EVALUATING QUANTILE RESERVE FOR EQUITY-LINKED INSURANCE IN A STOCHASTIC VOLATILITY MODEL: LONG VS. SHORT MEMORY

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ABSTRACT

This paper evaluates the long-term risk for equity-linked insurance products. We consider a specific type of equity-linked insurance product with guaranteed minimum maturity benefits (GMMBs), and assume that the underlying equity follows the stochastic volatility model which allows the return's latent volatility component to be short- or long-memory. The explicit form of the quantile reserve or the Value at Risk and its confidence intervals are derived for both the long-memory and short-memory stochastic volatility models. To illustrate the effect of long-memory volatility, we use the S&P 500 index as an example of linked equity. Simulation studies are performed to examine the accuracy of the quantile reserve and to demonstrate the consequence of low coverage probability if model misspecification takes place. The empirical results show that the confidence interval of quantile reserve could be severely underestimated if the long-memory effect in equity volatility is ignored.

KEYWORDS

Equity-linked Insurance; Value at Risk; Stochastic Volatility Model; Long-memory Stochastic Volatility Model.

1. INTRODUCTION

Stochastic modeling of assets and liabilities for equity-linked insurance has received much attention in the actuarial professional. Unlike traditional insurance contracts, equity-linked insurance enables the policyholder to participate in an underlying index or combination of funds. The segregated fund contract in Canada, unit-linked insurance in the Britain, and variable annuity (VA) in the United States represent typical equity-linked insurance products in different insurance markets. In a recent and popular market development, for equity-linked insurance products, a design requires the insurer to guarantee the investment return on the policy's account value, called an investment guarantee.

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There are many forms of investment guarantees, including guaranteed minimum death benefits (GMDBs), guaranteed minimum maturity benefits (GMMBs), guaranteed minimum income benefits (GMIBs), and guaranteed minimum withdrawal benefits (GMWBs)[1], all of which are common guarantee features with variable annuity (VA) products in the U.S. market. These innovative guarantee features account for the growing popularity of VA, such that the more than $1.35 trillion currently invested represents a 50% increase over the past five years (Condron 2008). Ledlie et al. (2008) point out that VA products also have enjoyed significant international success. They believe that the prospects for the product in the United Kingdom and throughout Europe are very favorable.

However, significant financial risk is involved in issuing investment guarantees with equity-linked insurance. If the equity market crashes (e.g., the 2008 financial crisis), the insurer must pay for the guarantee for each in-the-money policy. It affects many policies at the same time. Such risk is systematic and catastrophic[2]. The insurer may face insolvency if the reserves and capital it holds are inadequate[3]. Regulators in North American countries have required the actuary to evaluate the reserve and capital requirements based on stochastic equity return model. In 2001, the Canada Institute of Actuaries Task Force on Segregated Funds (CIA, 2001) proposed a stochastic methodology for estimating reserves and capital requirement. In 2005, the American Academy of Actuaries (AAA, 2005) followed and set up a methodology for calculating the risk-based capital of the equity risk for its variable annuity contracts in its C3 Phase II report. The insurer are allowed to develop or choose the stochastic asset return model for their evaluation, which is called internal model[4]. Therefore, the kind of equity return model that can best model the guarantee risk is a hot topic and still under development.

The Maturity Guarantee Working Party (MGWP, 1980) first proposed a stochastic simulation approach to calculate the quantile reserve for the product with GMMB[5]. This concept of quantile reserve is identical to the

[1] The features of the corresponding investment guarantees are, briefly, as follow: The GMDB guarantees the policy-holder a specific monetary sum upon death during the term of the contract. The GMMBs guarantees the policy holder a specific monetary at the maturity of the contract. The GMIB ensures that the lump sum accumulated under a separate account contract may be converted to an annuity at a guaranteed rate. For more details of these guarantees, please refer to Hardy (2003).

[2] Systematic risk is undiversified and therefore it will cause catastrophic loss to the insurer.

[3] For example, guaranteed annuity options (GAOs) had been a problem for some British life offices, especially Equitable Life. Under guaranteed annuity options, the insurer guarantees to cover the policyholder's accumulated fund value to a life annuity at a fixed rate when the policy matures. It happened however that some insurers did not reserve for GAOs or the reserve is not sufficient (Wilkie et al, 2003).

[4] The use of internal models for determining liability and capital requirement can be referred to Brender (2002).

[5] In the 1960s and early 1970s, the unit-linked insurance with a GMMBs of 100 percent of the premium were typically sold in the U.K. market. The insurer suffered financial problems and the benefit fell into disfavor because of the equity crisis in 1973.

Value-at-Risk (VaR) concept of finance but generally applied over longer time periods by the insurance companies rather than by the banks. Thus, to calculate the quantile reserves, the insurer assesses an appropriate quantile of the loss distribution based on an equity return model, for example, 99 percent. The earliest return models for long-term applications were developed by Wilkie (Wilkie 1986, 1995) and have been widely applied in the U.K. actuarial practice. They use autoregression to model long-term yield, short-term yield, property, share dividend, and share dividend yield. In more recent actuarial work, Hardy (2001) investigates the regime-switching lognormal (RSLN) model to reserve for equity-linked insurance and compares it with other econometric models, such as autoregressive conditional heteroscedasticity (ARCH) and generalized ARCH (GARCH). She finds that the RSLN model performs better than ARCH and GARCH for modeling long-term series. In addition, the CIA (2001) and AAA (2005) have investigated a plethora of models and set up stochastic methodologies for actuarial practice. Hardy et al. (2006) further illustrate various models proposed by the AAA (2005), including the stochastic volatility (SV) model in addition to the econometric models and RSLN model. They conclude that different models lead to different results regarding reserves for investment guarantees. Therefore, finding a proper equity return model for evaluating the equity risk has become a task of imminent importance for both researchers and practitioners in actuarial sciences.

There are two widely documented phenomena, among others, concerning the returns' volatility dynamics that the task of modeling equity returns attempts to address. The first is about the volatility clustering reported in Mandelbrot (1963) that large changes tend to be followed by large changes and small changes tend to be followed by small changes. This general tendency of volatility clustering also reflects that the returns are conditionally heteroscedastic. The second is due to Taylor (1986) who observed that strong positive serial correlation exists in nonlinear transformations of returns, such as square, logarithm of square, and absolute value, whereas the return series itself behaves almost like white noise. Stationary models which have been proposed to describe the aforementioned two properties include the ARCH (or GARCH) family (Engle, 1982; Bollershlev, 1986), exponential GARCH (EGARCH, Nelson, 1991), and the stochastic volatility model (e.g., Taylor, 1986; Harvey et al., 1994). In actuarial literature, a great deal of attention also centers on models such as ARCH, GARCH, and SV (e.g., Wilkie, 1995; Hardy et al., 2006). The AAA (2005) even recommends the use of SV models to assess the capital requirement for VA with guarantees. The main feature of these models is that there is a non-linear positive volatility process in the model specification designated to govern the volatility dynamics of the underlying returns. Recently, using the square or absolute value of returns as a proxy of the volatility process, empirical studies (e.g., Ding et al., 1993) have revealed that the volatility process is not only correlated but exhibits highly persistent autocorrelations that the ARCH or GARCH process cannot produce. The decay rate of the autocorrelations is so slow that it can be better extrapolated by a positive sequence, which is not summable. On the contrary, the volatility of processes such as the ARCH or GARCH model are usually characterized by rapidly decaying and summable autocovariances.

A real-valued stationary time-series process $\{X_t, t = 0, \pm 1, \pm 2, \ldots\}$ with autocovariance of such a decay rate is usually called long-memory, and its autocovariance function is represented as

$$\gamma(j) = \operatorname{cov}(X_t, X_{t+j}) = j^{2d-1}L(j) \sim cj^{2d-1}, \quad j \to \infty, \tag{1}$$

for some positive constant c, and $0 < d < 1/2$, with a positive function $L(.)$ that is slowly varying at infinity (i.e., $\lim_{x\to\infty} L(\tilde{a}x)/L(x) \to 1$ for all $\tilde{a} > 0$); in addition, "$g_n \sim h_n$" signifies $\lim_{n\to\infty} g_n/h_n = 1$. More statistical properties about a long-memory process are provided in Section 2 and Appendix F.

The effect of long memory in stochastic volatility has been documented in share indices, individual stocks, and foreign exchanges. Ding et al. (1993) report that there exists strong autocorrelations for long lags in the power transformation of absolute returns of the S&P 500 index. Lobato and Savin (1998) further provide convincing evidence of persistent correlation in the square or absolute value of returns based on the daily returns of the S&P 500 index for the period July 1962 to December 1994. Breidt et al. (1998) study the value-weighted common stock index and indicate that the long-memory stochastic volatility (LMSV) model reproduces closely the empirical autocorrelations structure of the conditional volatilities. In addition to index returns, the long-memory phenomenon of volatility has been documented in individual stocks (Ray and Tsay, 2000), as well as in some high-frequency data such as minute-by-minute stock returns (Ding and Granger, 1996) and foreign exchange rate returns (Bollerslev and Wright, 2000). To provide a better fit for data with strong dependence in volatility, Breidt et al. (1998) suggest the long-memory stochastic volatility model, which is a natural extension of the traditional SV model. The LMSV process is a stationary time series formed by martingale differences that contain a latent volatility process exhibiting long memory. It can be seen that the LMSV model exhibits the desirable property that the return series itself is white noise and the square (or logarithm of square) of return is long-memory, which coincides with the stylized effect mentioned above. The main message the LMSV structure reveals is that the magnitude of the shocks to the returns dissipates at a much slower rate than short-memory conditional variance models like the ARCH or GARCH can depict. The specific form of the LMSV model is presented in the next section.

