PREDICTIVE DISTRIBUTIONS FOR RESERVES WHICH SEPARATE TRUE IBNR AND IBNER CLAIMS

BY

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ABSTRACT

This paper considers the model suggested by Schnieper (1991), which separates the true IBNR claims from the IBNER. Stochastic models are defined, using both recursive and non-recursive procedures, within the framework of the models described in England and Verrall (2002). Approximations to the prediction error of the reserves are derived analytically.

Some extensions to the original Schnieper model are also discussed, together with other possible applications of this type of model.

KEYWORDS

Bornhuetter-Ferguson; chain ladder; claims reserving; predictive distribution.

1. INTRODUCTION

Schnieper (1991) proposed a method of claims reserving, which explicitly separated the incurred data into new claims amounts and changes in incurred amounts for existing claims. The method proposed by Schnieper has not received much attention since then, although Mack (1993) used some of the ideas, indirectly, to derive a method to calculate estimates of the prediction error for the chain ladder technique. As far as we are aware, there has been no further work in the literature following up the specific modelling structure proposed by Schnieper (1991). We believe that the ideas in Schnieper (1991) should be considered again, and so we present in this paper a number of additions to the work of Schnieper, taking into account the advances in stochastic claims reserves models which have taken place since the paper was published. In particular, we derive recursive formulae for the estimates of the prediction errors. We also make some proposals for extending the Schnieper models, and discuss ideas for extending the application of this type of approach to other types of data.

A useful summary of stochastic claims reserving models can be found in England and Verrall (2002). That paper, and the references given in it, supply

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a background to the development of the stochastic framework used in this paper. Other useful references include Taylor (2000) and Wuthrich and Merz (2008) It is assumed that the data have been aggregated by development year and accident year (noting that the methods can be extended straightforwardly to quarterly or monthly data). England and Verrall (2002) considers both recursive and non-recursive models for claims run-off triangles, and both approaches will be used in this paper.

The paper is set out as follows. Section 2 contains a brief description of the model proposed in Schnieper (1991), including the estimates of the parameters and of outstanding claims. Section 3 shows how to derive the estimates of the prediction errors analytically. In Section 4, the prediction errors are estimated for the data used in Schnieper (1991). Finally, in Section 5, a discussion is given of the model proposed by Schnieper, suggesting some possible extensions to the model. We also examine in this section whether a similar approach could usefully be applied to other data sets. Thus, as well as investigating stochastic models for the Schnieper method, another purpose of this paper is to show how the basic ideas of stochastic claims reserving can now be extended in order to provide further useful practical approaches to reserving.

2. The Schnieper Model

This section describes the model suggested by Schnieper (1991), and also gives details of the estimates of the parameters and of outstanding claims derived by Schnieper. The method was specifically designed with reinsurance data in mind, but it is possible that it could be useful for other types of data as well.

2.1. The data

The Schnieper model deals with very volatile incurred data for reinsurance business, and the idea is to separate the data into two more detailed, separate parts so that they become more stable and easy to model. These two parts are the new claims that arise at each development period, and the change in the incurred amounts for claims that arose at previous development periods. Clearly, whether or not this is possible depends on the information available, and the original application in Schnieper was to reinsurance data. While recognising that there will be many cases where data is not available at this level of detail, we also believe that the approach is useful when the data are available. It is also our opinion that the general approach may also be adapted to other situations when other types of data are available, also split into two parts.

Without loss of generality, we assume that the data are available in triangular form, indexed by accident year, i, and development year, j. The cumulative incurred data are denoted by $\{X_{ij} : 1 \leq i \leq n;\ 1 \leq j \leq n-i+1\}$:

$$
\begin{array}{llll}
X_{11} & X_{12} & \cdots & X_{1n} \\
X_{21} & \cdots & X_{2,n-1} & \\
\vdots & & & \\
X_{n1} & & &
\end{array}
$$

It is assumed that the incremental incurred claims ($X_{ij} - X_{i,j-1}$) are the sum of incremental incurred from the old claims ($-D_{ij}$) and the new claims (N_{ij}). In other words, $-D_{ij}$ represents the change in the cumulative incurred claims for claims reported in previous development periods, and $\mathrm{N_{ij}}$ is the new claims reported in development period j. Thus,

$$X_{ij} - X_{i,j-1} = -D_{ij} + N_{ij} \tag{2.1}$$

and for cumulative claims:

$$X_{ij} = X_{i,j-1} - D_{ij} + N_{ij}. \tag{2.2}$$

As was discussed in Verrall (2000) and England and Verrall (2002), the stochastic models for claims reserving can be formulated either for incremental or cumulative data, with no difference in the results. It is therefore a matter of convenience which is used, and there are advantages to each in different circumstances. For example, when deriving expressions for the estimation error, it is usually easier to use the cumulative claims.

Schnieper also assumes that a measure of the exposure, E_i, is also available for each accident year i, and it will be seen that this leads to estimation that has some similarities with the Bornhuetter-Ferguson method (Bornhuetter and Ferguson, 1972). Also, it is noted that introducing external information can be especially useful for unstable data, such as newly developed data with little or no history. We continue with this assumption, in order to be consistent with Schnieper (1991), but also discuss how the approach may be adapted when exposure data are not available.

In common with Schnieper (1991), we do not attempt to forecast beyond development year n. We refer to cumulative claims at development year n as "Ultimate Claims".

We define the information up to payment year k by H_k and the information up to development year k by F_k, where

$$
\begin{aligned}
& H_k = \{N_{ij}, D_{ij} : 1 \le i, j \le n;\ i+j-1 \le k\} \\
\text{and} \quad & F_k = \{N_{ij}, D_{ij} : 1 \le i, j \le n;\ j \le k\}.
\end{aligned}
$$

F_k corresponds to B_k in Mack (1993).

2.2. The model assumptions

The general model assumptions concern the independence between accident years, the uncorrelatedness between development years and the mean and variance of the incremental incurred claims amounts from old claims and from new claims. These are given as follows:

Assumption 1: There exist constants λ_j and δ_j, such that for known exposure E_i we have that,

$$E[N_{ij} \mid H_{i+j-2}] = E_i \lambda_j, \quad 1 \leq i, j \leq n, \tag{2.3}$$

$$E[D_{ij} \mid H_{i+j-2}] = X_{i,j-1} \delta_j, \quad 1 \leq i \leq n, 2 \leq j \leq n. \tag{2.4}$$

According to Assumption 1, the model structure for the incremental incurred claims from new claims, N_{ij}, is non-recursive in format, with column parameters, λ_j, which have to be estimated from the data and row parameters, E_i, which are assumed known. Therefore, it also can be seen to be similar to the Bornhuetter-Ferguson method with the row parameters assumed to be known *a priori*. A discussion of the Bornhuetter-Ferguson method in a Bayesian context can be found in Verrall (2004).

The model structure for the incremental incurred claims amounts from existing claims has some similarities with the chain ladder model, where δ_j is similar to a development factor. The difference here is that $X_{i,j-1}$ is taken from a different triangle, rather than being the previous cumulative claims. For this reason, Mack (1993) describes this method as "a mixture of the Bornhuetter-Ferguson technique and the chain ladder method".

Assumption 2: There exist constants σ_j^2 and τ_j^2, such that

$$Var[N_{ij} \mid H_{i+j-2}] = E_i \sigma_j^2, \quad 1 \leq i, j \leq n, \tag{2.5}$$

$$Var[D_{ij} \mid H_{i+j-2}] = X_{i,j-1} \tau_j^2, \quad 1 \leq i \leq n, 2 \leq j \leq n. \tag{2.6}$$

Assumption 2 defines only the variances of the random variables, but not the full distribution. This approach was also used by Mack (1993) in his stochastic model for the chain ladder technique, and has been discussed in a series of recent papers (Buchwalder et al, 2006, Mack et al, 2006, Venter, 2006, and Gisler, 2006). Buchwalder et al (2006) use a time series approach, and raise some difficulties concerning the estimation of the estimation error in the chain ladder model. Mack et al (2006) discuss this further, showing that the time series approach also presents some difficulties. In these papers, approximations to the estimates of the estimation error are discussed in some detail. This point was first discussed by England and Verrall (2002), in paragraph 7.6.2, which empirically compared two possible approximations to the estimates of the estimation error, pointing out that they were very similar but not exactly identical. Mack et al (2006) has clarified this theoretically, and we discuss the alternatives in section 3. See also sections 3.2.3 and 3.3.1 of Wuthrich and Merz (2008).

Assumption 3: Independence between accident years

As in Schnieper (1991), it is assumed that $\{N_{1j}, D_{1j} : 1 \leq j \leq n\} \ldots \{N_{nj}, D_{nj} : 1 \leq j \leq n\}$, are independent between accident years.

Assumption 4: Uncorrelatedness between development years

$\{N_{ij} \mid H_{i+j-2} : 1 \leq i, j \leq n\}$ and $\{D_{ij} \mid H_{i+j-2} : 1 \leq i \leq n, 2 \leq j \leq n\}$ are uncorrelated.

