p,$$

where the first inequality follows from (2.5) and (3.32).

(ii) $0 < p < \alpha$ case: In this situation, we have

$$\mathbb{P}(R_f(X) > VaR_p(R_{f_\lambda^*}(X))) \le \mathbb{P}(X > VaR_p(X)) = \bar{F}_X(VaR_p(X)) \le p.$$

As a result, the definition of VaR in (2.3) implies

$$VaR_p(R_f(X)) \le VaR_p(R_{f_\lambda^*}(X)), \ \forall 0 < p < 1.$$

Moreover, it follows from (3.31) and (3.32) that the inequality in the above equation is strict over $[\alpha, \alpha + \varepsilon]$ for some $\varepsilon > 0$. Consequently, this leads to a contradiction:

$$\lambda = \mathbb{E}[R_f(X)] = \mathbb{E}[VaR_U(R_f(X))] < \mathbb{E}[VaR_U(R_{f_\lambda^*}(X))] = \mathbb{E}[R_{f_\lambda^*}(X)] = \lambda$$

by using the fact that $R_f(X)$ and $VaR_U(R_f(X))$ are equal in distribution. Thus, through (3.30), we know f_λ^* is a solution to the modified optimal reinsurance problem (3.29).

Second, the study of the effect of λ on the optimal reinsurance f_λ^* is the same as that of Theorem 3.3 and hence the remaining proof is omitted. □

Remark 3.4. *We emphasize that the optimal ceded loss function f^{*3} given in* (3.25) *is a truncated stop-loss reinsurance. This implies that for the loss less than $VaR_\alpha(X)$, the reinsurer is responsible for the loss in excess of the deductible γ. However, if the incurred loss X is greater than $VaR_\alpha(X)$, then the reinsured amount reduces drastically to zero from a maximum of $VaR_\alpha(X) - \gamma$. It suggests that the insurer is only concerned with reinsuring moderate losses but not large losses. This seems counterintuitive since the insurer should care about tail risk. One possible explanation accounting for such phenomenon is that in an extreme event with loss exceeding VaR (i.e. a catastrophic event with a very small probability α of occurrence), the insurer will be in trouble, regardless of the reinsurance arrangement. Consequently, it is prudent for the insurer not to reinsure any loss beyond its VaR to reduce the reinsurance premium. See also Froot (2001) for detailed discussion of such a reinsurance policy in the context of catastrophe risk market.*

Remark 3.5. *Interestingly, when $\vartheta \geq \bar{F}_X(0)$, the deductible γ reduces to* 0 *so that f^{*3} simplifies to $f^{*3}(x) = x\mathbb{I}(x \leq VaR_\alpha(X))$. This implies that the insurer is fully insured for loss up to $VaR_\alpha(X)$. Once the incurred loss exceeds the threshold level $VaR_\alpha(X)$, the insurer is responsible for the entire loss amount and hence the insurer only suffers in such an extreme event.*

Remark 3.6. *By contrasting the three VaR-based reinsurance models analyzed in the preceding subsections, it clearly highlights the sensitivity of the optimal solutions with respect to the constraints imposed on the reinsurance models. More specifically, the optimal reinsurance policy could be in the form of stop-loss, limited stop-loss, or truncated stop-loss depending on the conditions imposed on both ceded and retained loss functions. We also emphasize that while the reinsurance model in Subsection 3.1 has been analyzed in Cai et al. (2008), the latter two models are totally new.*

4. Optimal reinsurance under CVaR risk measure

In this section, we focus on the CVaR-based optimal reinsurance model (2.8) with the admissible set of the ceded loss functions

$$\mathcal{C} = \{f : f(x) \text{ is a measurable function with } 0 \leq f(x) \leq x\} \supsetneq \mathcal{C}^{3'}. \quad (4.1)$$

Similar to Subsection 3.2, we derive the optimal reinsurance solution by dividing our analysis into two steps. In the first step, we study the following modified optimization problem:

$$\min_{h \in C_\mu} CVaR_\alpha(T_h(X)), \tag{4.2}$$

where $C_\mu = \{f \in C : \mathbb{E}[f(X)] = \mu\}$ with $0 \leq \mu \leq \mathbb{E}[X]$. The second step is then applied to analyze the effect of μ on the optimal reinsurance.

To proceed, let us reproduce, without proof, the Ohlin's Lemma (see Asmussen (2000)):

Lemma 4.1. *Let Y_1 and Y_2 be random variables such that*

$$\mathbb{E}[Y_1] = \mathbb{E}[Y_2],\ \ F_{Y_1}(x) \leq F_{Y_2}(x),\ \ x < b \ \ \textit{and}\ \ \bar{F}_{Y_1}(x) \leq \bar{F}_{Y_2}(x),\ \ x \geq b \tag{4.3}$$

for some b, then $Y_1 \leq_{cx} Y_2$, i.e.

$$\mathbb{E}[w(Y_1)] \leq \mathbb{E}[w(Y_2)]$$

for any convex function w provided that the expectations exist.