In the recent development of risk management for market risk, incorporating long-memory effect in asset volatility to calculate the Value-at-Risk has received attentions in financial literature (Guermat and Harris, 2002; Härdle and Mungo, 2008; Caporin, 2008). However, the effect of LMSV has not been addressed in actuarial literature. The present paper introduces the LMSV model with an attempt to study the effect of LMSV on evaluating the Value-at-Risk or quantile reserve for equity-linked insurance. VaR is one of the most popular risk measures used in finance and actuarial science by both researchers and practitioners, despite some discussions about its shortcomings (Down and Blake, 2006). Equity-linked insurance in general has a long-duration feature, because it is popular in the defined contribution pension market. Therefore, the criteria for selecting an equity return model should capture the future dynamics of the equity return for a long duration, that is, the persistent correlation in volatility. Our approach differs from existing actuarial literature (e.g., Hardy, 2001, Hardy et al., 2006) in that it considers the time-series model of returns that takes into account the persistent correlation in volatility. We first examine the evidence of LMSV in long-term share indices using daily data from the S&P 500 index for the period January 1977 to December 2006. We then analyze the effects of the LMSV on quantile reserves for equity-linked insurance. We illustrate it using a specific type of equity-linked insurance with GMMBs. Under the setting that returns follow the stochastic volatility model, we derive the analytic forms of quantile reserves and its confidence intervals for both short- or long-memory volatility process. To study the effect of LMSV, we compare the confidence intervals of the quantile reserves, whose accuracy we assess using simulations. In the numerical study, we find that ignoring the LMSV leads to a misspecification that produces low coverage probability. In addition, we show that the confidence interval of the quantile reserve could be severely underestimated if the long-memory effect in equity volatility is ignored.

Therefore, this paper makes a threefold contribution relative to previous works on investment guarantees. First, we introduce and examine the stochastic volatility model in a general form for asset returns, which could be short- or long-memory. Second, using the above SV model for returns, we derive a closed-form formula of quantile reserve for the GMMB-type equity-linked insurance. Such a formula for quantile reserves with a stochastic volatility model pertaining to long-horizon returns has not been discussed in prior literature. Third, we construct the confidence interval for the quantile reserves, considering both the short- and long-memory volatility cases in the SV framework. The derivation of the confidence intervals can benefit insurers by providing a robust check to examine the accuracy of the quantile reserves, especially for long-term equity-linked insurance products.

The structure of this paper is as follows: Section 2 describes the long-memory process and the test for the existence of long memory in the volatility of stock returns. In Section 3, we derive an asymptotic formula of the quantile reserve for equity-linked contracts with GMMBs and give an estimate for it. By considering the SV model whose volatility process may be short- or long-memory, we prove a central limit theorem to establish confidence intervals with respect to the quantile reserve. In Section 4, numerical results on coverage probability are presented to address the importance of the use of the LMSV model for long-duration insurance products. Section 5 concludes. Proofs of the two main theorems and the formal definition of several econometric time series models mentioned in this paper are collected in Appendices A to F.

2. A Stochastic Volatility Model for Equity Returns

2.1. Long Memory in Stochastic Volatility

While both the ARCH-type (including GARCH family) process and the stochastic volatility (SV) process have been studied widely and enjoy equal success, in the present paper, we concentrate on the SV model because it has the modeling flexibility to allow the volatility process to be either short-memory or long-memory. Denote r_t as the equity return at time t. The SV model is defined by

$$r_t = \mu + v_t u_t, \quad v_t = \bar{\sigma}\exp(Z_t/2), \tag{2}$$

where μ is the mean of r_t, and $\{u_t\}$ is the sequence of shocks comprised of independent identically distributed (i.i.d.) random variables with mean 0 and unit variance, $Eu_t = 0$ and $Eu_t^2 = 1$; the whole random sequence $\{u_t\}$ is assumed independent of the latent volatility component $\{Z_t\}$. Furthermore, $\{Z_t\}$ is a linear process defined as

$$Z_t = \sum_{i=0}^{\infty} a_i \eta_{t-i}, \tag{3}$$

where $\{\eta_t\}$ is i.i.d. random variables (Gaussian or non-Gaussian) with 0 mean and finite variance σ_η^2. If the mean of $\{Z_t\}$ is non-zero, we let it be absorbed by the positive $\bar{\sigma}$. For notation convenience, r_t defined in Equation (2) will be used to denote the random variable of returns.

The traditional SV model (Taylor, 1986; Harvey et al., 1994), which is the short-memory version of Equation (2), requires that the coefficient sequence $\{a_i\}$ be summable, that is,

$$\sum_{i=0}^{\infty} \left|a_i\right| < \infty, \tag{4}$$

which implies that the autocovariance sequence is also summable. The long-memory stochastic volatility model introduced by Breidt et al. (1998) also follows the structure specified in Equation (2) but uses innovation coefficients $\{a_i\}$ of $\{Z_t\}$ in the form of $a_i \sim C \cdot i^{-\beta^*}$ for some $\beta^* \in (1/2, 1)$ and a positive constant C. Set $d = 1 - \beta^*$. Note that with $a_i \sim C \cdot i^{-\beta^*}$, which is clearly not summable, the j-th autocovariance $\gamma(j)$ of $\{Z_t\}$ can be expressed as $\gamma(j) = \sigma_\eta^2 \sum_{i=0}^{\infty} a_i a_{i+j}$, which has the same asymptotic form as in Equation (1). In Equation (1) the autocovariance function decays to 0 at a polynomial rate such that the function is not summable. Under some regularity conditions, Equation (1) is equivalent to the fact that spectral density of $\{Z_t\}$ diverges to infinity at the zero frequency with a rate related to d. The most important feature that distinguishes the long-memory process from its short-memory counter part is the non-$\sqrt{n}$ normalization constant. Take the sample mean for example, for a size n sample, the usual scaling factor $\sqrt{n}$ for independent or weakly dependent data will fail under long memory because the variance of the sample mean decays to 0 at a rate of $O(n^{-d-1/2})$, which is both slower than $O(n^{-1})$ and is usually unknown in practice.

An important special class of the long-memory process is the fractional autoregressive integrated moving average (FARIMA) process (see Appendix F). Note that if Z_t is Gaussian, the autocovariance function of $\{r_t^2\}$ has the same decay rate as $\{Z_t\}$ regardless of whether $\{Z_t\}$ is short- or long-memory (Taylor, 1986).

A useful property of the SV model is that if the mean is known, taking logarithm of the square of the series centered by its mean yields the following simple additive model:

$$\begin{aligned} y_t &= \log(r_t - \mu)^2 = \log \bar{\sigma}^2 + E(\log u_t^2) + Z_t + \left(\log u_t^2 - E(\log u_t^2)\right) \\ &\equiv \mu^* + Z_t + \varepsilon_t, \end{aligned} \tag{5}$$

where $\{\varepsilon_t\}$ is i.i.d. with mean zero and variance σ_ε^2. If u_t is standard normal, as we shall assume for the rest of this paper, then $E[\log u_t^2] = -1.27$, and $\sigma_\varepsilon^2 = \pi^2/2$ (Wishart, 1947). From Equation (5), it is evident that the lagged autocovariances of $\{y_t\}$ are identical to those of $\{Z_t\}$, because $\{\varepsilon_t\}$ is an i.i.d. sequence independent of $\{Z_t\}$. Therefore, although $\{Z_t\}$ is not observable, finding empirical evidence about long memory in $\{Z_t\}$ can be achieved by performing statistical tests on a volatility proxy, namely, the logarithm of the squared series.

2.2. Evidence of Long Memory in Stock Volatility

In this section, we formally test for the existence of long memory in the volatility of stock markets' series by analyzing the logarithm of the squared series, one of the commonly used volatility proxies. The memory parameter, d, is estimated by regressing the logarithm of the periodogram at the first f Fourier frequencies on a function of the frequencies, as proposed by Geweke and Porter-Hudak (1983). For the LMSV model, Deo and Hurvich (2001) derive the limiting distribution of the Geweke and Porter-Hudak estimator $\hat{d}$ and show that $\hat{d}$ is asymptotically consistent with d if $f = KT^\delta$ for any $0 < \delta < 1$, where T is the length of the return series.

We perform the tests for long memory on S&P 500 daily returns, starting on the first trading day of January 1977 and ending on the last trading day of December 2006, which consists of 30 years of data. The mean of the daily log returns is 0.034%, and the daily volatility using the same data is 1%. To estimate the memory parameter, we regress the logarithm of the periodogram at the first $f = T^{0.56}$ Fourier frequencies. The estimated memory parameter $\hat{d}$ is 0.41, and the standard error is 0.06. According to the standard errors obtained from the output of the regression, the estimated memory parameter $\hat{d}$ is significantly different from 0, which offers strong evidence that the volatility of the S&P 500 daily returns is long-memory. Figure 1 presents the empirical and fitted autocorrelations for the log squares series. The empirical autocorrelations of the logarithm of squared returns show a slow decay with non-negligible values for hundreds of lags.