2.3. Unbiased estimates of the parameters

Following the assumptions in section 2.2, the (conditionally) unbiased estimates of the parameters were provided by Schnieper as follows.

$$\hat{\lambda}_j = \frac{\sum_{i=1}^{n+1-j} N_{ij}}{\sum_{i=1}^{n+1-j} E_i}, \quad 1 \leq j \leq n, \tag{2.7}$$

and

$$\hat{\delta}_j = \frac{\sum_{i=1}^{n+1-j} D_{ij}}{\sum_{i=1}^{n+1-j} X_{i,j-1}}, \quad 2 \leq j \leq n. \tag{2.8}$$

Also

$$\hat{\sigma}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} \frac{1}{E_i} \left(N_{ij} - \hat{\lambda}_j E_i\right)^2, \quad 1 \leq j \leq n-1, \tag{2.9}$$

and

$$\hat{\tau}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} \frac{1}{X_{i,j-1}} \left(D_{ij} - \hat{\delta}_j X_{i,j-1}\right)^2, \quad 2 \leq j \leq n-1. \tag{2.10}$$

Following Schnieper (1991), we assume that claims are developed enough to use the approximation that $\sigma_n^2 = 0$ and $\tau_n^2 = 0$. Note that these are approximations rather than estimators.

Schnieper (1991) also provides estimators for the conditional variances of $\hat{\lambda}_j$ and $\hat{\delta}_j$. These can be obtained from the following identities by using estimates of σ_j^2 and τ_j^2

$$Var\left(\hat{\lambda}_j \,\middle|\, F_{j-1}\right) = \frac{\sigma_j^2}{\sum_{i=1}^{n-j+1} E_i}, \quad 1 \le j \le n,$$

and

$$Var\left(\hat{\delta}_j \,\middle|\, F_{j-1}\right) = \frac{\tau_j^2}{\sum_{i=1}^{n-j+1} X_{i,j-1}}, \quad 2 \le j \le n.$$

2.4. Ultimate claims

The main purpose of claims reserving is to predict outstanding claims. In order to do this, we require an estimate of ultimate claims for each accident year, given the observed data. We are also interested in the prediction of claim development at different development period prior to the ultimate.

For both of the purposes, it is clear that we require the "t-steps-ahead" forecast of cumulative claims, in the terminology of time series. A recursive forecasting formula is given in equation (2.11), where n is the number of rows in the triangles, and k is the latest diagonal ($k = n - i + 1$).

$$E\left[X_{i,k+t} \,\middle|\, H_n\right] = (1 - \delta_{k+t}) E\left[X_{i,k+t-1} \,\middle|\, H_n\right] + E_i \lambda_{k+t}. \tag{2.11}$$

PROOF:

$$\begin{aligned} E\left[X_{i,k+t} \,\middle|\, H_n\right] &= E\left[E\left(X_{i,k+t} \,\middle|\, H_{i+k+t-2}\right) \,\middle|\, H_n\right] \\ &= E\left[E\left(X_{i,k+t-1} - D_{i,k+t} + N_{i,k+t} \,\middle|\, H_{i+k+t-2}\right) \,\middle|\, H_n\right] \\ &= E\left[\left(X_{i,k+t-1} - X_{i,k+t-1}\,\delta_{k+t} + E_i \lambda_{k+t} \,\middle|\, H_n\right)\right] \\ &= (1 - \delta_{k+t}) E\left[X_{i,k+t-1} \,\middle|\, H_n\right] + E_i \lambda_{k+t}. \end{aligned}$$

Thus, for row 2 we require the 1-step-ahead forecast, for row 3 we require the 2-steps-ahead forecast, and so on. The number of steps ahead from the latest observed time (the leading diagonal) to ultimate claims only depends on which accident year we are considering.

Using this recursive derivation method, a formula for the expected ultimate claim for accident year i is obtained as follows, which is the same as Schnieper (1991):

$$E[X_{in} | H_n] = X_{i,n+1-i}(1-\delta_{n+2-i})\dots(1-\delta_n)$$
$$+ E_i[\lambda_{n+2-i}(1-\delta_{n+3-i})\dots(1-\delta_n) + \lambda_{n+3-i}(1-\delta_{n+4-i})\dots(1-\delta_n) + \dots + \lambda_n]$$
$$= X_{i,n+1-i} \prod_{j=n+2-i}^{n} (1-\delta_j) + E_i\left[\sum_{j=n+2-i}^{n-1} \left(\lambda_j \prod_{l=j+1}^{n}(1-\delta_l)\right) + \lambda_n\right]. \tag{2.12}$$

Since these expressions contain unknown parameters, it is necessary to use estimators:

$$\hat{E}[X_{i,k+t} | H_n] = (1-\hat{\delta}_{k+t})\,\hat{E}[X_{i,k+t-1} | H_n] + E_i\,\hat{\lambda}_{k+t}.$$

$\hat{E}[X_{i,k+t} | H_n]$ is the prediction of $X_{i,k+t}$, and we use the notation of $\hat{X}_{i,k+t}$ for this: $\hat{E}[X_{i,k+t} | H_n] = \hat{X}_{i,k+t}$.

Hence

$$\hat{X}_{i,k+t} = (1-\hat{\delta}_{k+t})\,\hat{X}_{i,k+t-1} + E_i\,\hat{\lambda}_{k+t}. \tag{2.13}$$

Note that it is straightforward to show that this is a conditionally unbiased estimator, given $X_{i,n+i-1}$.

3. Prediction errors

One of the principle reasons for using stochastic models is so that prediction errors and predictive distributions can be estimated. In the other words, a range around the best estimate can be provided, which is essential for solvency requirements, and for capital modelling and risk measurement. In this section, we show how prediction errors may be estimated using two approaches, analytically.

When using recursive models, the prediction errors for the reserves are the same as the prediction errors for the ultimate claims, since we always condition on the latest cumulative claims. Thus, we may consider whichever is easier to deal with. In most cases, this means using ultimate claims, especially when the models are set up in the form of recursive models.

In general, we require the conditional Mean Squared Error of Prediction (MSEP). Consider $n-i+1 < m \leq n$, so that X_{im} is a future observation in the lower triangle.

$$MSEP[\hat{X}_{im} | H_n] = E[(X_{im} - \hat{X}_{im})^2 | H_n]$$
$$= [((X_{im} - E[X_{im} | H_n]) - (\hat{X}_{im} - E[X_{im} | H_n))^2 | H_n]$$
$$= E[(X_{im} - E[X_{im} | H_n])^2 | H_n] + (\hat{X}_{im} - E[X_{im} | H_n])^2$$
$$= \textit{process variance} + \textit{estimation error}.$$

The process variance is straightforward due to the assumptions of independence between accident years and uncorrelatedness between development years. However, the estimation error has been discussed in detail by Buchwalder et al (2006), Mack et al (2006), Venter (2006) and Gisler (2006). In brief, there are essentially two approaches which can be used, of which Mack et al (2006) argue that one is better than the other. In addition, Gisler (2006), using a Bayesian approach, makes a case for using the other approach. See also Gisler and Wuthrich (2008) for a further discussion of this issue in a Bayesian context. For completeness, we include both approaches in this paper.

The first method, which was used by England and Verrall (2002), approximates the estimation error by the estimation variance, $Var[\hat{X}_{im} \mid F_{n-i+1}]$. This is equivalent to Approach 3 of Buchwalder et al (2006). Mack et al (2006) use a slightly different approach, and section 6 of Mack et al (2006) discusses the difference between the two approaches and conjectures that they should give similar results (as was also concluded in England and Verrall, 2002).

In this paper, we derive alternative approximations for the estimation error for the Schnieper model. These are derived in sections 3.2 and 3.3, but first we consider the other component of the MSEP, the process variance.

3.1. Process variance for accident year (row) *i*

The process variance is derived recursively, which makes it particularly straightforward to implement in a spreadsheet. The t-steps-ahead formula for the expectation and variance of ultimate claims may be obtained recursively as follows.

Recall that $k = n - i + 1$, denoting the latest observations in the leading diagonal. The process variance of the t-step-ahead forecast in the lower triangle can be recursively written as follows,

$$Var[X_{i,k+t} \mid H_n] = (1-\delta_{k+t})^2 Var[X_{i,k+t-1} \mid H_n] + \tau_{k+t}^2 E[X_{i,k+t-1} \mid H_n] + E_i \sigma_{k+t}^2. \tag{3.1}$$

The proof of (3.1) is given in Appendix A.

