ON CHARACTERIZATION OF DISTORTION PREMIUM PRINCIPLE*

BY

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ABSTRACT

In this paper, based on the additive measure integral representation of a non-additive measure integral, it is shown that any comonotonically additive premium principle can be represented as an integral of the distorted decumulative distribution function of the insurance risk. Furthermore, a sufficient and necessary condition that a premium principle is a distortion premium principle is given.

KEY WORDS AND PHRASES

Choquet Integral, Comonotonicity, Distortion Premium Principle, Non-Additive Measure.

INTRODUCTION

Due to its many good properties, considerable attention has been given to the distortion premium principle, for example, Wang (1995), Wang (1996), Wang et al (1997), Wang (1998), Wang and Dhaene (1998) and Young (1999) and so on. A characterization of the distortion premium principle was given by Wang et al (1997). In their paper, a condition that the collection of risks contains all Bernoulli variables, which can take only the values 0 and 1, is required. We will discuss the characterization of the distortion premium principle again. In this paper, the above condition need not be required. Based on the additive measure integral representation of a non-additive measure integral, we show that any comonotonically additive premium principle can be represented as an integral of the distorted decumulative function of the insurance risk. An alternative characterization of the distortion premium principle is derived.

This paper is arranged as follows. In section 1, after some concepts introduced firstly, two propositions will be proved. One is on the additive measure integral representation of a non-additive measure integral, and the other is on the positive homogeneity of a comonotonically additive premium principle.

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The well-known Greco Theorem will be modified in section 2. Then the main results, that the distorted integral representation of a comonotonically additive premium principle and a sufficient and necessary condition that a premium principle is a distortion premium principle, are given in section 3.

1. Preliminaries

Let (Ω, F, P) be a probability space where Ω represents the outcomes of the world and F is a σ-algebra of some subsets of Ω, and P is a probability measure defined on the measurable space (Ω, F). The following concepts of the comonotonicity and the comonotonic additivity can be found in Yaari (1987), Denneberg (1994), Wang et al (1997), Wang (1998) and others.

Definition 1.1. *(Comonotonicity). $X(\omega)$ and $Y(\omega)$ are two random variables defined in (Ω, F, P), they are called comonotonic if*

$$[X(\omega_1) - X(\omega_2)][Y(\omega_1) - Y(\omega_2)] \geq 0, \quad a.s.$$

Definition 1.2. *(Comonotonically additive). L is a collection of some insurance risks. If a premium principle H satisfies*

$$H[X + Y] = H[X] + H[Y]$$

for all pairs of comonotonic risks X and Y in L, we say that H is comonotonically additive in L.

Lemma 1.1. *Two random variables $X(\omega)$ and $Y(\omega)$, defined in (Ω, F, P), are comonotonic if and only if there exists a random variable Z defined in (Ω, F, P) and two non-decreasing functions f and g such that $X = f(Z)$ and $Y = g(Z)$ almost surely.*

We use 2^{Ω} to represent the class of all subsets of Ω, and let γ be a set function from 2^{Ω} onto the extended non-negative space $\bar{R}^{+} = [0, \infty]$. If $\gamma(A) \geq \gamma(B)$ for all $A \subseteq B$, γ is called a non-negative monotonic set function or a non-additive measure on Ω.

Definition 1.3. *(Choquet Integral). Let X be a non-negative random variable defined on the measurable space (Ω, F). The Choquet integral of X with respect to non-additive measure γ is defined as*

$$\int_{\Omega} X d\gamma = \int_{0}^{\infty} \gamma\{\omega : X(\omega) > t\}\, dt$$

The Choquet integral is different from the traditional additive measure integral, but the following proposition shows that a non-additive measure integral can be represented as an additive measure integral.

Proposition 1.1. *Let X be a non-negative random variable defined on* (Ω, F). *For any non-additive measure γ in* (Ω, F), *there exists an additive measure γ^* on* (Ω, F) *such that*

$$\int_{\Omega} X d\gamma = \int_{\Omega} X d\gamma^*$$

Proof. Obviously, in order to prove this proposition we only need to prove that for any non-increasing function $g(t)$ there exists a non-increasing function $g^*(t)$, which is continuous from the right, such that

$$\int_0^{\infty} g(t)dt = \int_0^{\infty} g^*(t)dt \tag{1.1}$$

Let $g^*(t) = g(t+)$, the right-side limit of $g(t)$. Since $g^*(t) = g(t)$ almost everywhere, equation (1.1) holds. Obviously, what remains is to prove that $g^*(t)$ is continuous from the right.