To compare the performance of the LMSV models with other commonly used models, we also fit GARCH, EGARCH, short-memory SV, and IGARCH models to the same series of S&P 500 returns. Among them, the EGARCH model allows for asymmetric effects between positive and negative asset returns. We select GARCH (1,1), EGARCH (2,1), and IGARCH (1,1) based on the SIC criterion. For simplicity, we set $\{Z_t\}$ to be a Gaussian AR(1) process for the short-memory SV model. All parameters of the corresponding models were estimated by maximizing the conditional log-likelihood functions. To compare the performance of fitting models with the real observations, we compute the autocorrelations of the log squares for fitted models and plotted them against the empirical autocorrelations for the log-squared return series. Using GARCH parameter estimates, we simulate 1000 realizations and compute the sample autocorrelations of the log squares for each realization. The same simulations apply to EGARCH, IGARCH, and short-memory SV. The average of the sample autocorrelations of 1000 replications is plotted as the autocorrelations for the fitted ARCH family and short-memory SV models. Fitted autocorrelations of the LMSV model are obtained from Equations (31) and (33) with the Geweke and Porter-Hudak estimator $\hat{d}$. In other words, we

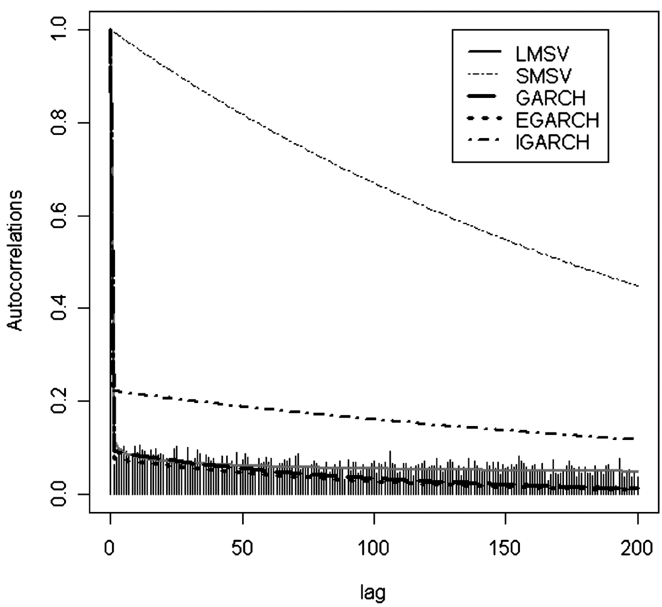

FIGURE 1: Empirical and fitted autocorrelation functions for 30 years of S&P 500 return series. Estimated long-memory dependence parameter is $\hat{d} = 0.41$.

use autocorrelations of an FARIMA (0,d,0) process to fit the log squares of S&P 500 returns. As shown in Figure 1, none of the GARCH, EGARCH, or IGARCH models fits the data reasonably well; their autocorrelations either decay too rapidly (GRACH and EGARCH) or are too persistent (IGARCH). Only the LMSV model is able to reproduce the empirical autocorrelation structure of the returns' volatility reasonably well. The empirical result is similar to that by Breidt et al. (1998), who demonstrate the superiority of the LMSV model over existing (short-memory) volatility models using the daily returns for the value-weighted market index from the CRSP tapes for July 1962 - December 1987.

3. Derivation of Quantile Reserve and Confidence Interval for Equity-linked Insurance Under a SV model

3.1. Description of Policy Setting

To study the effect of volatility persistence, we consider the LMSV model for modeling the equity risk and calculate the quantile reserve for equity-linked insurance contracts. The quantile reserve was first suggested by the maturity guarantees working party (MGWP, 1980) to deal with the unit-linked insurance with maturity guarantee. The calculation of quantile reserve is based on the quantile of the guaranteed liability distribution[6]. The principle of the quantile reserve is identical to the concept of value at risk defined in finance. The regulator of Financial Services Authority (FSA) set the standard that the insurer must hold capital of at least 99.5% probability of its firm's survival over one year[7]. In addition to the risk measures of VaR, the conditional tail expectation (CTE) is also used to determine the reserve and capital requirements for equity-linked contracts for segregated fund guarantees and VA guarantees, which is CTE(95)[8] in Canada and CTE(90) in the United States[9].

To illustrate the effect of LMSV analytically, we focus on the VaR risk measure or quantile reserves for a specific type of guarantee, namely, the guaranteed minimum maturity benefits (GMMBs). For different types of guarantees, the analytic form may not be available, which may require a different approach to address the issue. We use the argument of conditioning on the volatility process

[6] The liability is the amount that the insurer has to pay if the insured event happens. For equity-linked insurance contracts, the guaranteed liability is the expected difference of the guaranteed amount and the actual account value.

[7] The assessment of capital that a firm needs to hold is set by FSA in the individual capital requirement (ICA 2007).

[8] CTE(α) is the notation for the conditional tailed expectation at the α% confidence level.

[9] In Canada, the Office of the Superintendent of Financial Institutions (OSFI) set the mandatory minimum continuing capital and surplus (MCCSR) for segregated fund guarantees and implement in December 2000 (CIA 2001). In the United States, the capital requirement for VA guarantee is regulated by the risk-based capital requirement of C3 risk, beginning in 2005 (AAA, 2005).

to derive explicit forms for the quantile reserve and its confidence interval for both short-memory and long-memory cases.

We assume a single premium policy with an amount P invested in an equity whose daily returns follow the SV model given in Equation (2) with known mean μ. We permit the latent volatility component to be short- or long-memory. To model the guaranteed risk, we first define the notations as follows:

- G is the amount of the guarantee, $G > 0$.
- P is the single premium.
- S_t is the price process of the underlying equity on day t; S_0 represents the price of the initial day.
- $F(t)$ is the account value at time t in days, such that $F(0)$ represents the initial account value. The account value is decided according to the underlying share index invested by the policyholder.
- m is the management fee including the charge for the guarantee.
- T is maturity date of the equity-linked insurance contracts.
- r_t is the log return at time t $\left(\text{i.e. } r_t = \frac{S_t}{S_{t-1}}\right)$.
- V_a is the $100\alpha\%$ quantile of the liability distribution.

The GMMBs option gives the policyholder the right to receive the maturity benefit, depending on the greater of the accumulated account value or the amount of guarantee. The guaranteed benefit upon maturity can be expressed as $\text{Max}(G, F(T))$. Thus, the liability of GMMBs for the insurer is calculated as $\text{Max}(G - F(T), 0)$ on maturity. Including the management fee, the account value at maturity date T can be expressed as

$$F(T) = P \cdot \frac{S_T}{S_0} \cdot e^{-Tm},$$

which is modeled on a daily basis. Based on the equity return model, we can obtain the account value by projecting the return on selected share index. The liability distribution can then be determined. If we set the initial premium equal to S_0, the account value can be expressed in simplified form as $F(T) = S_T e^{-Tm}$.

3.2. Quantile Reserve

The quantile reserve for the GMMBs is calculated according to the quantile of the liability distribution. Let ζ denote the probability that there is no liability for the insurer (i.e., the fund value is greater than the guaranteed amount.). That is, $\zeta = P(S_T e^{-Tm} - G \geq 0)$. Then, for all $\alpha \leq \zeta$, the $100\alpha\%$ quantile reserves of V_α is equal to 0. For $\alpha > \zeta$, V_α satisfies the following equation:

$$1 - \alpha = P\left(G - S_T e^{-Tm} > V_\alpha\right) = P\left(S_T e^{-Tm} < G - V_\alpha\right). \tag{6}$$

Let σ^2 denote the variance of the returns r_t. Then, Equation (6) implies

$$\begin{aligned} 1-\alpha &= P\left(S_0 \exp\left(\sum_{t=1}^{T} r_t\right)\cdot e^{-Tm} < G - V_\alpha\right) \\ &= P\left(\frac{\sum_{t=1}^{T} r_t - T\mu}{\sqrt{T}\sigma} < \frac{\ln(G-V_\alpha)-\ln S_0 + Tm - T\mu}{\sqrt{T}\sigma}\right). \end{aligned} \tag{7}$$

Using the SV model for the underlying asset return r_t, we derive an estimate for the quantile reserve. We first observe that the central limit theorem holds for the sample mean of the SV model, that is,

$$\frac{\sum_{t=1}^{T} r_t - T\mu}{\sqrt{T}\sigma} \xrightarrow{d} N(0,1), \tag{8}$$

where $\sigma^2 = E(r_t)^2$. The proof of Equation (8) is offered in Appendix A. The central limit theorem given in Equation (8) can be used to derive an estimate of the quantile reserves for the GMMBs guarantee. From Equations (7) and (8), it follows that

$$\frac{\ln(G-V_\alpha)-\ln S_0 + Tm - T\mu}{\sqrt{T}\sigma} \xrightarrow{P} \Phi^{-1}(1-\alpha), \text{ as } T\to\infty. \tag{9}$$

Therefore, a natural estimate of V_α, denoted $\hat{V}_\alpha$, would be

$$\hat{V}_\alpha = G - S_0 \exp\left\{\Phi^{-1}(1-\alpha)\sqrt{T}\,\hat{\sigma} + T\left(\mu - m\right)\right\}, \tag{10}$$

which is obtained by solving

$$\left[\ln(G-V_\alpha)-\ln S_0 + Tm - T\mu\right] / \sqrt{T}\sigma = \Phi^{-1}(1-\alpha),$$

where σ is replaced by the sample standard deviation $\hat{\sigma}$, and $\hat{\sigma} = (\sum_{t=1}^{T}(r_t-\mu)^2/T)^{1/2}$.

Before proving the limiting distribution for the deviation of $\hat{V}_\alpha$ from the true V_α, we need to accomplish two preparatory tasks: deriving the true value of V_α, and showing the asymptotic distribution of normalized $(\hat{\sigma}^2 - \sigma^2)$. For the first task, by utilizing the normality property of r_t, conditional on the sigma field generated by $\{Z_1, Z_2, \ldots, Z_T\}$ and $T^{\frac{1}{2}-d}(\sigma_T^{*2} - \sigma^2) = O_P(1)$, where $\sigma_T^{*2} = \sum_{t=1}^{T} v_t^2/T$, we achieve

$$1-\alpha = \Phi\left(\frac{\ln(G-V_\alpha)-\ln S_0 - T\mu + Tm}{\sigma\sqrt{T}}\right) + o\left(T^{-\left(\frac{1}{2}-d\right)}\right). \tag{11}$$

Then, solving Equation (11) for V_α gives the following theorem.

Theorem 1. Suppose that the returns $\{r_t\}$ follow the SV model defined in Equation (2) and that $\{u_t\}$ is an independent sequence of standard normal random variables. The general formula for the true quantile reserves V_α, applicable for both short-memory and long-memory cases in the SV model, is

$$V_\alpha = G - S_0 \exp\left(\sigma\sqrt{T}\,\Phi^{-1}\left(1-\alpha-o\left(T^{-\left(\frac{1}{2}-d\right)}\right)\right) + T\mu - Tm\right) \tag{12}$$

as T tends to infinity. In Equation (12), when $d = 0$, it represents the true quantile reserve for the short-memory SV process.

The proofs of Equation (11) and Theorem 1 are in Appendix C. The formula in Equation (12) provides a closed-form solution for calculating the α-th quantile reserve for the GMMBs guarantees with respect to the integrated SV returns $\sum_{t=1}^{T} r_t$. The novelty of this result lies in expressing V_α in an explicit form without any parametric assumption about the marginal distribution of the dependent returns $\{r_t\}$.