Since this expression contains unknown parameters, it is necessary to use estimators:

$$\hat{V}ar[X_{i,k+t} \mid H_n] = (1-\hat{\delta}_{k+t})^2 \hat{V}ar[X_{i,k+t-1} \mid H_n] + \hat{\tau}_{k+t}^2 \hat{E}[X_{i,k+t-1} \mid H_n] + E_i \hat{\sigma}_{k+t}^2. \tag{3.2}$$

A non-recursive formula can also be derived for the process variance:

$$\begin{aligned} Var[X_{in} \mid H_n] = {} & X_{i,n-i-1} \sum_{v=n-i+2}^{n} \left[\tau_v^2 \prod_{j=n-i+2}^{v-1} \left(1-\delta_j\right) \prod_{j=v+1}^{n} \left(1-\delta_j\right)^2 \right] \\ & + E_i \sum_{r=n-i+2}^{n} \left[\sigma_r^2 \prod_{j=r+1}^{n} \left(1-\delta_j\right)^2 + \lambda_r \sum_{v=r+1}^{n} \left(\tau_v^2 \prod_{j=r+1}^{v-1} \left(1-\delta_j\right) \prod_{j=v+1}^{n} \left(1-\delta_j\right)^2 \right) \right]. \end{aligned}$$

[Note that here and elsewhere in this paper we use the convention that $\prod_{j=a}^{b} x_j = 1$ if $a > b$, in order to keep the equations as concise as possible.]

The proof of this expression is also given in Appendix A. Again, the estimator of this can be obtained by using estimators of the parameters on the right hand side.

Note that the second approach provides the same process variance of the ultimate loss prediction as the first approach. However, the first approach has the advantage that it gives the process variance of loss prediction in every single development period.

3.2. Estimation error for accident year (row) *i*

We consider the estimation error in a recursive format. Again, it is simpler to consider ultimate claims rather than the reserve. From section 2, the recursive formula for the t-steps-ahead forecast is

$$\hat{X}_{i,k+t} = (1 - \hat{\delta}_{k+t})\,\hat{X}_{i,k+t-1} + E_i\,\hat{\lambda}_{k+t}. \tag{3.3}$$

We consider 2 approaches for approximating the estimation error. Firstly, we approximate the estimation error using the estimation variance, as was done by England and Verrall (2002). The variance of the t-steps-ahead forecast can be obtained recursively using the following formula:

$$\begin{aligned} &Var[\hat{X}_{i,k+t} \mid F_k] \\ &= \left(E[(1-\hat{\delta}_{k+t} \mid F_k])^2\, Var[\hat{X}_{i,k+t-1} \mid F_k] + Var[\hat{\delta}_{k+t} \mid F_k]\left(E[\hat{X}_{i,k+t-1} \mid F_k]\right)^2 \\ &+ Var[\hat{\delta}_{k+t} \mid F_k]\,Var[\hat{X}_{i,k+t-1} \mid F_k] + Cov[(1-\hat{\delta}_{k+t})^2, \hat{X}^2_{i,k+t-1} \mid F_k] + E_i^2\,Var[\hat{\lambda}_{k+t} \mid F_k]. \end{aligned} \tag{3.4}$$

The derivation of this expression can be found in Appendix B. It is not straightforward to derive an expression for $Cov[(1-\hat{\delta}_{k+t})^2, \hat{X}^2_{i,k+t-1} \mid F_k]$, and approximations to (3.4) are used here. The first approximation follows England and Verrall (2002) and simply omits this covariance (this is also the approximation used in Approach 3 of Buchwalder et al, 2006):

$$\begin{aligned} &Var[\hat{X}_{i,k+t} \mid F_k] \\ &\approx \left(E[(1-\hat{\delta}_{k+t} \mid F_k])^2\, Var[\hat{X}_{i,k+t-1} \mid F_k] + Var[\hat{\delta}_{k+t} \mid F_k]\left(E[\hat{X}_{i,k+t-1} \mid F_k]\right)^2 \\ &\qquad + Var[\hat{\delta}_{k+t} \mid F_k]\,Var[\hat{X}_{i,k+t-1} \mid F_k] + E_i^2\,Var[\hat{\lambda}_{k+t} \mid F_k]. \end{aligned}$$

Note that an estimate of this can be obtained by using suitable estimates on the right hand side. Since the estimators are unbiased, $E[\hat{X}_{i,k+t-1} \mid F_k] = E[X_{i,k+t-1} \mid F_k]$, and we use $\hat{E}[X_{i,k+t-1} \mid F_k]$ as the estimate of this. Similarly, we estimate $E[1-\hat{\delta}_{k+t} \mid F_k]$ by $(1-\hat{\delta}_{k+t})$.

Denote $\hat{Var}[\hat{X}_{i,k+t} \mid F_k]$ as the estimate for $Var[\hat{X}_{i,k+t} \mid F_k]$, then the estimate for the estimation variance is as follows:

$$\hat{Var}[\hat{X}_{i,k+t} \mid F_k] = (1-\hat{\delta}_{k+t})^2 \hat{Var}[\hat{X}_{i,k+t-1} \mid F_k] + \hat{Var}(\hat{\delta}_{k+t} \mid F_k) \hat{Var}[\hat{X}_{i,k+t-1} \mid F_k] + \hat{Var}(\hat{\delta}_{k+t} \mid F_k) \hat{X}^2_{i,k+t-1} + E_i^2 \hat{Var}(\hat{\lambda}_{k+t} \mid F_k). \tag{3.5}$$

As suggested in Mack et al (2006), section 6, it is possible also to adjust equation (3.4), by noting that often $Var(\hat{\delta}_{k+t} \mid F_k)\, Var[\hat{X}_{i,k+t-1} \mid F_k] \approx -Cov(\hat{\delta}^2_{k+t}, \hat{X}^2_{i,k+t-1} \mid F_k]$. Thus, the omission of $Cov(\hat{\delta}^2_{k+t}, \hat{X}^2_{i,k+t-1} \mid F_k]$ can be approximately compensated for by omitting $Var(\hat{\delta}_{k+t} \mid F_k)\, Var[\hat{X}_{i,k+t-1} \mid F_k]$ as well. This would mean using an approximation to (3.5):

$$\hat{Var}[\hat{X}_{i,k+t} \mid F_k] \approx (1-\hat{\delta}_{k+t})^2 \hat{Var}[\hat{X}_{i,k+t-1} \mid F_k] + \hat{Var}(\hat{\delta}_{k+t} \mid F_k) \hat{X}^2_{i,k+t-1} + E_i^2 \hat{Var}(\hat{\lambda}_{k+t} \mid F_k). \tag{3.5a}$$

Note that $\hat{Var}(\hat{\delta}_{k+t} \mid F_k)$ and $\hat{Var}(\hat{\lambda}_{k+t} \mid F_k)$ can be obtained using the results in section 2.3.

In section 4, where the results are compared, we refer to (3.5) as "L and V Original" and (3.5a) as "L and V with Adjustment".

A different approach follows Mack et al (2006) and approximates the estimation error directly, rather than the estimation variance. The resulting approximation for the estimation is derived in Appendix B. The estimator of this approximation is:

$$\begin{aligned} & X^2_{i,n-1+i} \prod_{j=n-i+2}^{n} \left(1-\hat{\delta}_j\right)^2 \sum_{v=n-i+2}^{n} \frac{Var\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \\ & + E_i^2 \sum_{r=n-i+2}^{n} \left\{ \hat{\lambda}_r^2 \prod_{j=r+1}^{n} \left(1-\hat{\delta}_j\right)^2 \left[\sum_{v=r+1}^{n} \left(\frac{Var\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right) + \frac{Var\left(\hat{\lambda}_r \mid F_{r-1}\right)}{\hat{\lambda}_r^2} \right] \right\} \\ & + 2X_{n-i+1} E_i \sum_{r=n-i+2}^{n} \left(\hat{\lambda}_r \prod_{j=n-i+2}^{r-1} \left(1-\hat{\delta}_j\right) \prod_{j=r+1}^{n} \left(1-\hat{\delta}_j\right)^2 \sum_{v=r+1}^{n} \frac{Var\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right). \end{aligned} \tag{3.6}$$

Although expression (3.6) appears complicated, it is not too difficult to implement in a spreadsheet.

The results from each of these approaches are compared in section 4.

3.3. MSEP for the Total Loss Reserve

The final step is to obtain an approximation to the conditional Mean Squared Error of Prediction (MSEP) for the overall total reserve.