Firstly, we note that $g^*(t) = \lim_{\epsilon \to 0+} g(t+\epsilon) \le g(t)$. Therefore

$$\lim_{\epsilon \to 0+} g^*(t+\epsilon) \le \lim_{\epsilon \to 0+} g(t+\epsilon) = g^*(t) \tag{1.2}$$

For any fixed $\epsilon > 0$, we take δ and η such that $0 < \delta, \eta < \epsilon/2$. Then $g^*(t+\delta) = \lim_{\eta \to 0+} g(t+\delta+\eta) \ge g(t+\epsilon)$. Let $\epsilon \to 0+$, and then $\delta \to 0+$, we have

$$\lim_{\delta \to 0+} g^*(t+\delta) \ge g^*(t) \tag{1.3}$$

It follows from (1.2) and (1.3) that $g^*(t)$ is continuous from the right. The proposition is proved. ■

Proposition 1.2. *Let L be a set of some risks satisfying the criterion that for any $X \in L$ and any $a \ge 0$, $aX \in L$, and let H be a comonotonically additive premium principle in L. Then H is positively homogeneous, that is, for any $a \ge 0$, we have*

$$H[aX] = aH[X] \tag{1.4}$$

Proof. The result is trivial for $a = 0$. When a is a positive integer, (1.4) follows from the comonotonicity of H. When a is a positive rational number, say $a = m/n$ where m and n are positive integers, then $nH[(m/n)X] = H[n(m/n)X] = H[mX] = mH[X]$. Thus (1.4) holds.

When a be a positive number. In the case $H[X] < \infty$, there exists a strictly decreasing series of positive rational numbers, $\{a_n\}$, such that $a_n \to a$. Then $\epsilon_n = a_n - a$ converges to 0 decreasingly. Therefore

$$\begin{aligned} aH[X] &= \lim_{n\to\infty} a_n H[X] = \lim_{n\to\infty} H[a_n X] = \lim_{n\to\infty} H[(a+\epsilon_n)X] \\ &= H[aX] + \lim_{n\to\infty} H[\epsilon_n X] \end{aligned} \tag{1.5}$$

Let $\{b_n\}$ be a series of positive rational numbers such that $0 \le \epsilon_n \le b_n$ and $b_n \to 0$. It is from the comonotonic additivity of H that $0 \le H[\epsilon_n X] \le H[b_n X] = b_n H[X]$, which implies $H[\epsilon_n X] \to 0$. Hence, (1.4) follows from (1.5).

In the case $H[X] = \infty$, there exists a positive rational number, say b, such that $b \le a$. It is from the comonotonic additivity of H that $H[aX] \ge H[bX] = bH[\mathrm{X}] = \infty = aH[\mathrm{X}]$.

To sum up, the proposition is proved. ∎

Wang (1996) once proved the positive homogeneity of a distortion premium principle. We know that the distortion premium principle is comonotonically additive, but a comonotonically additive premium principle may not be the distortion premium principle. So the above proposition is generalization of Wang's result.

The concepts of the distortion premium principle (see Wang et al (1997)) is given here. A general distortion function, say g, is a bounded non-decreasing and non-negative function defined on [0,1]. If g satisfies $g(0) = 0$ and $g(1) = 1$ then g is briefly called a distortion function. A distortion premium principle H of a non-negative risk X is defined as

$$H[X] = \int_0^\infty g[S_X(t)]\,dt$$

where $S_X(t) = P(X > t)$ is the decumulative distribution function (ddf) of X. If g only satisfies $g(0) = 0$, H is called a general distortion premium principle. Thus $\tilde{g}$ is a distortion function if

$$\tilde{g}[S_X(t)] = \frac{g[S_X(t)]}{g(1)}$$

In addition, the notation $X \wedge a$ means the minimum of X and a hereafter.