3.3. Confidence Interval for Quantile Reserve

In this section we construct confidence intervals of V_α under both the long-memory and the short-memory SV model. To calculate the confidence intervals, we first deal with the asymptotic distribution of normalized $(\hat{\sigma}^2 - \sigma^2)$ based on Equations (11) and (12). With SV models, we can express $(\hat{\sigma}^2 - \sigma^2)$ as

$$\begin{aligned}\hat{\sigma}^2 - \sigma^2 &= T^{-1}\sum_{t=1}^{T}\left[(r_t-\mu)^2 - \sigma^2\right] \\ &= T^{-1}\sum_{t=1}^{T} v_t^2(u_t^2 - 1) + T^{-1}\sum_{t=1}^{T}\left(v_t^2 - \sigma^2\right).\end{aligned} \tag{13}$$

With the short-memory SV model, the process $\{Z_t\}$ (cf. Equations (2) and (3)) is short-memory, such that the coefficients $\{a_i\}$ in Equation (3) are summable. In addition, $\mathrm{Var}(T^{-1}\sum_{t=1}^{T}(v_t^2 - \sigma^2)) = O(T^{-1})$, and the estimator $\hat{\sigma}^2$ obeys the central limit theorem (Ho, 2006), which is

$$\sqrt{T}\,(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} N(0, g^2), \tag{14}$$

where $g^2 = \lim_{T\to\infty} Var(T^{-\frac{1}{2}}\sum_{t=1}^{T} v_t^2(u_t^2 - 1)) + \lim_{T\to\infty} Var(T^{-\frac{1}{2}}\sum_{t=1}^{T}(v_t^2 - \sigma^2))$. To compute g^2 analytically, we must have an explicit form of the covariance function for $\{\exp(Z_t) - \sigma^2\}$, which is available when $\{Z_t\}$ follows a Gaussian process. In this circumstances the explicit form of the variance g^2 is derived in Appendix B.

Furthermore, for the long-memory process, Ho (2006) shows that $\hat{\sigma}^2$ is still asymptotically normal but converges to the true variance at a rate slower than $\sqrt{T}$. More specifically, as $T \to \infty$,

$$\begin{aligned} T^{\frac{1}{2}-d}\left(\hat{\sigma}^2 - \sigma^2\right) &= T^{\frac{1}{2}-d}\left(\sum_{t=1}^{T}\left(v_t^2 - \sigma^2\right)/T\right) + o_p(1) \\ &\xrightarrow{d} \sigma^2 \cdot N(0, \xi^2), \end{aligned} \tag{15}$$

where ξ^2 depends on the distribution of $\{Z_t\}$, d, and c (cf. Equation (1)). To obtain a analytic forms of the confidence interval for the quantile reserve (V_α), we first show that a central limit theorem for the logarithm of the ratio of $\frac{V_\alpha - G}{\hat{V}_\alpha - G}$, where $\hat{V}_\alpha$ and V_α are given in Equations (10) and (12), respectively. By using the central limit theorem, the $100(1-\beta)\%$ confidence interval for V_α for both the short- and long-memory cases are obtained in the next theorem.

Theorem 2. Assume that the returns $\{r_t\}$ follow the SV model specified in Equation (2), with $\{u_t\}$ being a sequence of i.i.d. standard normal random variables, and $E\eta_1^8 + Ev_1^4 < \infty$. Suppose the mean μ of the return process r_t is known, and the variance σ^2 of r_t is bounded below by a known positive constant δ^*, such that $0 < \delta^* < \sigma^2$. The confidence intervals for both short-memory and long-memory are expressed as follows:

(i) If $\{Z_t\}$ is long-memory with the memory parameter $0 < d < 1/2$, then

$$\left(-\Phi^{-1}(1-\alpha)\,T^d\right)^{-1} \ln\left(\frac{V_\alpha - G}{\hat{V}_\alpha - G}\right) \xrightarrow{d} N\left(0, \frac{\sigma^2\xi^2}{4}\right), \tag{16}$$

where $N(0, \frac{\sigma^2\xi^2}{4})$ is the asymptotic distribution of $T^{\frac{1}{2}-d}(\hat{\sigma} - \sigma)$. The $100(1-\beta)\%$ confidence interval for V_α or the $100\alpha\%$ quantile of liability distribution is

$$\begin{aligned} G + \left(\hat{V}_\alpha - G\right)\exp\left(-UT^d\,\Phi^{-1}(1-\alpha)\right) &\leq V_\alpha \\ \leq G + \left(\hat{V}_\alpha - G\right)\exp\left(-LT^d\,\Phi^{-1}(1-\alpha)\right)&, \end{aligned} \tag{17}$$

where L and U are the $100(\beta/2)\%$ and the $100(1-\beta/2)\%$ quantile of $N(0, \frac{\sigma^2\xi^2}{4})$, respectively.

(ii) If $\{Z_t\}$ is short-memory, such that the coefficients a_i in Equation (3) are summable as defined in Equation (4), then

$$\left(-\Phi^{-1}(1-\alpha)\right)^{-1} \ln\left(\frac{V_\alpha - G}{\hat{V}_\alpha - G}\right) \xrightarrow{d} N\left(0, \frac{g^2}{4\sigma^2}\right), \tag{18}$$

where g^2 depends on the second moment and autocovariance of $\{v_t^2\}$. Here $N(0, g^2/4\sigma^2)$ is the asymptotic distribution of $T^{\frac{1}{2}}(\hat{\sigma}-\sigma)$. The confidence interval in Equation (17) of part (i) holds for the short-memory case by setting d equal to zero 0 and revising L and U to represent the $100(\frac{\beta}{2})\%$ and the $100(1-\frac{\beta}{2})\%$ quantile of $N(0, g^2/4\sigma^2)$, respectively.

The proof of Theorem 2 appears in Appendix C.

When implementing Equation (17) to construct the confidence intervals, we need to have the convergence rate d (under the long-memory case) and the limiting variances derived in Equations (16) and (18). Note that the explicit form of g^2 is only available when $\{Z_t\}$ is a Gaussian process. Alternatively, we can use a resampling scheme, called the sampling window method, to calculate the limiting variance under different return process. The details of the sampling method are in Appendix D.

4. Numerical Studies

4.1. Analysis of Coverage Accuracy

In this section we conduct two numerical studies. First, we investigate finite-sample performance of the confidence intervals derived in Theorem 2. Second, We use the S&P 500 index as the GMMBs' linked equity to illustrate the LMSV effect on the actual construction of confidence intervals for quantile reserves. We use simulated time series to examine the accuracy of the confidence intervals obtained in equation (17) for both short- and long-memory SV processes. The parameters underlying the SV model in equation (2) are set to be $\mu = 0.00034$ and $\bar{\sigma} = 1$. For the short-memory SV case, we assume that $\{Z_t\}$ follows an AR(1) process defined as $Z_t = \phi_0 + \phi Z_{t-1} + \varepsilon_t'$ and $\varepsilon_t' \overset{i.i.d.}{\sim} N(0, \bar{\beta}^2(1-\phi^2))$. For the long-memory SV case, $\{Z_t\}$ is assumed to be an FARIMA $(0, d, 0)$ process, given by $(1-B)^d Z_t = \eta_t$, where $\{\eta_t\}$ is a Gaussian white noise independent of $\{u_t\}$ with variance $\sigma_\eta^2 = 1$, The parameters used to generate the $\{Z_t\}$ process are estimated from the daily returns of the S&P 500; the estimates are: $\phi_0 = -0.02$, $\phi = 0.996$ and $\bar{\beta} = 0.4$ for the short-memory case, and $d = 0.41$ for the long-memory case. For the equity-linked insurance product GMMBs under consideration, we assume that the single premium is 100, or $P = 100$. The guarantee amount is 100% of the premium paid, $G = 100$. The charge is $m = 2.2$ b.p. per day, or 554.4 b.p. annually.

We first examine the coverage accuracy of the confidence intervals based on realizations that are simulated by (2) for the short- and long-memory SV model separately. Three projection lengths of T (2520, 5040, and 7560) are considered and 1,000 replications are carried out for each of the three different lengths. For each replication, we use Equation (17) to construct confidence intervals for a fixed quantile, and then calculate the coverage probabilities based for 1,000 replications. Here we consider the 95% confidence interval for the 95% quantile reserve.

Table 1 presents the coverage probability for the short-memory framework (i.e., $d = 0$). Using Equation (17) to construct the confidence intervals for quantile reserves, we adopt two approaches. The first is to use the analytic solution derived in section 3.3 and Appendix B. Because $\{Z_t\}$ is assumed to follow a Gaussian AR(1) process, the analytic form for the limiting variance of $\sqrt{T}(\hat{\sigma} - \sigma)$ is available. In fact, L and U are $-1.96\sqrt{g^2/4\sigma^2}$ and $+1.96\sqrt{g^2/4\sigma^2}$, respectively, where g^2 is expressed as in Equation (19). To approximate $\sum_{j=1}^{\infty}(\exp(4\bar{\beta}^2\phi^j) - 1)$ in $\underline{g}^2$, we sum $(4\bar{\beta}^2\phi^j) - 1$ for j from 1 to a sufficiently large k until $\sum_{j=1}^{k}(\exp(4\bar{\beta}^2\phi^j) - 1)$ converges. The second approach is to employ a resampling scheme called sampling window method describe in Appendix D. In Table 1 figures in brackets represent the coverage probability that are calculated by using the sampling method. Among various sets of parameters for the AR(1) process, the coverage probability is at least 0.92 for a 95% confidence interval for the autoregressive coefficient ϕ equals to 0.4 and 0.7. However, when ϕ is set to coincide with the estimated value 0.996 based on the S&P 500 sample, the coverage probability for 95% confidence interval is only 0.7, and the bias gets greater if the projection length becomes longer. This is due to the effect of misspecification that the short-memory SV model can not capture the volatility persistence exhibited by $\{Z_t\}$ which is a nearly integrated non-stationary AR(1) with ϕ close to 1.