As in section 3.1, the approximation to the conditional MSEP is obtained as the sum of process variance and the estimation error. Under the independence assumptions between accident years, the process variance of the overall reserve is simply the sum of the process variances of the individual row reserves. However, it becomes more complicated for the estimation error of the overall reserve. It is straightforward to see that the predictions of the row reserves are not independent since they are estimated by the same column parameters. Again, there are two possible approaches here. The first considers the estimation variance. In this case, we need to consider the covariance between row reserves, and a recursive procedure is again used in this context. Consider the covariance of the reserve estimates in column $m+t$ for rows s and p, where $m = n - s + 1$ and $2 \leq s < p \leq n$, $Cov[\hat{X}_{s,m+t}, \hat{X}_{p,m+t} | F_m]$.

$$
\begin{aligned}
& Cov[\hat{X}_{s,m+t}, \hat{X}_{p,m+t} | F_m] \\
&= Var(\hat{\delta}_{m+t} | F_m)\, Cov[\hat{X}_{s,m+t-1}, \hat{X}_{p,m+t-1} | F_m] \\
&\quad + E[\hat{X}_{s,m+t-1} | F_m]\, E[\hat{X}_{p,m+t-1} | F_m]\, Var(\hat{\delta}_{m+t} | F_m) \\
&\quad + Cov[\hat{X}_{s,m+t-1}, \hat{X}_{p,m+t-1} | F_m]\left(E(1-\hat{\delta}_{m+t} | F_m)\right)^2 \\
&\quad + Cov[\hat{X}_{s,m+t}\, \hat{X}_{p,m+t}, (1-\hat{\delta}_{m+t+1})^2 | F_m] + E_s E_p Var(\hat{\lambda}_{m+t} | F_m).
\end{aligned} \tag{3.7}
$$

The proof of (3.7) is given in Appendix B.

We denote the estimate of $Cov[\hat{X}_{s,m+t}, \hat{X}_{p,m+t} | F_m]$ by $\hat{C}ov[\hat{X}_{s,m+t}, \hat{X}_{p,m+t} | F_m]$. Also, we use $\hat{X}_{s,m+t}$ and $\hat{X}_{p,m+t}$ as the estimates for $E[\hat{X}_{s,m+t} | F_m]$ and $E[\hat{X}_{p,m+t} | F_m]$, respectively, and $(1-\hat{\delta}_j)$ as the estimate for $E(1-\hat{\delta}_j | F_m)$. Thus, the estimate of the covariance is as follows:

$$
\begin{aligned}
& \hat{C}ov[\hat{X}_{s,m+t}, \hat{X}_{p,m+t} | F_m] \\
&= \hat{V}ar(\hat{\delta}_{m+t} | F_m)\, \hat{C}ov[\hat{X}_{s,m+t-1}, \hat{X}_{p,m+t-1} | F_m] \\
&\quad + \hat{X}_{s,m+t-1}\, \hat{X}_{p,m+t-1}\, \hat{V}ar(\hat{\delta}_{m+t} | F_m) \\
&\quad + \hat{C}ov[\hat{X}_{s,m+t-1}, \hat{X}_{p,m+t-1} | F_m]\,(1-\hat{\delta}_{m+t})^2 \\
&\quad + \hat{C}ov[\hat{X}_{s,m+t}\, \hat{X}_{p,m+t}, (1-\hat{\delta}_{m+t+1})^2 | F_m] + E_s E_p \hat{V}ar(\hat{\lambda}_{m+t} | F_m).
\end{aligned} \tag{3.8}
$$

Note that a similar argument as was used following equation (3.4) can be applied here for the treatment of $\hat{C}ov[\hat{X}_{s,m+t}\, \hat{X}_{p,m+t}, (1-\hat{\delta}_{m+t+1})^2 | F_m]$, and we follow the same choices as were described earlier.

For ultimate claims, we put $m + t = n$, and derive the required covariances using this recursive formula. For row 2 we use the iterative formula once, and in general for row s, we use formula (3.7) $(s-1)$ times. This enables us to derive the covariances between the estimates of any of the row totals, as required when calculating the estimation variance for the overall total. Thus, we can go on and derive the conditional mean square error of prediction for the overall reserves.

The estimator of the conditional mean square error of prediction is approximated as the sum of process variance and the estimation variance:

$$\hat{MSEP}\left[\sum_{i=1}^{n} X_{in} \mid H_n\right] = \hat{Var}\left[\sum_{i=1}^{n} X_{in} \mid H_n\right] + \hat{Var}\left[\sum_{i=1}^{n} \hat{X}_{in} \mid H_n\right]$$
$$= \sum_{i=1}^{n} \hat{Var}\left[X_{in} \mid H_n\right] + \sum_{i=1}^{n} \hat{Var}\left[\hat{X}_{in} \mid H_n\right] + 2\sum_{s=1}^{n-1} \sum_{p=s+1}^{n} \hat{Cov}\left[\hat{X}_{sn}, \hat{X}_{pn} \middle| H_n\right].$$

This is derived recursively, using equations (3.3), (3.5) or (3.5a), and (3.7). This is straightforward to implement in a spreadsheet, which is available on request from the corresponding author.

The alternative approach is to follow the approximation used by Mack et al (2006). This results in a complicated formula (A1), which is derived in Appendix D. Again, it is possible to implement this formula in a spreadsheet, in spite of its complex appearance.

The following section contains an example which illustrates the use of the results derived in this section. The results for the two approaches are compared.

4. Example

In this section we use the data from Schnieper (1991) and calculate the prediction errors using both approximations derived in section 3. The data used by Schnieper consisted of an IBNR triangle, X_{ij}, and exposure, E_i, which are shown in Table 1. Tables 2 and 3 show the more detailed data, consisting of the new claims, N_{ij}, and the changes in the existing claims, $-D_{ij}$. These data were taken from a practical motor third party liability excess-of-loss pricing problem.

Tables 4 and 5 show the parameters estimates for each development year for the models. These are obtained by using equations (2.7) and (2.8), as derived in Schnieper (1991).

Tables 6 and 7 show the estimates of the variance parameters, obtaining using equations (2.9) and (2.10).

The reserve estimates, obtained using equation (2.12), are shown in Table 8.

In Table 9, we show the approximations to the estimation error using the different methods derived in sections 3.2 and 3.3. The methods are as follows. "L and V Original" refers to the first approach, which approximates the

TABLE 1

CUMULATIVE IBNR (X_{ij}) AND EXPOSURE (E_i) FOR BOTH NEW AND EXISTING CLAIMS.

Accident year \ Dev year	1	2	3	4	5	6	7	Exposure
1	7.5	28.9	52.6	84.5	80.1	76.9	79.5	10,224
2	1.6	14.8	32.1	39.6	55.0	60.0		12,752
3	13.8	42.4	36.3	53.3	96.5			14,875
4	2.9	14.0	32.5	46.9				17,365
5	2.9	9.8	52.7					19,410
6	1.9	29.4						17,617
7	19.1							18,129

TABLE 2

INCREMENTAL INCURRED CLAIMS FROM NEW CLAIMS (N_{ij}).

Accident year \ Dev year	1	2	3	4	5	6	7
1	7.5	18.3	28.5	23.4	18.6	0.7	5.1
2	1.6	12.6	18.2	16.1	14	10.6	
3	13.8	22.7	4	12.4	12.1		
4	2.9	9.7	16.4	11.6			
5	2.9	6.9	37.1				
6	1.9	27.5					
7	19.1						

TABLE 3

INCREMENTAL INCURRED CLAIMS FROM EXISTING CLAIMS (D_{ij}).

Accident year \ Dev year	1	2	3	4	5	6
1	–3.1	4.8	–8.5	23	3.9	2.5
2	–0.6	0.9	8.6	–1.4	5.6	
3	–5.9	10.1	–4.6	–31.1		
4	–1.4	–2.1	–2.8			
5	0	–5.8				
6	0					

TABLE 4

Estimates of the parameters in the N triangle.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
0.0005	0.0011	0.0014	0.0012	0.0012	0.0005	0.0005

TABLE 5

Estimates of the parameters in the D triangle.

δ_2	δ_3	δ_4	δ_5	δ_6	δ_7
–0.3595	0.0719	–0.0476	–0.0536	0.0703	0.0325

TABLE 6

Estimates of the variance parameters for N triangle.

σ_1^2	σ_2^2	σ_3^2	σ_4^2	σ_5^2	σ_6^2
0.002895	0.005433	0.011851	0.006314	0.003131	0.003302

TABLE 7

Estimates of the variance parameters for D triangle.

τ_2^2	τ_3^2	τ_4^2	τ_5^2	τ_6^2
0.150082	1.609408	1.38487	11.97382	0.092046

TABLE 8

Schnieper Reserves.

Accident Year	**2**	**3**	**4**	**5**	**6**	**7**	**Overall Total**
Reserve	4.410	4.796	32.914	60.303	77.188	104.326	283.938

estimation error using the estimation variance. This was the approach used by England and Verrall (2002) for the chain ladder technique. "L and V with Adjustment" follows the same approach as "L and V Original", but omits the products of the variances as suggested by Mack et al (2006) – see the end of

TABLE 9

A COMPARISON OF THE ESTIMATES OF THE ESTIMATION ERROR.

Method \ Accident Year	2	3	4	5	6	7	Overall Total
L & V Original	7.057	10.172	16.626	24.325	24.299	28.493	100.396
L & V with adjustment	7.057	10.172	16.623	24.242	24.137	28.282	100.276
Mack's Approximation	7.059	10.176	16.624	24.244	24.139	28.284	96.267

TABLE 10

A COMPARISON OF THE PREDICTION ERROR.