2. The Greco Theorem

The Greco theorem (see, for example, Denneberg (1994)) plays a key role in the characterization of the distortion premium principle. Here we give a new version of the Greco theorem on the space of non-negative random variables.

Theorem 2.1. *(Modified Greco Theorem). Let L be a non-empty collection of some non-negative risks in Ω. For any given $X \in L$ and any non-negative numbers a, a_1 and a_2, $a_1 \le a_2$, both aX and $X \wedge a_2 - X \wedge a_1$ also belong to L. Suppose that a premium principle $H : L \to \bar{R}^+$ satisfies the following conditions:*

(1) Monotonicity (preserving order). If $X(\omega) \le Y(\omega)$ a.s., $H[X] \le H[Y]$;

(2) Additivity of comonotonic risks. If X and Y are comonotonic, $H[X+Y] = H[X] + H[Y]$;

(3) Continuity. Let a be a non-negative number, then $\lim_{a\to+\infty} H[X \wedge a] = H[X]$;

(4) Finiteness. If $X \in L$ is a bounded risk, that is, there exists a constant C such that $X \leq C$ a.s., $H[X] < \infty$.

Then there exists a monotone set function $\gamma : 2^{\Omega} \to \bar{R}+$ such that for all $X \in L$

$$H[X] = \int_{\Omega} X d\gamma = \int_{\Omega}^{\infty} \gamma\{\omega : X(\omega) > x\}\, dx \tag{2.1}$$

Proof. Two set functions on 2^{Ω} are defined as follows:

$$\alpha(A) = \sup\{H[X] \mid X \in L, X \leq I_A\},$$
$$\beta(A) = \inf\{H[X] \mid X \in L, X \geq I_A\},$$

where I_A is the indicator of set $A \in 2^{\Omega}$. Obviously, $\alpha \leq \beta$. Since H is monotone, both α and β are also monotone. Let γ be some monotone set function on 2^{Ω} satisfying $\alpha \leq \gamma \leq \beta$. For example, we can select $\gamma = \beta$ (or α). For any fixed $X \in L$, $X \wedge x$ is non-decreasing. Because of the comonotonicity of H, $H[X \wedge x]$ is a non-decreasing function with respect to x, and has finite derivative almost everywhere. It is clear that for any $\epsilon > 0$

$$\frac{X \wedge (t+\epsilon) - X \wedge t}{\epsilon} \leq I_{\{X(\omega) > t\}} \leq \frac{X \wedge t - X \wedge (t-\epsilon)}{\epsilon} \leq I_{\{X(\omega) > t-\epsilon\}}$$

According to the definitions of α, β and the positive homogeneity of H (see Proposition 1.2), we have

$$\begin{aligned}\frac{H[X \wedge (t+\epsilon)] - H[X \wedge t]}{\epsilon} &\leq \alpha\{\omega : X(\omega) > t\} \leq \gamma\{\omega : X(\omega) > t\} \\ &\leq \beta\{\omega : X(\omega) > t\} \leq \frac{H[X \wedge t] - H[X \wedge (t-\epsilon)]}{\epsilon} \\ &\leq \alpha\{\omega : X(\omega) > t-\epsilon\} \leq \gamma\{\omega : X(\omega) > t-\epsilon\}\end{aligned}$$

Let $0 = t_0 \leq t_1 \leq \ldots \leq t_n = t$. It is from

$$\gamma\{\omega : X(\omega) > t\} \leq \frac{H[X \wedge t] - H[X \wedge (t-\epsilon)]}{\epsilon} \leq \gamma\{\omega : X(\omega) > t-\epsilon\}$$

that

$$\sum_{1}^{n} \gamma\{\omega : X(\omega) > t_i\}(t_i - t_{i-1}) \leq H[X \wedge t] \leq \sum_{1}^{n} \gamma\{\omega : X(\omega) > t_{i-1}\}(t_i - t_{i-1})$$

Since $\gamma\{\omega : X(\omega) > t\}$ is monotonic, it is integrable with respect to t. Hence, as n approaches infinite and $\max_{1 \leq i \leq n}(t_i - t_{i-1})$ approaches zero, we have

$$H[X \wedge t] = \int_{0}^{t} \gamma\{\omega : X(\omega) > x\}\, dx \tag{2.2}$$

Let $t \to \infty$ and notice the continuity of H, (2.1) follows from (2.2). Theorem 2.1 is proved. ■

Compared with the original version of the Greco Theorem, Theorem 2.1 (Modified Greco Theorem) required that H is finite for bounded risks. We think that such a modification is feasible when it is applied to insurance premium calculation principle.