TABLE 1

COVERAGE PROBABILITIES OF 95% CONFIDENCE INTERVAL FOR V_α BASED ON SHORT-MEMORY STOCHASTIC VOLATILITY SEQUENCES. THE DATA-GENERATING PROCESS IS $Z_t = \phi_0 + \phi Z_{t-1} + \varepsilon_t'$, WHERE $\varepsilon_t' \overset{iid}{\sim} N(0, \bar{\beta}^2(1-\phi^2))$. THE COVERAGE PROBABILITY OF THE SUBSAMPLING CONFIDENCE INTERVAL IS LISTED IN BRACKETS.

Coverage accuracy of equation (17) with $d = 0$	S&P 500 parameters $\phi_0 = -0.02$, $\bar{\beta} = 0.4$					
	$\phi = 0.4$		$\phi = 0.7$		$\phi = 0.996$	
$T = 2520$	0.94	(0.92)	0.93	(0.92)	0.80	(0.70)
$T = 5040$	0.93	(0.92)	0.93	(0.93)	0.80	(0.69)
$T = 7560$	0.92	(0.92)	0.93	(0.93)	0.80	(0.68)

We further investigate the confidence interval of the quantile reserve for the long-memory framework in Table 2. We begin with analyzing the coverage probability with the long-memory $\{Z_t\}$ being assumed to be an FARIMA $(0, d, 0)$ process. For several different memory parameters d, the 95% confidence interval of the quantile reserves is calculated using Equation (27), which is a realized version of Equation (17) using the sampling window method. The coverage probability performs reasonably well, especially for the long-duration projection. The coverage accuracy improves with smaller d. For example, the coverage probability for a 95% confidence interval is 0.91 with $d = 0.45$ and

TABLE 2

COVERAGE PROBABILITIES OF 95% CONFIDENCE INTERVAL FOR V_α BASED ON LONG-MEMORY STOCHASTIC VOLATILITY SEQUENCES. THE DATA-GENERATING PROCESS IS $Z_t = \sum_{i=0}^{\infty} a_i \eta_{t-i}$, WHERE $a_i \sim C \cdot i^{d-1}$ and $\eta_t \overset{iid}{\sim} N(0,1)$ (*i.e.*, $Z_t \sim$ FARIMA$(0, d, 0)$).

	$d = 0.45$		$d = 0.41$		$d = 0.3$	
	(27)	(17) with $d = 0$	(27)	(17) with $d = 0$	(27)	(17) with $d = 0$
$T = 2520$	0.91	0.13	0.92	0.22	0.94	0.48
$T = 5040$	0.91	0.11	0.93	0.21	0.95	0.45
$T = 7560$	0.92	0.11	0.94	0.18	0.95	0.42

0.94 with $d = 0.3$, based on $T = 2520$; it is 0.92 with $d = 0.45$ and 0.95 with $d = 0.3$, for $T = 7560$. We now turn to examine the risk on evaluating the quantile reserve with misspecified models. To illustrate the effect of model misspecification, we use Equation (17) with $d = 0$ to calculate the confidence interval of the quantile reserves for returns which are actually generated by a LMSV model. As Table 2 shows, ignoring long memory in asset volatility will cause serious undercoverage for the confidence interval. For example, assume Z_t is an FARIMA$(0, 0.45, 0)$ process. For a 95% confidence interval, the model risk causes the coverage probability to be much lower than expected, or 0.13 with a projection length of 2520 and 0.11 with a projection length of 7560. Given the evidence that the volatility of the S&P 500 index is long-memory, if the insurer disregards this fact index and treats it as traditional short memory, the corresponding confidence interval of quantile reserve for the equity-linked products would be far from reliable as a measure to manage the risk of insolvency.

4.2. Quantile Reserve and Confidence Interval for Equity-Linked GMMBs

We consider the GMMB contract specified in section 3.1 with the daily S&P 500 index being the underlying linked asset. The maturity is set for 25 and 30 years. For the former the sample period is from January 1982 to the last trading day of December 2006, and for the latter from January 1977 to the last trading day of December 2006. Following the sampling method described in Appendix D, we calculate the contract's quantile reserves (0.9%, 0.95% and 0.99%) and their confidence intervals (90% and 95%) by using Equations (27) and (17) (with $d = 0$) for the long- and short-memory case, respectively. The results are reported in Tables 3 and 4 for the durations of 25 and 30 years. For both of the two maturities, the long-memory volatility effect is apparent since the confidence interval under the LMSV model is much wider than that for the short-memory SV model; and the scales of changes for the upper limits are considerably greater than those for the lower limits, which is due to the fact that convergence of the central limit theorem given in (15) is slow under the LMSV and the distribution of $T^{1/2-d}(\hat{\sigma}^2 - \sigma^2)$ is skewed. It is worth noting that while the insurer bears more risk for the 25-year policy than for the 30-year

TABLE 3

CONFIDENCE INTERVALS OF QUANTILE RISK MEASURE FOR 25-YEAR EQUITY-LINKED FUND CONTRACT CONNECTED WITH THE S&P 500.

Quantile Risk Measure $\hat{V}_{0.9}$ = 24.58 (Percentage of Fund Value)		
Confidence Interval for $V_{0.9}$	90% C.I.	95% C.I.
Long-memory SV (27)	(12.88, 79.39)	(9.39, 84.15)
(17) with $d = 0$	(19.12, 29.04)	(16.41, 29.5)
Quantile Risk Measure $\hat{V}_{0.95}$ = 44.14 (Percentage of Fund Value)		
Confidence Interval for $V_{0.95}$	90% C.I.	95% C.I.
Long-memory SV (27)	(30.1, 92.4)	(17.69, 96.44)
(17) with $d = 0$	(38.9, 48.35)	(36.26, 48.77)
Quantile Risk Measure $\hat{V}_{0.99}$ = 68.2 (Percentage of Fund Value)		
Confidence Interval for $V_{0.99}$	90% C.I.	95% C.I.
Long-memory SV (27)	(56.33, 98.11)	(44.97, 99.35)
(17) with $d = 0$	(63.89, 71.53)	(61.67, 71.86)

TABLE 4

CONFIDENCE INTERVALS OF QUANTILE RISK MEASURE FOR 30-YEAR EQUITY-LINKED FUND CONTRACT CONNECTED WITH THE S&P 500.

Quantile Risk Measure $\hat{V}_{0.9}$ = 17.13 (Percentage of Fund Value)		
Confidence Interval for $V_{0.9}$	90% C.I.	95% C.I.
Long-memory SV (27)	(11.43, 49.26)	(8.8, 58.83)
(17) with $d = 0$	(10.97, 21.76)	(9, 22.26)
Quantile Risk Measure $\hat{V}_{0.95}$ = 39.68 (Percentage of Fund Value)		
Confidence Interval for $V_{0.95}$	90% C.I.	95% C.I.
Long-memory SV (27)	(34.3, 67.86)	(31.78, 75.42)
(17) with $d = 0$	(33.86, 43.97)	(31.98, 44.42)
Quantile Risk Measure $\hat{V}_{0.99}$ = 66.75 (Percentage of Fund Value)		
Confidence Interval for $V_{0.99}$	90% C.I.	95% C.I.
Long-memory SV (27)	(62.48, 82.35)	(60.43,90.66)
(17) with $d = 0$	(62.13, 70.05)	(60.6, 70.39)

policy as can be seen by comparing their $\hat{V}_\alpha$, the former has longer widths of confidence intervals than the latter. This is due to the fact that the estimates of quantile reserves are more accurate if the duration is longer. The overall message delivered in the two tables is clear that the insurer may severely underestimate the variation of the quantile reserve for the equity-linked insurance with guarantees if the effect of the return's long-memory volatility is not taken into account.

5. Conclusions

As equity-linked insurance products have developed rapidly and vastly in recent years, there is a strong need for actuaries to model equity returns for pricing and valuation. The main risk inherent in the equity-linked insurance products is that the insurer offers investment guarantees to the policyholder. Therefore, evaluating the equity risk associated with setting the reserve and capital is very critical. Such tasks have been regulated by insurance supervision bodies and gained great attention. However, unlike ordinary financial instruments, insurance products tend to be long-duration by nature. To reflect the equity risk, an equity return model should capture the dynamics of the equity process for a long projection period. This requirement creates difficulty in valuing long-duration insurance products rather than short-duration financial instruments. In this research, we introduce the long-memory stochastic volatility model to tackle the problem. The LMSV process is a stationary time series formed by martingale differences that contain a latent volatility process that exhibits long memory. Compared with short-memory models like ARCH or GARCH, the LMSV model reveals that the magnitude of the shocks to the returns dissipate at a much slower rate. Therefore, we examine the existence of LMSV for the long-duration data period using the most recent S&P 500 index data. The empirical study supports the use of the LMSV model for long-duration insurance products, because it can better capture the dynamics of equity returns for long durations than can many other existing models.

To evaluate the long-term risk for the insurer of equity-link insurance products, we derive a closed-form expression for quantile reserves of the underlying equity and propose a natural estimate for the quantile reserves. By assuming the returns to follow a short- or long-memory stochastic volatility process, we prove a central limit theorem by which the confidence intervals for the quantile reserve can be constructed. We also show how to use a resampling scheme, the sampling window method, to implement the construction of the confidence interval. The theory we establish is supported by the simulation studies we conduct. In addition, we use the proposed LMSV model to calculate quantile reserves for a specific type of GMMBs guarantees linked to S&P 500 index. The results show that if the insurer disregards long-memory persistence in the volatility of the S&P 500 index and treats it as traditional short memory, the corresponding confidence intervals of quantile reserve will suffer severely low coverage probability and as a result become unreliable in terms of solvency. This lack of reliability has significant implications in risk management for insurance products offering investment guarantees.