Method \ Accident Year	2	3	4	5	6	7	Overall Total
L & V Original	9.475	14.297	29.814	41.199	43.507	49.202	121.859
L & V with adjustment	9.475	14.297	29.812	41.150	43.417	49.080	121.761
Mack's Approximation	9.477	14.300	29.813	41.151	43.418	49.081	118.481

section 3.2 for details. Finally, "Mack's approximation" gives the results using the second approach described in sections 3.2 and 3.3. This was the approach used by Mack (1993) to approximate the estimation error for the chain ladder technique.

In a similar way, Table 10 shows the different approximations for the prediction error, again using the 2 approaches derived in section 3.

It can be seen that there is a reasonably good agreement between the three sets of results. For the Schnieper method, there does not appear to be an ordering of the prediction errors, as was found for the chain ladder technique by Mack et al (2006). For the chain ladder technique, it was found that Mack's approximation gave consistently lower values for the prediction error. We surmise that the same ordering does not apply in this case since Schnieper's method is rather more complicated and uses 2 triangles of data.

5. DISCUSSION

This paper has extended the analysis of the Schnieper model to include prediction errors using analytical methods. We believe that the model deserves to be reconsidered in the context of its original, using the new stochastic framework. One limitation of the application of the Schnieper model is that it requires a relatively detailed data set. i.e. the exposure of every accident year, the time when the claims occur and how they develop in every calendar year. For this reason, the model can not be used in every application. The model was also originally used in a specific context, and it is likely that this is a further reason why it has not been considered any further since it was published. However, the ideas from Schnieper (1991) of modelling two sets of data have some similarities with other slightly different problems, for example, the consideration of paid and incurred run-off triangles.

We believe that this paper may pave the way for new approaches to paid and incurred data, which may use the results derived here for the predictive distributions and the prediction errors. For example, we could suggest a straightforward extension to paid and incurred data, which may have some practical appeal. This is to use the model for incremental incurred amounts from new claims, N_{ij}, for incremental paid data for all development years apart from the first. In fact, the claims amount from first development years can always be ignored as they represent, in effect, non-random elements of the model. Similarly, incremental case reserves can be fitted by the model for the incremental decreases in incurred amount from existing claims, D_{ij}, proposed by Schnieper (1991). The underlying requirement of the consistent ultimate losses projection between paid and incurred can be reflected by the exposure assumption.

Further, a prior distribution could be applied to model parameters, such as λ_j and δ_j, and a Bayesian approach adopted. This would have the advantages that the flexibilities of the stochastic models are improved by introducing expert opinion: the prediction errors for the ultimate losses parameters could also be calculated, even when the model parameters are intuitively adjusted by experts under certain circumstances. For example, an appropriate prior distribution for the development factor parameters could be used in order to reflect significant changes observed in the data, which may be caused by changes in the management of claims.

As discussed before, the Schnieper model is a mixture of a chain ladder model and the Bornhuetter-Ferguson method. Another possible application of the Schnieper model is to change the Bornhuetter-Ferguson model for the losses from new claims to a chain ladder model type, so that we can drop the exposure requirement. This could be done by replacing (2.3) and (2.5) by

$$E\left[N_{ij} \middle| H_{i+j-2}\right] = N_{i,j-1}\lambda_j, \;\; 1 \le i \le n, \, 2 \le j \le n,$$

and

$$Var\left[N_{ij} \middle| H_{i+j-2}\right] = N_{i,j-1}\sigma_j^2, \;\; 1 \le i \le n, \, 2 \le j \le n.$$

Notice that this model is exactly the same as the chain ladder stochastic model.

In summary, we suggest that the Schnieper model, in the stochastic framework given in this paper, deserves further consideration, now that these types of stochastic model are gaining practical acceptance. We would also suggest that this provides a useful avenue for further research in models that have useful practical applications.

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Appendix A. Process Variance

Proof of (3.1)

From the model assumptions, it is easy to get the formulae for *one*-step-ahead:

$$Var[X_{i,k+1} \mid H_{i+k-1}] = Var[X_{ik} - D_{i,k+1} + N_{i,k+1} \mid H_{i+k-1}] = \tau_{k+1}^2 X_{ik} + E_i \sigma_{k+1}^2.$$

This shows that the recursive formulae are (trivially) correct for $t = 1$, using the fact that the mean and variance of $X_{ik} \mid X_{ik}$ are X_{ik} and 0, respectively.

We assume that the above formula is true for t and show that it is correct for $t + 1$. Double expectation theory is used in order to condition on the information up to last development year. In this way, the mean and variance assumptions can be used in the process variance derivation.

$$\begin{aligned}
Var[X_{i,k+t+1} \mid H_n] &= Var[E(X_{i,k+t+1} \mid H_{i+k+t-1}) \mid H_n] + E[Var(X_{i,k+t+1} \mid H_{i+k+t-1}) \mid H_n] \\
&= Var[E(X_{i,k+t} - D_{i,k+t+1} + N_{i,k+t+1} \mid H_{i+k+t-1}) \mid H_n] \\
&\qquad + E[Var(X_{i,k+t} - D_{i,k+t+1} + N_{i,k+t+1} \mid H_{i+k+t-1}) \mid H_n] \\
&= Var\big[(1-\delta_{k+t+1}) X_{i,k+t} + E_i \lambda_{k+t+1} \mid H_n\big] \\
&\qquad + E\big[\tau_{k+t+1}^2 X_{i,k+t} + E_i \sigma_{k+t+1}^2 \mid H_n\big] \\
&= (1-\delta_{k+t+1})^2 Var[X_{i,k+t} \mid H_n] + \tau_{k+t+1}^2 E[X_{i,k+t} \mid H_n] + E_i \sigma_{k+t+1}^2
\end{aligned}$$

which concludes the proof.

Proof of the non-recursive expressions for the process variances for both the individual row total reserve and the overall total reserve:

For the individual row total reserve,

$$\begin{aligned}
Var[X_{in} \mid H_n] &= (1-\delta_n)^2 (1-\delta_{n-1})^2 Var[X_{i,n-2} \mid H_n] + (1-\delta_n)^2 \tau_{n-1}^2 E[X_{i,n-2} \mid H_n] \\
&\qquad + (1-\delta_n)^2 E_i \sigma_{n-1}^2 + \tau_n^2 E[X_{i,n-1} \mid H_n] + E_i \sigma_n^2 \\
&= (1-\delta_n)^2 (1-\delta_{n-1})^2 (1-\delta_{n-2})^2 Var[X_{i,n-3} \mid H_n] \\
&\qquad + (1-\delta_n)^2 (1-\delta_{n-1})^2 \tau_n^2 E[X_{i,n-3} \mid H_n] \\
&\qquad + (1-\delta_n)^2 (1-\delta_{n-1})^2 E_i \sigma_{n-2}^2 + (1-\delta_n)^2 \tau_{n-1}^2 E[X_{i,n-2} \mid H_n] \\
&\qquad + (1-\delta_n)^2 E_i \sigma_{n-1}^2 + \tau_n^2 E[X_{i,n-1} \mid H_n] + E_i \sigma_n^2 \\
&= \dots \\
&= X_{i,n-i+1} \sum_{v=n-i+2}^{n} (1-\delta_{n-i+2}) \dots (1-\delta_{v-1}) \tau_v^2 (1-\delta_{v+1})^2 \dots (1-\delta_n)^2
\end{aligned}$$

$$+ E_i \sum_{r=n-i+2}^{n} \left(\begin{aligned} &\sigma_r^2 (1-\delta_{r+1})^2 \dots (1-\delta_n)^2 \\ &+ \sum_{v=r+1}^{n} \left(\lambda_r (1-\delta_{r+1}) \dots (1-\delta_{v-1}) \tau_v^2 (1-\delta_{v+1})^2 \dots (1-\delta_n)^2 \right) \end{aligned} \right).$$

$$= X_{i,n-i+1} \sum_{v=n-i+2}^{n} \left[\tau_v^2 \prod_{j=n-i+2}^{v-1} (1-\delta_j) \prod_{j=v+1}^{n} (1-\delta_j)^2 \right]$$

$$+ E_i \sum_{r=n-i+2}^{n} \left[\sigma_r^2 \prod_{j=r+1}^{n} (1-\delta_j)^2 + \lambda_r \sum_{v=r+1}^{n} \left(\tau_v^2 \prod_{j=r+1}^{v-1} (1-\delta_j) \prod_{j=v+1}^{n} (1-\delta_j)^2 \right) \right].$$

For the overall total, the squared prediction error is simply the sum of the squared prediction error for individual row totals.