By virtue of the above lemma, the following proposition gives the solutions to (4.2).

Proposition 4.1. *The optimal ceded loss function h_μ^* that solves* (4.2) *is given by*

$$h_\mu^*(x) \triangleq (x-d)_+, \tag{4.4}$$

where $d \geq 0$ is determined by $\mathbb{E}[(X-d)_+] = \mu$. Moreover,

$$\begin{aligned} CVaR_\alpha(T_{h_\mu^*}(X)) &= \min_{h \in C_\mu} CVaR_\alpha(T_h(X)) \\ &= \frac{1}{\alpha}\int_0^\alpha (VaR_s(X) \wedge d)\,\mathrm{d}s + (1+\rho)\,\mathbb{E}[(X-d)_+]. \end{aligned} \tag{4.5}$$

Proof. Theorem 2.58 in Föllmer and Schied (2004) demonstrates

$$\begin{aligned} &\mathbb{E}[(t-Y_1)_+] \leq \mathbb{E}[(t-Y_2)_+],\ \forall t \in \mathbb{R} \\ &\Leftrightarrow \int_0^p VaR_{1-s}(Y_1)\,\mathrm{d}s \geq \int_0^p VaR_{1-s}(Y_2)\,\mathrm{d}s, \forall p \in (0,1]. \end{aligned}$$

Straightforward algebra implies that CVaR risk measure preserves the convex order, i.e.

$$Y_1 \leq_{cx} Y_2 \Rightarrow CVaR_\alpha(Y_1) \leq CVaR_\alpha(Y_2).$$

Moreover, by verifying (4.3) in Lemma 4.1, we have

$$T_{h_\mu^*}(X) = R_{h_\mu^*}(X) + (1+\rho)\mu \leq_{cx} R_h(X) + (1+\rho)\mu = T_h(X)$$

for any $h \in C_\mu$, then $CVaR_\alpha(T_{h_\mu^*}(X)) \leq CVaR_\alpha(T_h(X))$. Consequently, h_μ^* is a solution to the optimization problem (4.2).

Furthermore, the translation invariance property of CVaR risk measure implies

$$\begin{aligned}
CVaR_\alpha(T_{h_\mu^*}(X)) &= CVaR_\alpha(R_{h_\mu^*}(X)) + (1+\rho)\,\mathbb{E}[h_\mu^*(X)] \\
&= \frac{1}{\alpha}\int_0^\alpha VaR_s(X \wedge d)\mathrm{d}s + (1+\rho)\,\mathbb{E}[(X-d)_+] \\
&= \frac{1}{\alpha}\int_0^\alpha \big(VaR_s(X) \wedge d\big)\mathrm{d}s + (1+\rho)\,\mathbb{E}[(X-d)_+],
\end{aligned}$$

where the last equality is derived by (2.6), then the proof is complete. □

Remark 4.1. *In the proof of the above proposition, we only use the convex order preserving property of CVaR risk measure. Therefore, the results of the above proposition could be naturally generalized to the cases when the optimality criteria preserve the convex order. It is important to point out that the theory of stochastic orders has been shown to be powerful in analyzing the optimal reinsurance problems. For example, Van Heerwaarden et al. (1989) use it to study several optimal reinsurance problems under the constraint* C_μ^2.

Given a reinsurance premium, the above proposition shows that under the CVaR criterion, it is optimal to choose the stop-loss reinsurance among C_μ. We next study the effect of the retention level d on the optimal reinsurance, and obtain the main result of this section.

Theorem 4.1. *The optimal ceded loss function h_C^* that solves* (2.8) *with constraint C in* (4.1) *is given by*

$$h_C^*(x) \triangleq \begin{cases} (x-d^*)_+, & \alpha < 1/(1+\rho); \\ 0, & \textit{otherwise}, \end{cases} \tag{4.6}$$

where d^* is defined in (3.1). *Moreover,*

$$CVaR_\alpha(T_{h_C^*}(X)) = \min_{h \in C^j} CVaR_\alpha(T_h(X)) = \begin{cases} \beta, & \alpha < 1/(1+\rho); \\ CVaR_\alpha(X), & \textit{otherwise} \end{cases}$$

for any $j = 1,2,3$, where β is defined in (3.2).