We present a statistical analysis of quantile reserves for a particular type of guarantees GMMBs. The methodology proposed in this research can help insurers study the quantile reserves for other types of guarantees under the SV framework. On the financial front there are some areas and issues worthy of further research. For example, the guarantee underlying the equity-linked contracts can be regarded as a put option. Pricing such embedded options is also important for the insurer to decide the fair value of guarantee cost, which is an interesting research topic and has been widely studied (Brennan and Schwartz, 1976; Boyle and Schwartz, 1977; Boyle and Hardy, 1997; Persson and Aase, 1997; Miltersen and Persson, 1999; Grosen and Jorgensen, 1997, 2000; Hansen and Miltersen, 2002; Schrager and Pelsser, 2004). To price the value of the option, these studies assume the underlying asset follows a geometric Brown Motion process. It would be a challenging problem to study how the presence of long-memory volatility affects valuing the long-duration insurance liability. In addition to reserving, from a risk management perspective, hedging and reinsurance are alternatives to managing equity-linked insurance with guarantee. It would also be important to find suitable hedging strategies for those insurance products to help ensure the financial safety of the insurer.

Appendix A

Lemma. Let $\{r_t\}$ be a SV process (2) with mean μ and variance σ^2. Then

$$\frac{\sum_{t=1}^{T} r_t - T\mu}{\sqrt{T}\sigma} \xrightarrow{d} N(0,1) \quad \text{as } T \to \infty.$$

Proof of Lemma. Under $\mathcal{H}_t$, the sigma field or information generated by $\{U_s : 1 \le s \le t\} \cup \{Z_s : 1 \le s \le \infty\}$ ($\mathcal{H}_0 = \phi$), it is clear that the SV returns $\{r_t - \mu, 1 \le t \le T\}$ forms a sequence of martingale differences, because $E(r_{t+1} - \mu \mid \mathcal{H}_t) = 0$. Moreover, $\{r_t - \mu\}$ satisfies the conditional Linderberg condition, such that for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{t=1}^{T} E\Big(\big[(r_t - \mu) / \sqrt{T}\big]^2 I\Big(\big|(r_t - \mu) / \sqrt{T}\big| > \varepsilon \big|\mathcal{H}_{t-1}\Big)\\
&\quad = T^{-1} \sum_{t=1}^{T} v_t^2 P\Big(\big|r_t - \mu\big| / \sqrt{T}\big| > \varepsilon \big|\mathcal{H}_{t-1}\Big)\\
&\quad \le C^* \cdot T^{-3/2} \sum_{t=1}^{T} v_t^3 \longrightarrow 0
\end{aligned}$$

in probability as $T \to \infty$. In the preceding inequality, C^* is a constant independent of T. By Corollary 3.1 of Hall and Heyde (1980), the following central limit theorem holds:

$$\frac{\sum_{t=1}^{T} r_t - T\mu}{\sqrt{T}\sigma} \xrightarrow{d} N(0,1) \text{ as } T \to \infty.$$

Appendix B

Lemma. Assume that the returns $\{r_t\}$ follow the SV model specified in Equation (2), with $\{u_t\}$ being a sequence of i.i.d. standard normal random variables, and $E\eta_1^8 + Ev_1^4 < \infty$. If $\{Z_t\}$ is the short-memory process and expressed as a Gaussian AR(1) process determined by the dynamic equation of $Z_t = \phi_0 + \phi Z_{t-1} + \varepsilon_t'$ and $\varepsilon_t' \overset{i.i.d.}{\sim} N(0, \bar{\beta}^2(1-\phi^2))$. Then, as in Ho (2006), the estimator $\hat{\sigma}^2$ obeys the central limit theorem, which is $\sqrt{T}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} N(0, g^2)$. We have the analytical form of g^2, which is

$$g^2 = \exp\left(\frac{2\phi_0}{1-\phi} + \bar{\beta}^2\right)\left\{3\exp(\bar{\beta}^2) - 1 + 2\,\frac{\exp(\bar{\beta}^2)-1}{\exp(4\bar{\beta}^2)-1}\sum_{j=1}^{\infty}\left(\exp(4\bar{\beta}^2\phi^j)-1\right)\right\}. \tag{19}$$

Proof of Lemma. Write $\sqrt{T}(\hat{\sigma}^2-\sigma^2) = T^{-\frac{1}{2}}\sum_{t=1}^{T} v_t^2(u_t^2-1) + T^{-\frac{1}{2}}\sum_{t=1}^{T}(v_t^2-\sigma^2)$. Because Z_t is normally distributed with mean $\phi_0/(1-\phi)$ and variance $\bar{\beta}^2$, e_t^Z is distributed as lognormal with variance $[\exp(\bar{\beta}^2)-1]\exp(\frac{2\phi_0}{1-\phi}+\bar{\beta}^2)$. Also, because $\{u_t\}$ is an i.i.d. normal sequence and is stochastically independent of $\{Z_t\}$, we have

$$\begin{aligned}
&Var\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} e^{Z_t}\left(u_t^2-1\right)\right)\\
&= \frac{1}{T}E\left[\sum_{t=1}^{T} e^{2Z_t}\left(u_t^2-1\right)^2\right] + \frac{1}{T}\sum\sum_{s\neq t} E\left[e^{Z_s}\left(u_s^2-1\right)e^{Z_t}\left(u_t^2-1\right)\right]\\
&= E\left[e^{2Z_t}\right]E\left[\left(u_t^2-1\right)^2\right]\\
&= 2\exp\left(\frac{2\phi_0}{1-\phi}+2\bar{\beta}^2\right).
\end{aligned} \tag{20}$$

Let $\rho_v^2(.)$ denote the autocorrelation of $\{v_t^2\}$, whose exact form has been derived by Taylor (1986, pp. 74-75). Applying Taylor's result and the Cesaro mean, the we obtain the limiting variance of $T^{-\frac{1}{2}}\sum_{t=1}^{T}(v_t^2-\sigma^2)$ as follows:

$$\begin{aligned}
&Var\left(T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(v_t^2-\sigma^2\right)\right)\\
&= Var\left(e^{Z_t}\right) + \frac{2}{T}\sum_{k=1}^{T-1}\sum_{j=1}^{k} Var\left(e^{Z_t}\right)\rho_{v^2}(j)\\
&= Var\left(e^{Z_t}\right)\left[1 + \frac{2}{T}\sum_{k=1}^{T-1}\sum_{j=1}^{k}\frac{\exp(4\bar{\beta}^2\phi^j)-1}{\exp(4\bar{\beta}^2)-1}\right]
\end{aligned}$$

$$= Var\left(e^{Z_t}\right)\left[1 + \frac{2}{\exp(4\bar{\beta}^2 - 1)} T^{-1} \sum_{k=1}^{T-1} \sum_{j=1}^{k} \left(\exp(4\bar{\beta}^2\phi^j) - 1\right)\right]$$
$$\to \left[\exp(\bar{\beta}^2) - 1\right]\exp\left(\frac{2\phi_0}{1-\phi} + \bar{\beta}^2\right)\left[1 + \frac{2}{\exp(4\bar{\beta}^2 - 1)} \sum_{j=1}^{\infty} \left(\exp(4\bar{\beta}^2\phi^j) - 1\right)\right] \quad (21)$$

as $T \to \infty$. Then Equation (19) follows by combining Equations (20) and (21).

Appendix C

Proof of Theorems 1 and 2. We begin with proving Theorem 2, and, during the course of the proof, establish Theorem 1 as an interim step. To prove Theorem 2 we show parts (i) and (ii) simultaneously. Define $\sigma_T^* = \left(\sum_{t=1}^T v_t^2/T\right)^{1/2} = \left(\sigma^2 \sum_{t=1}^T e^{Z_t}/T\right)^{1/2}, \mathcal{F}_T$ the sigma field generated by $\{Z_1, Z_2, ..., Z_T\}$, and $A = \left[\ln(G - V_\alpha) - \ln S_0 - T\mu + Tm\right]/\sqrt{T}$. Choose ε such that $0 < \varepsilon \leq \delta^*$, where δ^* is a known positive constant that is less than the variance σ^2 of r_t. Then,

$$\begin{aligned}
1 - \alpha &= P\left(\sum_{t=1}^{T} r_t < \ln(G - V_\alpha) - \ln S_0 + Tm\right) \\
&= E\left[P\left(\frac{\sum_{t=1}^T r_t - T\mu}{\sigma_T^* \sqrt{T}} < \frac{A}{\sigma_T^*} \middle| \mathcal{F}_t\right)\right] \\
&= \Phi\left(\frac{A}{\sigma}\right) + E\left[\Phi\left(\frac{A}{\sigma_T^*}\right) - \Phi\left(\frac{A}{\sigma}\right)\right] \\
&= \Phi\left(\frac{A}{\sigma}\right) + E\left[\left(\Phi\left(\frac{A}{\sigma_T^*}\right) - \Phi\left(\frac{A}{\sigma}\right)\right) I\left(\left|\sigma_T^{*2} - \sigma^2\right| > \varepsilon\right)\right] \\
&\quad + E\left[\left(\Phi\left(\frac{A}{\sigma_T^*}\right) - \Phi\left(\frac{A}{\sigma}\right)\right) I\left(\left|\sigma_T^{*2} - \sigma^2\right| \leq \varepsilon\right)\right] \\
&\equiv \Phi\left(\frac{A}{\sigma}\right) + E_{T,1} + E_{T,2},
\end{aligned}$$

where $I(Q)$ is the indicator function of the set Q. Now we analyze $E_{T,1}$ and $E_{T,2}$ separately. To handle both the short-memory and long-memory cases at the same time, we introduce a variable d^* whose value is zero if Z_t is short-memory or is the memory parameter $d \in (0, 1/2)$ if Z_t is long-memory. Using $|\,\Phi(\frac{A}{\sigma_T^*}) - \Phi\left(\frac{A}{\sigma}\,|\right) \leq 1$ and the Chebyshev inequality, we have

$$
\begin{aligned}
\left|E_{T,1}\right| &\leq E\left[I\left(\left|\sigma_T^{*2}-\sigma^2\right|>\varepsilon\right)\right] \\
&= P\left(\left|\sum_{t=1}^{T}\left(\bar{\sigma}^2 e^{Z_t}-\sigma^2\right)/T\right|>\varepsilon\right) \\
&\leq \varepsilon^{-2}E\left(\sum_{t=1}^{T}\left(\bar{\sigma}^2 e^{Z_t}-\sigma^2\right)/T\right)^2 \\
&= \varepsilon^{-2}T^{-2(\frac{1}{2}-d^*)}\,E\left(\sum_{t=1}^{T}\left(\bar{\sigma}^2 e^{Z_t}-\sigma^2\right)/T^{\frac{1}{2}+d^*}\right)^2 = O(T^{-2(\frac{1}{2}-d^*)}).
\end{aligned} \tag{22}
$$