Appendix B

Proof of (3.4)

$$\begin{aligned}
&Var[\hat{X}_{i,k+t} | F_k] \\
&= Var[(1-\hat{\delta}_{k+t})\hat{X}_{i,k+t-1} + \hat{\lambda}_{k+t} E_i | F_k] \\
&= Var[(1-\hat{\delta}_{k+t})\hat{X}_{i,k+t-1} | F_k] + Var[\hat{\lambda}_{k+t} E_i | F_k] \\
&= \left(E[(1-\hat{\delta}_{k+t}) | F_k]\right)^2 Var[\hat{X}_{i,k+t-1} | F_k] + Var[\hat{\delta}_{k+t} | F_k] \left(E[\hat{X}_{i,k+t-1} | F_k]\right)^2 \\
&\quad + Var[\hat{\delta}_{k+t} | F_k] Var[\hat{X}_{i,k+t-1} | F_k] + Cov[(1-\hat{\delta}_{k+t})^2, \hat{X}_{i,k+t-1}^2 | F_k] \\
&\quad - 2E[(1-\hat{\delta}_{k+t}) | F_k] Cov[(1-\hat{\delta}_{k+t}), \hat{X}_{i,k+t-1} | F_k] E[\hat{X}_{i,k+t-1} | F_k] \\
&\quad - \left(Cov[(1-\hat{\delta}_{k+t}), \hat{X}_{i,k+t-1} | F_k]\right)^2 + E_i^2 Var[\hat{\lambda}_{k+t} | F_k]
\end{aligned}$$

where $Cov[(1-\hat{\delta}_{k+t}), \hat{X}_{i,k+t-1} | F_k] = 0$ since $\hat{\delta}_{k+t}$ and $\hat{X}_{i,k+t-1}$ are uncorrelated.

Hence

$$\begin{aligned}
&Var[\hat{X}_{i,k+t} | F_k] \\
&= \left(E[(1-\hat{\delta}_{k+t}) | F_k]\right)^2 Var[\hat{X}_{i,k+t-1} | F_k] + Var[\hat{\delta}_{k+t} | F_k] \left(E[\hat{X}_{i,k+t-1} | F_k]\right)^2 \\
&\quad + Var[\hat{\delta}_{k+t} | F_k] Var[\hat{X}_{i,k+t-1} | F_k] + Cov[(1-\hat{\delta}_{k+t})^2, \hat{X}_{i,k+t-1}^2 | F_k] + E_i^2 Var[\hat{\lambda}_{k+t} | F_k]
\end{aligned}$$

which completes the proof.

Note that this proof also uses the fact that $\hat{\lambda}_j$ and $\hat{\delta}_j$ are uncorrelated, which is easy to see from model assumption 3.

Proof of (3.7)

Covariance for a row total

This proof is also in a recursive format. We start from the *one*-step-ahead estimation covariance, i.e. $t = 1$.

$$
\begin{aligned}
&Cov\left[\hat{X}_{s,m+1}, \hat{X}_{p,m+1} \,|\, F_m\right] \\
&\quad = Cov\left[X_{sm}(1-\hat{\delta}_{s+m+1}) + E_s\hat{\lambda}_{m+1}, \hat{E}(X_{pm}\,|\,F_m)(1-\hat{\delta}_{s+m+1}) + E_p\hat{\lambda}_{m+1} \,|\, F_m\right] \\
&\quad = Var(\hat{\delta}_{s+m+1}\,|\,F_m)\,X_{sm}\,\hat{E}(X_{pm}\,|\,F_m) + E_s E_p Var(\hat{\lambda}_{m+1}\,|\,F_m).
\end{aligned}
$$

Again, this shows that the recursive formula (3.7) is (trivially) correct for the one-step-ahead case.

In the same way as for the proof of the recursive formulae for the process error, we assume that the above formula is true for t and show that it is correct for $t + 1$.

We now consider the covariance for $t + 1$, using this *one*-step-ahead prediction formula,

$$
\hat{X}_{ij} = (1-\hat{\delta}_j)\,\hat{X}_{i,j-1} + E_i\hat{\lambda}_j.
$$

$$
\begin{aligned}
&Cov\left[\hat{X}_{s,m+t+1}, \hat{X}_{p,m+t+1} \,|\, F_m\right] \\
&= Cov\left[\hat{X}_{s,m+t}(1-\hat{\delta}_{m+t+1}) + E_s\hat{\lambda}_{m+t+1}, \hat{X}_{p,m+t}(1-\hat{\delta}_{m+t+1}) + E_p\hat{\lambda}_{m+t+1} \,|\, F_m\right] \\
&= Cov\left[\hat{X}_{s,m+t}(1-\hat{\delta}_{m+t+1}), \hat{X}_{p,m+t}(1-\hat{\delta}_{m+t+1}) \,|\, F_m\right] + E_s E_p Var(\hat{\lambda}_{m+t+1}\,|\,F_m) \\
&= Var(\hat{\delta}_{m+t+1}\,|\,F_m)\, Cov\left[\hat{X}_{s,m+t}, \hat{X}_{p,m+t} \,|\, F_m\right] \\
&\quad + E\left[\hat{X}_{s,m+t}\,|\,F_m\right] E\left[\hat{X}_{p,m+t}\,|\,F_m\right] Var(\hat{\delta}_{m+t+1}\,|\,F_m) \\
&+ Cov\left[\hat{X}_{s,m+t}, \hat{X}_{p,m+t}\,|\,F_m\right]\left(E(1-\hat{\delta}_{m+t+1})\right)^2 + Cov\left[\hat{X}_{s,m+t}, \hat{X}_{p,m+t}, (1-\hat{\delta}_{m+t+1})^2 \,|\, F_m\right] \\
&+ E_s E_p Var(\hat{\lambda}_{m+t+1}\,|\,F_m).
\end{aligned}
$$

which completes the proof.

Appendix C

Proof of (3.6), the estimation variances for the individual row total reserve using Mack's approximation approach.

We use the same approach as used in the proof of Theorem 3 on page 218, Mack (1993). Mack considers expressions similar to those in this proof and we write these expressions in such a way that appropriate approximations may be derived.

Let $M_{uv} = \prod_{l=u}^{v-1} \left(1 - \hat{\delta}_l\right)\left(\delta_v - \hat{\delta}_v\right) \prod_{l=v+1}^{n} (1 - \delta_l)$

and $J_r = \begin{cases} \left(\lambda_r - \hat{\lambda}_r\right) \prod_{l=r+1}^{n} (1 - \delta_l) & r = 1, 2, ..., n-1 \\ \left(\lambda_n - \hat{\lambda}_n\right) & r = n. \end{cases}$

Then, from (2.12), $\left(E[X_{in} | H_n] - \hat{X}_{in}\right)^2$ can be written as

$$\left\{X_{i,n-1+1} \sum_{v=n-i+2}^{n} M_{n-i+2,v} + E_i\left[\sum_{r=n-i+2}^{n} \left(\sum_{v=r+1}^{n} \hat{\lambda}_r M_{r+1,v} + J_r\right)\right]\right\}^2.$$

Let $P_i = \sum_{v=n-i+2}^{n} M_{n-i+2,v}$ and $G_r = \sum_{v=r+1}^{n} \hat{\lambda}_r M_{r+1,v} + J_r$. Then

$$\left(E[X_{in} | H_n] - \hat{X}_{in}\right)^2 = \left\{X_{i,n-i+1} P_i + E_i \sum_{r=n-i+2}^{n} G_r\right\}^2$$

$$= X_{i,n-i+1}^2 \underbrace{P_i^2}_{(1)} + E_i^2\left[\underbrace{\sum_{r=n-i+2}^{n} G_r^2}_{(2)} + 2\underbrace{\sum_{r_1 > r_2} G_{r_1} G_{r_2}}_{(3)}\right] + 2X_{i,n-i+1} \underbrace{E_i P_i\left[\sum_{r=n-i+2}^{n} G_r\right]}_{(4)}.$$

We consider each of the terms labelled (1) to (4) in turn.

Considering first $P_i^2 = \sum_{v=n-i+2}^{n} M_{n-i+2,v}^2 + 2\sum_{v_1 < v_2} M_{n-i+2,v_1} M_{n-i+2,v_2}$. We estimate this using

$$\sum_{v=n-i+2}^{n} E\left(M_{n-i+2,v}^2 \,\middle|\, F_{v-1}\right) + 2\sum_{v_1 < v_2} E\left(M_{n-i+2,v_1} M_{n-i+2,v_2} \,\middle|\, F_{v_2-1}\right)$$

Note that $\sum_{v_1 < v_2} E\left(M_{n-i+2,v_1} M_{n-i+2,v_2} | F_{v_2} - 1\right) = 0$ and $E[(\delta_v - \hat{\delta}_v)^2 | F_{v-1}] = Var(\hat{\delta}_v | F_{v-1})$, since $E[\hat{\delta}_v | F_{v-1}] = \delta_v$.