3. Characterization of Distortion Premium Principle

Theorem 2.1 says that a comonotonically additive premium principle can be represented as a Choquet integral. In this section we will study how to change a Choquet integral into a distortion integral.

Theorem 3.1. *Suppose the conditions in Theorem 2.1 are satisfied. Then for any* $X \in L$*, there exists a bounded non-decreasing and non-negative function* g_X *with* $g_X(0) = 0$ *such that*

$$H[X] = \int_0^\infty g_X[S_X(t)]\,dt \tag{3.1}$$

Proof. Let $G_{\gamma,X}(x) = \gamma\{\omega : X(\omega) > x\}$. It follows from the proof of Proposition 1.1 that there exists a non-increasing function $G^*_{\gamma,X}(x)$, which is continuous from the right, such that $G^*_{\gamma,X}(x) = G_{\gamma,X}(x)$ almost everywhere. Then by Theorem 2.1, we have

$$H[X] = \int_0^\infty G^*_{\gamma,X}(x)dx$$

Now we will prove that $S_X(x_1) = S_X(x_2)$ implies $G^*_{\gamma,X}(x_1) = G^*_{\gamma,X}(x_2)$. Obviously, if $S_X(x_1) = S_X(x_2)$, $P\{\omega : X(\omega) \in (x_1, x_2)\} = 0$ for some $x_1 < x_2$. Let γ^* be an additive measure in (Ω, F) generated by $G^*_{\gamma,X}$, then by the proof of Proposition 1.1, $\gamma^*\{\omega : X(\omega) > x\} = G^*_{\gamma,X}(x)$ and $\int_\Omega X d\gamma = \int_\Omega X d\gamma^*$. Let

$$Z = \frac{X \wedge x_2 - X \wedge x_2}{x_2 - x_1}$$

Then $Z \in L$, and since $P\{\omega : X(\omega) \in (x_1, x_2)\} = 0$, it holds almost surely that

$$Z = \begin{cases} 0, & X \le x_2 \\ 1, & X > x_2 \end{cases} = \begin{cases} 0, & X \le x_1 \\ 1, & X > x_1 \end{cases}$$

Therefore

$$H[Z] = \int Z d\gamma^* = \gamma^*\{\omega : X(\omega) > x_1\} = \gamma^*\{\omega : X(\omega) > x_2\}$$

Thus $G^*_{\gamma,X}(x_1) = G^*_{\gamma,X}(x_2)$. Then it is proved that $S_X(x_1) = S_X(x_2)$ implies $G^*_{\gamma,X}(x_1) = G^*_{\gamma,X}(x_2)$. Note that this results in that there exists a map g_X from the set $S_X(t) | t \geq 0\}$ onto the set $\{G^*_{\gamma,X}(t) | t \geq 0\}$ such that $G^*_{\gamma,X}(x) = g_X[S_X(x)]$. Hence (3.1) holds.

Obviously, g_X is a bounded and non-negative function. Below we will prove that g_X is a non-decreasing function with $g_X(0) = 0$. Let $S_X(x_1) > S_X(x_2)$. Then $x_1 < x_2$ which implies $\gamma^*\{\omega : X(\omega) > x_1\} \geq \gamma^*\{\omega : X(\omega) > x_2\}$, i.e. $G^*_{\gamma,X}(x_1) \geq G^*_{\gamma,X}(x_2)$. Therefore g_X is non-decreasing. Since $\{S_X(t) : t \geq 0\}$ is a subset of [0, 1], we can extend the domain of definition of g_X onto [0, 1] such that the extended g_X is still non-decreasing. Obviously, $G^*_{\gamma,X}(\infty) = 0$ which implies $g_X(0) = 0$. Thus Theorem 3.1 is proved. ∎

Notice that the subscript X in g_X means that the bounded non-decreasing and non-negative function g_X may be related to the risk X. The remaining problem is what conditions the risk set L and the premium principle H should satisfy so that g_X is unrelated to the risk $X \in L$. In Wang et al (1997), the condition is that the risk set L includes all Bernoulli variables. The following two Corollaries say that such a condition can be weakened.