The upper bound of the last equality in Equation (22) comes from Ho and Hsing (1997). For $E_{T,2}$, by applying mean value theorem to $\Phi(\frac{A}{\sigma_T^*})-\Phi\left(\frac{A}{\sigma}\right)$, we obtain

$$
\begin{aligned}
E_{T,2} &= E\left[A\phi(A^*)\left(\frac{1}{\sigma_T^*}-\frac{1}{\sigma}\right)I\left(\left|\sigma_T^{*2}-\sigma^2\right|\leq\varepsilon\right)\right] \\
&= E\left[A\phi(A^*)\left(\frac{\sigma^2-\sigma_T^{*2}}{\sigma_T^*\sigma(\sigma_T^*+\sigma)}\right)I\left(\left|\sigma_T^{*2}-\sigma^2\right|\leq\varepsilon\right)\right] \\
&= T^{-(\frac{1}{2}-d^*)}A\cdot E\left[\phi(A^*)\frac{-I\left(\left|\sigma_T^{*2}-\sigma^2\right|<\varepsilon\right)}{\sigma_T^*\sigma(\sigma_T^*+\sigma)}\left(\sum_{t=1}^{T}\left(\bar{\sigma}^2 e^{Z_t}-\sigma^2\right)/T^{\frac{1}{2}+d^*}\right)\right] \\
&= o\left(T^{-(\frac{1}{2}-d^*)}\right),
\end{aligned} \tag{23}
$$

where A^* depends on A and σ_T^*, and its value lies between A/σ and A/σ_T^*. Write

$$
W_T = \phi(A^*)\frac{-I\left(\left|\sigma_T^{*2}-\sigma^2\right|<\varepsilon\right)}{\sigma_T^*\sigma(\sigma_T^*+\sigma)}T^{\frac{1}{2}-d^*}\left(\sum_{t=1}^{T}\left(\bar{\sigma}^2 e^{Z_t}-\sigma^2\right)/T\right).
$$

Because σ_T^* converges to σ and A/σ converges to $\Phi^{-1}(1-\alpha)$, it is easy to see that $\phi(A^*)\xrightarrow{P}\phi(\Phi^{-1}(1-\alpha))$ and $I(|\sigma_T^{*2}-\sigma^2|<\varepsilon)/[\sigma_T^*\sigma(\sigma_T^*+\sigma)]\xrightarrow{P}1/(2\sigma^3)$. In addition, by Theorem 3.1 and Corollary 3.3 of Ho and Hsing (1997), $T^{\frac{1}{2}-d^*}(\sum_{t=1}^{T}(\bar{\sigma}^2 e^{Z_t}-\sigma^2)/T)$ is asymptotically normal. Using Slusky's Theorem, we realize $W_T\xrightarrow{d}W'$, where W' is a normal distribution with mean zero. Equation (23) follows if we can show that $E(W_T)\xrightarrow{P}E(W')=0$. To do so, it suffices to prove that W_T is uniformly integrable (Corollary 5 of Billingsley, 1971, p. 9), which is guaranteed if EW_T^2 is bounded over T (see Exercise 8 of Chung, 2001). It is not difficult to see that for some constant C_2 independent of T,

$$EW_T^2 \leq C_2 \cdot E\left(\sum_{t=1}^{T}\left(\bar{\sigma}^2 e^{Z_t} - \sigma^2\right) / T^{\frac{1}{2}+d^*}\right)^2,$$

which implies that EW_T^2 is bounded over T. Hence, Equation (23) holds. Combining Equations (22) and (23) gives

$$1 - \alpha = \Phi\left(\frac{\ln(G - V_\alpha) - \ln S_0 - T\mu + Tm}{\sigma\sqrt{T}}\right) + o\left(T^{-(\frac{1}{2}-d^*)}\right), \tag{24}$$

which is precisely Equation (11) in Section 3.2. Rearranging Equation (24) to $1 - \alpha + o(T^{-(\frac{1}{2}-d^*)}) = \Phi(\frac{\ln(G - V_\alpha) - \ln S_0 - T_\mu + Tm}{\sigma\sqrt{T}})$ and taking the inverse of $\Phi(\cdot)$ on both sides of the equation, we achieve

$$\Phi^{-1}\left(1 - \alpha + o\left(T^{-(\frac{1}{2}-d^*)}\right)\right) = \frac{\ln(G - V_\alpha) - \ln S_0 - T\mu + Tm}{\sigma\sqrt{T}}.$$

Then Equation (12),

$$V_\alpha = G - S_0 \exp\left(\sigma\sqrt{T}\,\Phi^{-1}\left(1 - \alpha - o\left(T^{-(\frac{1}{2}-d^*)}\right)\right) + T\mu - Tm\right),$$

follows. Dividing $V_\alpha - G$ by $\hat{V}_\alpha - G$ yields

$$\frac{V_\alpha - G}{\hat{V}_\alpha - G} = \exp\left(\sigma\sqrt{T}\,\Phi^{-1}\left(1 - \alpha + o(T^{-(\frac{1}{2}-d^*)})\right) - \hat{\sigma}\sqrt{T}\,\Phi^{-1}(1-\alpha)\right). \tag{25}$$

Taking the logarithm of both side of Equation (25) and applying the mean value theorem to $\Phi^{-1}(1 - \alpha + o(T^{-(\frac{1}{2}-d^*)})) - \Phi^{-1}(1-\alpha)$, we have

$$\begin{aligned}\ln\left(\frac{V_\alpha - G}{\hat{V}_\alpha - G}\right) &= -\sqrt{T}(\hat{\sigma} - \sigma)\Phi^{-1}(1-\alpha) \\ &\quad + \sigma\sqrt{T}\left[\Phi^{-1}\left(1 - \alpha + o(T^{-(\frac{1}{2}-d^*)})\right) - \Phi^{-1}(1-\alpha)\right] \\ &= -\sqrt{T}(\hat{\sigma} - \sigma)\Phi^{-1}(1-\alpha) + \sigma o(T^{d^*})(\Phi^{-1})'(\alpha^*),\end{aligned}$$

where a^* satisfies $|\alpha^* - (1-\alpha)| < \left|o(T^{-(\frac{1}{2}-d^*)})\right|$. Consequently,

$$\frac{\ln\left(\frac{V_\alpha - G}{\hat{V}_\alpha - G}\right)}{-\Phi^{-1}(1-\alpha)T^{d^*}} = T^{\frac{1}{2}-d}(\hat{\sigma} - \sigma) + o(1),$$

which is asymptotically normal according to Ho (2006). Then the $100(1-\beta)\%$ confidence interval for V_α follows immediately.

Appendix D

Implementation of Confidence Interval of Quantile Reserves

To use Equation (17) to construct confidence intervals, we need to have the limiting variance derived in Equations (16) and (18), and the convergence rate d if the volatility is long-memory. Instead of estimating them directly, we use a resampling scheme, called the sampling window method, to achieve this purpose. The core of the sampling window method is to approximate the sampling distribution of a statistic by recomputing it with subsamples of the observed data. The statistic we are focusing on is simply the sample standard deviation of the returns, because the limiting distributions given in Theorem 2 (see Equations (16) and (18)) are generated by normalized $T^H(\hat{\sigma} - \sigma)$ with $H = 1/2$ or $1/2 - d$.

We first deal with the short-memory case that corresponds to part (ii) of Theorem 2. Let b denote the block length of the subsample, which is subject to $1 \leq b \leq T$. The ith subsample of block length b can be expressed as $B_i = (r_i, \ldots, r_{i+b-1})$, where $1 \leq i \leq N_b = T - b + 1$. Let τ_T be an known constant that represents the convergence rate of $\hat{\sigma}$ to σ, that is, $\tau_T = \sqrt{T}$ in the short-memory case. Define $\hat{\sigma}_{T,b,i} = [\sum_{t=i}^{i+b-1}(r_t - \mu)^2/b]^{\frac{1}{2}}$, the estimator of σ based on the subsample B_i. The distribution then can be defined as

$$L_{T,b}(x) = \frac{1}{T-b+1}\sum_{i=1}^{T-b+1} I\left\{\tau_b\left(\hat{\sigma}_{T,b,i} - \hat{\sigma}\right) \leq x\right\},$$

where τ_b is a normalized constant with sample size b. From Politis and Romano (1994), we know that under the mild assumptions, $L_{T,b}(x)$ converges in probability to the distribution function of $\sqrt{T}(\hat{\sigma} - \sigma)$, meaning that the specific quantile of $L_{T,b}(x)$ serves as a consistent estimate of the same quantile of the distribution function of $\sqrt{T}(\hat{\sigma} - \sigma)$.

Because $b = O(T^k)$ with $k \leq 1/3$ is usually optimal for weakly dependent data (Künsch, 1989; Hall et al., 1995; Hall and Jing, 1996), we use the block length of $b = 3T^{1/3}$ to investigate the coverage accuracy of Equation (17) for the short-memory framework (i.e. $d = 0$) in Section 4.2. In this case, L and U are the 2.5% and the 97.5% quantile of $L_{T,b}(x)$, respectively.