Hence we estimate P_i^2 by $\hat{P}_i^2$, where

$$\hat{P}_i^2 = \sum_{v=n-i+2}^{n} \left(1 - \hat{\delta}_{n-i+2}\right)^2 ... \left(1 - \hat{\delta}_{v-1}\right)^2 \hat{V}ar\left(\hat{\delta}_v \,\middle|\, F_{v-1}\right)\left(1 - \hat{\delta}_{v+1}\right)^2 ... \left(1 - \hat{\delta}_n\right)^2.$$

This can be rewritten as follows, using the same logic as Mack (1993), page 219:

$$\hat{P}_i^2 = \left(1 - \hat{\delta}_{n-i+2}\right)^2 \ldots \left(1 - \hat{\delta}_n\right)^2 \sum_{v=n-i+2}^{n} \frac{\hat{Var}\left(\hat{\delta}_v \,\middle|\, F_{v-1}\right)}{\left(1 - \hat{\delta}_v\right)^2}.$$

Next consider (2):

$$G_r^2 = \left(\sum_{v=r+1}^{n} \hat{\lambda}_r M_{r+1,v} + J_r\right)^2 = \sum_{v=r+1}^{n} \left(\hat{\lambda}_r M_{r+1,v}\right)^2$$
$$+ 2\sum_{v_1<v_2} \left(\hat{\lambda}_r M_{r+1,v_1}\right)\left(\hat{\lambda}_r M_{r+1,v_2}\right) + J_r^2 + 2J_r \sum_{v=r+1}^{n} \hat{\lambda}_r M_{r+1,v}.$$

To estimate this, we consider

$$\sum_{v=r+1}^{n} \underbrace{E\left[\left(\hat{\lambda}_r M_{r+1,v}\right)^2 \,\middle|\, F_{v-1}\right]}_{(1^*)} + 2\underbrace{\sum_{v_1<v_2} E\left[\left(\hat{\lambda}_r M_{r+1,v_1}\right)\left(\hat{\lambda}_r M_{r+1,v_2}\right) \,\middle|\, F_{v_2-1}\right]}_{(2^*)}$$
$$+ \underbrace{E\left[J_r^2 \,\middle|\, F_{r-1}\right]}_{(3^*)} + 2\underbrace{E\left[J_r \mid F_{r-1}\right] \sum_{v=r+1}^{n} E\left[\hat{\lambda}_r M_{r+1,v} \,\middle|\, F_{v-1}\right]}_{(4^*)}.$$

From the definition of M_{uv}, (1*) can be estimated as follows:

$$\sum_{v=r+1}^{n} \hat{\lambda}_r^2 \left(1 - \hat{\delta}_{r+1}\right)^2 \ldots \left(1 - \hat{\delta}_{k-1}\right)^2 \hat{Var}\left[\hat{\delta}_v \,\middle|\, F_{v-1}\right]\left(1 - \hat{\delta}_{v+1}\right)^2 \ldots \left(1 - \hat{\delta}_n\right)^2$$
$$= \sum_{v=r+1}^{n} \hat{\lambda}_r^2 \left(1 - \hat{\delta}_{r+1}\right)^2 \ldots \left(1 - \hat{\delta}_n\right)^2 \frac{\hat{Var}\left[\hat{\delta}_v \,\middle|\, F_{v-1}\right]}{\left(1 - \hat{\delta}_v\right)^2}.$$

Next, since $E\left[\delta_v - \hat{\delta}_v \mid F_{v-1}\right] = 0$, (2*) = 0.

For (3*), we note that $E\left[(\lambda_r - \hat{\lambda}_r)^2 \mid F_{r-1}\right] = Var(\hat{\lambda}_r \mid F_{r-1})$ and hence, (3*) is estimated using $\hat{Var}(\hat{\lambda}_r \mid F_{r-1})\,(1 - \hat{\delta}_{r+1})^2 \ldots (1 - \hat{\delta}_n)^2$.

Finally, we note that (4*) = 0, since $E\left[\lambda_r - \hat{\lambda}_r \mid F_{r-1}\right] = 0$.

Hence we estimate G_r^2 by $\hat{G}_r^2$, where

$$\hat{G}_r^2 = \sum_{v=r+1}^{n} \hat{\lambda}_r^2 \left(1-\hat{\delta}_{r+1}\right)^2 \ldots \left(1-\hat{\delta}_n\right)^2 \frac{Var\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2}$$

$$+ Var\left(\hat{\lambda}_r \mid F_{r-1}\right)\left(1-\hat{\delta}_{r+1}\right)^2 \ldots \left(1-\hat{\delta}_n\right)^2$$

$$= \left(1-\hat{\delta}_{r+1}\right)^2 \ldots \left(1-\hat{\delta}_n\right)^2 \left\{ \sum_{v=r+1}^{n} \left(\hat{\lambda}_r^2 \frac{Var\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right) + Var\left(\hat{\lambda}_r \mid F_{r-1}\right) \right\}.$$

Therefore, (2) is estimated as:

$$\sum_{r=n-i+2}^{n} \hat{G}_r^2 = \sum_{r=n-i+2}^{n} \left(1-\hat{\delta}_{r+1}\right)^2 \ldots \left(1-\hat{\delta}_n\right)^2 \left\{ \sum_{v=r+1}^{n} \left(\hat{\lambda}_r^2 \frac{Var\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right) + Var\left(\hat{\lambda}_r \mid F_{r-1}\right) \right\}.$$

Consider next (3):

$$\sum_{r_1 > r_2} G_{r_1} G_{r_2} = \sum_{r_1 > r_2} \left(\sum_{v=r_1+1}^{n} \hat{\lambda}_{r_1} M_{r_1+1,v} + J_{r_1} \right) \left(\sum_{v=r_2+1}^{n} \hat{\lambda}_{r_2} M_{r_2+1,v} + J_{r_2} \right)$$

$$= \sum_{r_1 > r_2} \left[\left(\sum_{v=r_1+1}^{n} \hat{\lambda}_{r_1} M_{r_1+1,v} \right) \left(\sum_{v=r_2+1}^{n} \hat{\lambda}_{r_2} M_{r_2+1,v} \right) + J_{r_1} J_{r_2} \right.$$

$$\left. + \left(\sum_{v=r_1+1}^{n} \hat{\lambda}_{r_1} M_{r_1+1,v} \right) J_{r_2} + \left(\sum_{v=r_2+1}^{n} \hat{\lambda}_{r_2} M_{r_2+1,v} + \right) J_{r_1} \right].$$

Again, we approximate this by its expectation, and note that this is 0: hence (3) disappears.

And finally (4): $P_i \left[\sum_{r=n-i+2}^{n} G_r \right] = P_i \left[\sum_{r=n-i+2}^{n} \sum_{v=r+1}^{n} \left(\hat{\lambda}_r M_{r+1,v} + J_r \right) \right]$

$$= \left[\sum_{r=n-i+2}^{n} \left(P_i \sum_{v=r+1}^{n} \hat{\lambda}_r M_{r+1,v} + P_i J_r \right) \right].$$

Using the results above, we estimate this by

$$\hat{P}_i \left[\sum_{r=n-i+2}^{n} \hat{G}_r \right] =$$

$$\sum_{r=n-i+2}^{n} \left(\hat{\lambda}_r \left(1-\hat{\delta}_{n-i+2}\right) \ldots \left(1-\hat{\delta}_{r-1}\right) \left(1-\hat{\delta}_{r+1}\right)^2 \ldots \left(1-\hat{\delta}_n\right)^2 \sum_{v=r+1}^{n} \frac{\hat{Var}\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right).$$

Therefore, the estimation error of row total reserve is estimated as:

$$X_{i,n-i+1}^2\left(1-\hat{\delta}_{n-i+2}\right)^2\ldots\left(1-\hat{\delta}_n\right)^2\sum_{v=n-i+2}^{n}\frac{\hat{Var}\left(\hat{\delta}_v\middle|F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2}$$

$$+E_i^2\sum_{r=n-i+2}^{n}\left\{\hat{\lambda}_r^2\left(1-\hat{\delta}_{r+1}\right)^2\ldots\left(1-\hat{\delta}_n\right)^2\left[\sum_{v=r+1}^{n}\left(\frac{\hat{Var}\left(\hat{\delta}_v\middle|F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2}\right)+\frac{\hat{Var}\left(\hat{\lambda}_r\middle|F_{r-1}\right)}{\hat{\lambda}_r^2}\right]\right\}$$

$$+2X_{n-i+1}E_i\sum_{r=n-i+2}^{n}\left(\hat{\lambda}_r\left(1-\hat{\delta}_{n-i+2}\right)\ldots\left(1-\hat{\delta}_{r-1}\right)\left(1-\hat{\delta}_{r+1}\right)^2\ldots\left(1-\hat{\delta}_n\right)^2\sum_{v=r+1}^{n}\frac{\hat{Var}\left(\hat{\delta}_v\middle|F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2}\right)$$

which complete the proof.