Corollary 3.1. *Suppose the conditions in Theorem 2.1 and conditional state independence are satisfied. If $I_{\{X>a\}} \in L$ for any $X \in L$ and any $a > 0$, there exists a general distortion function g with $g(0) = 0$ such that for any $X \in L$ we have*

$$H[X] = \int_0^\infty g[S_X(t)]\,dt$$

Proof. It follows from the proof of Theorem 3.1 that the only thing which needs to be proved is that for any X, Y in L and non-negative numbers t_1 and t_2, $S_X(t_1) = S_Y(t_2)$ implies $\gamma\{\omega : X(\omega) > t_1\} = \gamma\{\omega : Y(\omega) > t_2\}$. Define two Bernoulli variables

$$U_X = I_{\{X(\omega)>t_1\}}, \quad U_Y = I_{\{Y(\omega)>t_2\}}$$

Both U_X and U_Y belong to L. Because $S_X(t_1) = S_Y(t_2)$, $H[U_X] = H[U_Y]$. Noticing that

$$H[U_X] = \gamma\{X(\omega) > t_1\}, \quad H[U_Y] = \gamma\{Y(\omega) > t_2\}$$

so we have $\gamma\{\omega : X(\omega) > t_1\} = \gamma\{\omega : Y(\omega) > t_2\}$. Corollary 3.1 is proved. ∎

Corollary 3.2. *Suppose that X is a fixed non-negative risk, and*

$$L = \{X \wedge b - X \wedge a : a, b \in \bar{R}^+,\ a \leq b\}$$

Then there exists a general distortion function g with $g(0) = 0$ such that for any $X \in L$ we have

$$H[X] = \int_0^\infty g[S_X(t)]\,dt$$

Proof. Noticing that L consists of X, $X \wedge a$, $X - X \wedge a$ and $X \wedge b - X \wedge a$, the Proposition 3.2 can be easily proved by the proof of Theorem 2.1.

Now we are going to give a characterization of the distortion premium principle, i.e. a necessary and sufficient condition that the premium $H[X]$ of a risk $X \in L$ can be represented as $H[X] = \int_0^\infty g[S_X(t)]dt$ where g is a general distortion function with $g(0) = 0$. Let $\epsilon > 0$ and $x \geq 0$, and denote

$$X(x, \epsilon) = \frac{X \wedge (x+\epsilon) - X \wedge x}{\epsilon}$$

When ϵ approaches 0, $X(x, \epsilon)$ converges almost everywhere and we have that

$$\begin{aligned} \frac{d(X \wedge x)}{dx} &= \lim_{\epsilon \to 0+} \frac{X \wedge (x+\epsilon) - X \wedge x}{\epsilon} = \lim_{\epsilon \to 0+} X(x, \epsilon) \\ &= \begin{cases} 1, & X > x \\ 0, & X \leq x \end{cases} = I_{\{X > x\}} \end{aligned} \tag{3.2}$$

Moreover, we have

$$\frac{dH[(X \wedge x)]}{dx} = \lim_{\epsilon \to 0+} H\left[\frac{X \wedge (x+\epsilon) - X \wedge x}{\epsilon}\right] = \lim_{\epsilon \to 0+} H[X(x, \epsilon)] \tag{3.3}$$

Theorem 3.2. *Let L be a non-empty collection of some non-negative risks in Ω. For any given $X \in L$ and any non-negative numbers a, a_1 and a_2, $a_1 \leq a_2$, both aX and $X \wedge a_2 - X \wedge a_1$ also belong to L. Then a premium principle $H: L \to \bar{R}^+$ is a general distortion premium if and only if H satisfies the following conditions:*

(1) Monotonicity (preserving order). If If $X(\omega) \leq Y(\omega)$ a.s., $H[X] \leq H[Y]$;