To carry out the sampling window method for the LMSV process, we require extra efforts, because the convergence rate $T^{1/2-d}$ in Equation (15) is no longer $\sqrt{T}$, and the memory parameter d is unknown. From Equations (15) and (16) we have

$$\left(-\Phi^{-1}(1-\alpha)\sqrt{d_T^2}\right)^{-1}\sqrt{T}\ln\left(\frac{V_\alpha - G}{\hat{V}_\alpha - G}\right) = \frac{T(\hat{\sigma} - \sigma)}{\sqrt{d_T^2}} + o(1) \xrightarrow{d} N\left(0, (4\sigma^2)^{-1}\right), \tag{26}$$

where d_T^2 captures the underlying dependence structure as well as the limiting variance $\sigma^2\xi^2/4$ so that $d_T = O(T^{d+\frac{1}{2}})$. To use the SW method, we first need to estimate d_T^2. Toward this end, we follow the setup employed in Hall et al. (1998) and replace d_T^2 with a data-based estimate $\hat{d}_T^2$, which involves two subsampling variance estimates. Following the same procedure as in the short-memory case, we use

$$L_{T,b}(x) = \frac{1}{T-b+1} \sum_{i=1}^{T-b+1} I\left\{b\left(\hat{\sigma}_{T,b,i} - \hat{\sigma}\right) / \hat{d}_b^{(i)} \le x\right\},$$

where $\hat{d}_b^{(i)}$ denotes the version of $\hat{d}_T$ for the ith block B_i, to approximate the distribution function of $T(\hat{\sigma} - \sigma)/\sqrt{\hat{d}_T^2}$. Here $\hat{t}_{\beta/2}$ represents the $100(\beta/2)\%$ quantile of subsampling estimator $L_{T,b}(x)$ taken as an estimate of the same quantile of $T(\hat{\sigma} - \sigma)/\sqrt{d_T^2}$. Then, the two-sided $100(1-\beta)\%$ subsampling confidence interval of V_α is

$$\begin{aligned} G + (\hat{V}_\alpha - G)\exp(-\hat{t}_{1-\beta/2}\, T^{-\frac{1}{2}}\sqrt{\hat{d}_T^2}\,\Phi^{-1}(1-\alpha)) &\le V_\alpha \\ \le G + (\hat{V}_\alpha - G)\exp(-\hat{t}_{\beta/2}\, T^{-\frac{1}{2}}\sqrt{\hat{d}_T^2}\,\Phi^{-1}(1-\alpha)), & \end{aligned} \tag{27}$$

which merely expresses the same idea of Equation (17) in different terms. In Section 4.2, we use this equation to calculate the confidence intervals of the quantile reserve for the long-memory case. We use $b = T^{1/2}$, similar to Hall et al. (1998). The size $T^{1/2}$ is intuitive for calculating the 95% confidence interval of quantile reserves, because subsamples from long-memory series generally should be longer than those for weakly dependent data, for which $b = O(T^k)$, and $k \le 1/3$ is usually optimal.

Appendix E
ARCH-type Process

E.1. The GARCH Model

Following Engle (1982), Bollerslev (1986) proposes a useful extension known as the generalized ARCH (GARCH) model. For a return series $\{r_t\}$, let $r_t = \mu + x_t$, where $x_t = v_t u_t$, $\{u_t\}$ is a sequence of i.i.d. random variables with mean 0 and variance 1, and $\{v_t^2\}$ is the variance of $\{x_t\}$ given information at time $t-1$. A GARCH (p, q) specification is given by

$$v_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i x_{t-i}^2 + \sum_{j=1}^{q} \varphi_j v_{t-j}^2,$$

with $\alpha_0 > 0$, $\alpha_i \ge 0$, $\varphi_j \ge 0$, and $\sum_{i=1}^{\max(p,q)}(\alpha_i + \varphi_i) < 1$. The constraint on $\alpha_i + \varphi_i$ implies that the unconditional variance of $\{x_t\}$ is finite, whereas its conditional variance $\{v_t^2\}$ evolves over time. A GARCH model can be regarded as an application of the ARMA idea to the squared series $\{x_t^2\}$, with $\{x_t^2\}$ being driven by the noise $\zeta_t = x_t^2 - v_t^2$. From the ARMA representation, it is clear that the autocorrelation function for $\{x_t^2\}$ has a short-memory geometric decay.

E.2. The IGARCH Model

The IGARCH models are unit-root GARCH models. An IGARCH (1,1) model can be written as

$$x_t = v_t u_t,\ v_t^2 = \alpha_0 + \varphi_1 v_{t-1}^2 + (1-\varphi_1) x_{t-1}^2,$$

where $\{u_t\}$ is defined as we have previously, and $0 < \varphi_1 < 1$. The unconditional variance of $\{x_t\}$, and thus that of $\{r_t\}$, is not defined for this IGARCH (1,1) model. A key feature of IGARCH models is that the impact of past shocks on volatility is permanent.

E.3. The EGARCH Model

As an alternative to the GARCH specification, Nelson (1991) proposes the exponential GARCH (EGARCH) model,

$$x_t = v_t u_t,\ \ln v_t^2 = \psi_0 + \sum_{i=1}^{\infty} \psi_i g\left(\tilde{\zeta}_{t-i}\right),\ \psi_1 = 1, \tag{28}$$

where $\{\psi_i\}_{i=1,2,\ldots,\infty}$ are real, nonstochastic, scalar sequences. To allow for asymmetric effects between positive and negative asset returns, one choice is to make $g(\tilde{\zeta}_t)$ a linear combination of $\tilde{\zeta}_t$ and $\left|\tilde{\zeta}_t\right|$:

$$g(\tilde{\zeta}_t) = \theta^* \tilde{\zeta}_t + \gamma^* \left[\left|\tilde{\zeta}_t\right| - E\left(\left|\tilde{\zeta}_t\right|\right)\right],$$

where θ^* and γ^* are real constants. Both $(\tilde{\zeta}_t)$ and $\left\{\left|\tilde{\zeta}_t\right| - E\left(\left|\tilde{\zeta}_t\right|\right)\right\}$ are zero mean i.i.d. sequences with continuous distributions. In addition to the infinite moving average representation in Equation (28), $\ln v_t^2$ can be expressed in an ARMA form. Due to the similarity between the autocovariance functions of $\{\ln x_t^2\}$ and $\{\ln v_t^2\}$, the EGARCH model produces series with short-memory volatilities.

Appendix F
Fractional ARIMA models

The ARIMA models introduced by Box and Jenkins (1970) became very popular because of their simplicity and flexibility. Let us first recall the definition of the ARMA and ARIMA processes. The process $\{Z_t, t = 0, \pm 1, \pm 2, \ldots\}$ is an ARMA (p, q) process if $\{Z_t\}$ is stationary and if for every t,

$$\phi(B)Z_t = \theta(B)\eta_t,\ t = 0,\ \pm 1,\ \pm 2, ..., \tag{29}$$

where $\phi(\cdot)$ and $\theta(\cdot)$ are the p_{th} and q_{th} degree polynomials

$$\phi(B) = 1 - \phi_1 B - ... - \phi_p B^p$$

and

$$\theta(B) = 1 - \theta_1 B - ... - \theta_q B^q$$

and B is the backward shift operator defined by

$$B^i Z_t = Z_{t-i},\quad t = 0,\ \pm 1,\ \pm 2, ...$$

Let $\gamma(j)$ denote the autocovariance function of a ARMA (p, q) process. An ARMA process $\{Z_t\}$ is often referred to a short-memory process because the autocovariance function is geometrically bounded,

$$|\gamma(j)| \leq Cs^{-j},\quad j = 1, 2, ...,$$

where $C > 0$ and $0 < s < 1$. The autocovariance function is summable and decreases rapidly as $j \to \infty$. If instead Equation (29) holds for the dth difference $(1 - B)^d Z_t$, then Z_t is called an ARIMA (p, d, q) process. The corresponding equation is

$$\phi(B)(1 - B)^d Z_t = \theta(B)\eta_t,\ t = 0,\ \pm 1,\ \pm 2, ... \tag{30}$$

The fractional autoregressive integrated moving average (FARIMA (p, d, q)) process is the extension of the class of ARIMA process for non-integer values of d, whose autocovariance functions have the asymptotic behavior (1). It is formulated as the unique stationary solution of the difference Equation (30) for $0 < d < 1/2$ (Adenstedt, 1974; Granger and Joyeux, 1980; Hosking, 1981). When $p = q = 0$, an FARIMA (0, d, 0) is usually called a fractional noise. When $d = 0$, Equation (30) produces nothing but the standard ARMA(p, q) process. If Z_t is causal and invertible, then, for $d \neq 0$, Z_t admits the following infinite-order MA representation

$$Z_t = \frac{\theta(B)}{\phi(B)} \nabla^{-d} \varepsilon_t = \sum_{i=0}^{\infty} \psi_j \varepsilon_{t-j}$$

with

$$\psi_j \sim C_1 j^{d-1},\ C_1 > 0.$$

The spectral density $f(\cdot)$ of Z_t is, for $d \neq 0$,

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left|\frac{\theta(e^{i\lambda})}{\phi(e^- i\lambda)}\right|^2 \left|1 - e^{-i\lambda}\right|^{-2d} \sim \frac{\sigma^2}{2\pi}\left[\theta(1)/\phi(1)\right]^2 \lambda^{-2d}$$

as $\lambda \to 0$. Let $\gamma^*(.)$ and $\rho^*(.)$ denote the autocovariance and autocorrelation function of fractional noise, respectively. Then,

$$\gamma^*(0) = \sigma_\eta^2 \frac{\Gamma(1-2d)}{\Gamma^2(1-d)} \equiv \sigma_Z^2 \tag{31}$$

and

$$\gamma^*(j) = \sigma_\eta^2 \frac{\Gamma(j+d)\,\Gamma(1-2d)}{\Gamma(j-d+1)\,\Gamma(d)\,\Gamma(1-d)} \sim c_1 j^{2d-1} \text{ as } j \to \infty, \tag{32}$$

where

$$c_1 = \sigma_\eta^2 \frac{\Gamma(1-2d)}{\Gamma(d)\,\Gamma(1-d)} \equiv \frac{\sigma_Z^2\,\Gamma(1-d)}{\Gamma(d)},$$

and

$$\rho^*(j) = \frac{\Gamma(j+d)\,\Gamma(1-d)}{\Gamma(j-d+1)\,\Gamma(d)} = \prod_{0<k\leq j} \frac{k-1+d}{k-d} \sim \frac{\Gamma(1-d)}{\Gamma(d)}\, j^{2d-1} \text{ as } j \to \infty. \tag{33}$$

(e.g., Brockwell and Davis, 1991).

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