Appendix D

The estimation error for the overall total reserve.

$$\left(E\left[\sum_{i=2}^{n}X_{in}\mid H_n\right]-\sum_{i=2}^{n}\hat{X}_{in}\right)^2=\left(\sum_{i=2}^{n}\left(E[X_{in}\mid H_n]-\hat{X}_{in}\right)\right)^2$$
$$=\sum_{i=2}^{n}a_{ii}+2\sum_{2\le i<j\le n}a_{ij}$$

where $a_{ij}=\left(E[X_{in}\mid H_n]-\hat{X}_{in}\right)\left(E[X_{jn}\mid H_n]-\hat{X}_{jn}\right)$

Now

$$a_{ij}=\left(X_{i,n-i+1}P_i+E_i\sum_{r=n-i+2}^{n}G_r\right)\left(X_{j,n-j+1}P_j+E_j\sum_{r=n-j+2}^{n}G_r\right)$$
$$=X_{i,n-i+1}X_{j,n-j+1}\underbrace{P_iP_j}_{A}+E_iE_j\underbrace{\left[\sum_{r_1=n-i+2}^{n}\sum_{r_2=n-j+2}^{n}G_{r_1}G_{r_2}\right]}_{B}$$
$$+X_{i,n-i+1}E_j\underbrace{\left[P_i\sum_{r=n-j+2}^{n}G_r\right]}_{C}+X_{j,n-j+1}E_i\underbrace{\left[P_j\sum_{r=n-i+2}^{n}G_r\right]}_{C}.$$

Considering first A, we note that this can be treated in the same way as P_i^2 (the term labelled (1)) in Appendix C. Thus, denoting $\hat{P}_i\hat{P}_j$ as the estimator of P_iP_j, we have

$$\hat{P}_i \hat{P}_j \approx (1-\hat{\delta}_{n-j+2}) \ldots (1-\hat{\delta}_{n-i+1})(1-\hat{\delta}_{n-i+2})^2 (1-\hat{\delta}_{n-i+3})^2 \ldots (1-\hat{\delta}_n)^2$$
$$\sum_{v=n-i+2}^{n} \frac{\hat{V}ar\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2}.$$

Consider next B:

$$\sum_{r_1=n-i+2}^{n} \sum_{r_2=n-j+2}^{n} G_{r_1} G_{r_2} =$$
$$\sum_{r_1=n-i+2}^{n} \sum_{r_2=n-j+2}^{n} \left(\sum_{v=r_1+1}^{n} \hat{\lambda}_{r_1} M_{r_1+1,v} + J_{r_1} \right)\left(\sum_{v=r_2+1}^{n} \hat{\lambda}_{r_2} M_{r_2+1,v} + J_{r_2} \right)$$
$$= \sum_{r_1=n-i+2}^{n} \sum_{r_2=n-j+2}^{n} \left(\sum_{v=r_1+1}^{n} \hat{\lambda}_{r_1} M_{r_1+1,v} \sum_{v=r_2+1}^{n} \hat{\lambda}_{r_2} M_{r_2+1,v} + J_{r_2} \sum_{v=r_1+1}^{n} \hat{\lambda}_{r_1} M_{r_1+1,v} \right.$$
$$\left. + J_{r_1} \sum_{v=r_2+1}^{n} \hat{\lambda}_{r_2} M_{r_2+1,v} + J_{r_1} J_{r_2} \right)$$

Again, we can follow the same method as was used in Appendix C in the treatment of G_r^2 (the term labelled (2)). As before, when we use expectations in order to estimate this expression, the cross-product terms disappear. Denoting the estimator of $G_{r_1} G_{r_2}$ by $\hat{G}_{r_1} \hat{G}_{r_2}$, we have

$$\sum_{r_1=n-i+2}^{n} \sum_{r_2=n-j+2}^{n} \hat{G}_{r_1} \hat{G}_{r_1} =$$
$$\sum_{r_1=n-i+2}^{n} \sum_{r_2=n-j+2}^{n} \left(\begin{array}{l} \hat{\lambda}_{r_2}\left(1-\hat{\delta}_{r_2+1}\right)\ldots\left(1-\hat{\delta}_{r_1}\right)\hat{\lambda}_{r_1}\left(1-\hat{\delta}_{r_1+1}\right)^2\ldots\left(1-\hat{\delta}_n\right)^2 \\ \left[\sum_{v=r_1+1}^{n} \frac{\hat{V}ar\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} + \frac{\hat{V}ar\left(\hat{\lambda}_{r_1} \mid F_{r_1-1}\right)}{\hat{\lambda}_{r_1}\left(1-\hat{\delta}_{r_2+1}\right)\ldots\left(1-\hat{\delta}_{r_1}\right)\hat{\lambda}_{r_2}} \right] \end{array} \right).$$

For each of the terms labelled C, we refer back to the term labelled (4) in Appendix C, and note that the estimator of this is

$$\hat{P}_i \left[\sum_{r=n-j+2}^{n} \hat{G}_r \right] =$$
$$\sum_{r=n-j+2}^{n} \left(\left(1-\hat{\delta}_{n-i+2}\right)\ldots\left(1-\hat{\delta}_r\right)\hat{\lambda}_r\left(1-\hat{\delta}_{r+1}\right)^2\ldots\left(1-\hat{\delta}_n\right)^2 \sum_{v=r+1}^{n} \frac{\hat{V}ar\left(\hat{\delta}_v \mid F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right).$$

Therefore the squared estimation error of overall total reserve is estimated as follows:

$$
\begin{aligned}
& X_{i,n-i+1}^2 \left(1-\hat{\delta}_{n-i+2}\right)^2 \ldots \left(1-\hat{\delta}_n\right)^2 \sum_{v=n-i+2}^{n} \frac{\hat{V}ar\left(\hat{\delta}_v \middle| F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \\
& + E_i^2 \sum_{r=n-i+2}^{n} \left\{ \hat{\lambda}_r^2 \left(1-\hat{\delta}_{r+1}\right)^2 \ldots \left(1-\hat{\delta}_n\right)^2 \left[\sum_{v=r+1}^{n} \left(\frac{\hat{V}ar\left(\hat{\delta}_v \middle| F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right) + \frac{\hat{V}ar\left(\hat{\lambda}_r \middle| F_{r-1}\right)}{\hat{\lambda}_r^2} \right] \right\} \\
& + 2X_{n-i+1} E_i \sum_{r=n-i+2}^{n} \left\{ \hat{\lambda}_r \left(1-\hat{\delta}_{n-i+2}\right) \ldots \left(1-\hat{\delta}_{r-1}\right) \left(1-\hat{\delta}_{r+1}\right)^2 \ldots \left(1-\hat{\delta}_n\right)^2 \sum_{v=r+1}^{n} \frac{\hat{V}ar\left(\hat{\delta}_v \middle| F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right\} \\
& + 2 \sum_{2 \le i < j \le n} \left\{ \begin{array}{l}
X_{i,n-i+1} X_{j,n-j+1} \displaystyle\prod_{l=n-j+2}^{n-i+1} \left(1-\hat{\delta}_l\right) \prod_{l=n-i+2}^{n} \left(1-\hat{\delta}_l\right)^2 \sum_{v=n-i+2}^{n} \frac{\hat{V}ar\left[\hat{\delta}_v \middle| F_{v-1}\right]}{\left(1-\hat{\delta}_v\right)^2} \\
+ E_i E_j \displaystyle\sum_{r_1=n-i+2}^{n} \sum_{r_2=n-j+2}^{n} \left(\begin{array}{l} \hat{\lambda}_{r_1} \hat{\lambda}_{r_2} \displaystyle\prod_{l=r_2+1}^{r_1} \left(1-\hat{\delta}_l\right) \prod_{l=r_1+1}^{n} \left(1-\hat{\delta}_l\right)^2 \\ \left[\displaystyle\sum_{v=r_1+1}^{n} \frac{\hat{V}ar\left(\hat{\delta}_v \middle| F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} + \frac{\hat{V}ar\left(\hat{\lambda}_{r_1} \middle| F_{r_1-1}\right)}{\hat{\lambda}_{r_1} \hat{\lambda}_{r_2} \displaystyle\prod_{l=r_2+1}^{r_1} \left(1-\hat{\delta}_l\right)} \right] \end{array} \right) \\
+ X_{i,n-i+1} E_j \displaystyle\sum_{r=n-j+2}^{n} \hat{\lambda}_r \prod_{l=n-i+2}^{r} \left(1-\hat{\delta}_l\right) \prod_{l=r+1}^{n} \left(1-\hat{\delta}_l\right)^2 \sum_{v=r+1}^{n} \left(\frac{\hat{V}ar\left(\hat{\delta}_v \middle| F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right) \\
+ X_{j,n-j+1} E_i \displaystyle\sum_{r=n-i+2}^{n} \hat{\lambda}_r \prod_{l=n-j+2}^{r} \left(1-\hat{\delta}_l\right) \prod_{l=r+1}^{n} \left(1-\hat{\delta}_l\right)^2 \sum_{v=r+1}^{n} \left(\frac{\hat{V}ar\left(\hat{\delta}_v \middle| F_{v-1}\right)}{\left(1-\hat{\delta}_v\right)^2} \right)
\end{array} \right\}
\end{aligned}
\tag{A1}
$$

Note that the first part of the above expression leads to the prediction error of individual row totals, which is derived in Appendix C.

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