Proof. We first consider the minimization problem on $\mathcal{C}$. Proposition 4.1 implies that solving the reinsurance model (2.8) is equivalent to finding the optimal retention level d that minimizes $CVaR_\alpha(T_{h_\mu^*}(X))$ in (4.5). Here, $CVaR_\alpha(T_{h_\mu^*}(X))$ can be rewritten as

$$\begin{aligned} CVaR_\alpha(T_{h_\mu^*}(X)) = {} & CVaR_\alpha(X) - \frac{1}{\alpha}\int_0^\alpha (VaR_s(X) - d)_+ \,\mathrm{d}s \\ & + (1+\rho)\int_0^1 (VaR_s(X) - d)_+ \,\mathrm{d}s, \end{aligned} \tag{4.7}$$

since X equals to $VaR_U(X)$ in distribution.

If $\alpha \geq 1/(1+\rho)$, the above equation implies that $CVaR_\alpha(T_{h_\mu^*}(X))$ is decreasing in d, then it attains its minimum at $d = \infty$ with value $CVaR_\alpha(X)$ and the corresponding optimal ceded loss function is 0.

Next, we focus on the case $\alpha < 1/(1+\rho)$. To proceed with our analysis, we consider the following two subcases: $\bar{F}_X(d) \leq \alpha$ and $\bar{F}_X(d) > \alpha$.

1. If $\bar{F}_X(d) \leq \alpha$, which is equivalent to $d \geq VaR_\alpha(X)$ according to (2.5), then we have

   $$\mathbb{E}[(X-d)_+] = \int_0^1 (VaR_s(X) - d)_+ \,\mathrm{d}s = \int_0^\alpha (VaR_s(X) - d)_+ \,\mathrm{d}s,$$

   and hence (4.7) simplifies to

   $$CVaR_\alpha(T_{h_\mu^*}(X)) = CVaR_\alpha(X) + ((1+\rho) - 1/\alpha)\,\mathbb{E}[(X-d)_+],$$

   which implies that $CVaR_\alpha(T_{h_\mu^*}(X))$ increases in d. Therefore, on the interval $[VaR_\alpha(X), \infty]$, $CVaR_\alpha(T_{h_\mu^*}(X))$ attains its minimum at $d = VaR_\alpha(X)$.
2. If $\bar{F}_X(d) > \alpha$, then $0 \leq d < VaR_\alpha(X)$. In this case, (4.5) simplifies to

   $$CVaR_\alpha(T_{h_\mu^*}(X)) = d + (1+\rho)\,\mathbb{E}[(X-d)_+] = d + (1+\rho)\int_d^\infty \bar{F}_X(s)\,\mathrm{d}s.$$

   Recall that $\alpha < 1/(1+\rho)$, then $d^* \leq VaR_\alpha(X)$. Therefore, the above equation implies that $CVaR_\alpha(T_{h_\mu^*}(X))$ attains its minimum at $d = d^*$ on the interval $[0, VaR_\alpha(X)]$.

Collecting all the above results yields that $h_{\mathcal{C}}^*$ is a solution to the reinsurance model (2.8) with the constraint $\mathcal{C}$ defined in (4.1). Further, $h_{\mathcal{C}}^* \in \mathcal{C}^j$ for $j = 1,2,3$. Since the global optimal solution is also locally optimal, then $h_{\mathcal{C}}^*$ is also a solution to the optimal reinsurance model (2.8) with the constraints $\{\mathcal{C}^j; j = 1,2,3\}$. Hence the proof is complete. □

Remark 4.2. *It should be emphasized that Balbás et al. (2009) and Tan et al. (2011) also obtain the above result by using the Lagrangian approach. In contrast, we provide a different but straightforward approach. As noted in Remark 4.1,* *a key advantage of our method is that the above result can be easily extended to other optimality criteria that preserve convex order.*

5. Concluding remarks

In this paper, we analyze the solutions to the VaR- and CVaR-based optimal reinsurance models over different classes of ceded loss functions with increasing generality. The impact of the assumed feasible set of ceded loss functions on the optimal reinsurance design is highlighted in the case of VaR criterion. More specifically, the optimal reinsurance policy can be in the form of stop-loss, limited stop-loss, or truncated stop-loss, depending on the conditions imposed on the ceded and retained loss functions. This suggests a difference in risk management strategy depending on the adopted optimal reinsurance model. The different optimal reinsurance policies also suggest the differences in the insurer's style toward risk management and its attitude towards risk. In the case of limited stop-loss reinsurance, both the insurer and reinsurer are willing to pay more for larger loss so that there is a proportionate sharing of risk between the insurer and the reinsurer (though subjected to a limit for the reinsurer). In contrast, the truncated stop-loss reinsurance induces the insurer to reinsure against moderate loss but not large catastrophic loss. The CVaR-based optimal reinsurance model, on the other hand, is quite robust in the sense that the stop-loss reinsurance is always the optimal solution irrespective of the conditions on the ceded and retained loss functions.

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