(2) Additivity of comonotonic risks. If X and Y are comonotonic, $H[X+Y] = H[X] + H[Y]$;

(3) Continuity in distributions. Let X_n and Y_n be two arbitrary series of risks in L with cumulative functions F_n and G_n respectively, then $\lim_{n\to+\infty} H[X_n] = \lim_{n\to+\infty} H[Y_n]$ when $\lim_{n\to+\infty} F_n = \lim_{n\to+\infty} G_n$;

(4) Finiteness. If $X \in L$ is a bounded risk, that is, there exists a constant C such that $X \leq C$ a.s., $H[X] < \infty$.

Proof. For the proof of the sufficiency, it can be seen by the proof of Theorem 3.1 and Corollary 3.1 that the only thing which needs to be proved is that for any X, Y in L and non-negative numbers x and y, $S_X(x) = S_Y(y)$ implies $\gamma^*\{\omega : X(\omega) > x\} = \gamma^*\{\omega : Y(\omega) > y\}$ where γ^* is an additive measure such that $H[X] = \int_0^\infty \gamma^*\{X(\omega) > t\}dt$ by the proof of Theorem 3.1. Obviously, both $X(x, \epsilon)$ and $Y(y, \epsilon)$ are included in L. Because of the additivity of the comonotonic risks, we have that

$$H[X(x,\epsilon)] = H\left[\frac{X \wedge (x+\epsilon)] - H[X \wedge x]}{\epsilon}\right]$$
$$= \frac{1}{\epsilon}\left\{\int_0^\infty \gamma^*\{\omega : X(\omega) \wedge (x+\epsilon) > t\}\, dt - \int_0^\infty \gamma^*\{\omega : X(\omega) \wedge x > t\}\, dt\right\}$$
$$= \frac{1}{\epsilon}\int_0^{x+\epsilon} \gamma^*\{\omega : X(\omega) > t\}\, dt$$

It implies that

$$\gamma^*\{\omega : X(\omega) > x\} \geq H[X(x,\epsilon)] \geq \gamma^*\{\omega : X(\omega) > x+\epsilon\}$$

Therefore, by (3.3) we have

$$\left.\frac{dH[(X \wedge t)]}{dt}\right|_{t=x} = \lim_{\epsilon \to 0+} H[X(x,\epsilon)] = \gamma^*\{\omega : X(\omega) > x\} \tag{3.4}$$

Similarly, we have

$$\left.\frac{dH[(Y \wedge t)]}{dt}\right|_{t=y} = \lim_{\epsilon \to 0+} H[Y(y,\epsilon)] = \gamma^*\{\omega : Y(\omega) > y\} \tag{3.5}$$

Let F_ϵ and G_ϵ denote the distributions of $X(x,\epsilon)$ and $Y(x,\epsilon)$ respectively. Then by (3.2), $S_X(x) = S_Y(y)$ implies $\lim_{\epsilon\to 0+} F_\epsilon = \lim_{\epsilon\to 0+} G_\epsilon$. Therefore, it follows from condition 3 that $\lim_{\epsilon\to 0+} H[x,\epsilon] = \lim_{\epsilon\to 0+} H[Y(y,\epsilon)]$. Thus $\gamma^*\{\omega : X(\omega) > t_1\} = \gamma^*\{\omega : Y(\omega) > t_2\}$ from (3.4) and (3.5). The sufficiency is proved.

For the proof of the necessity, the monotonicity and the finiteness can be easy to check. As for the proof of additivity of comonotonic risks, one can see Denneberg (1994). If H is a general distortion principle, i.e. $H[X] = \int_0^\infty g\,[S_X(t)]dt$, because general distortion function g is bounded, the continuity in distributions can be easily proved by the dominated convergence theorem. The necessity is proved. Thus Theorem 3.2 is proved. ■

We know that $H[X \wedge t]$ is a non-decreasing function in t, and $dH[X \wedge t]/dt$ is the increasing rate of $H[X \wedge t]$ when t increases. From (3.4) (or (3.5)) we can see that if H is a distortion premium principle, the increasing rate of $H[X \wedge t]$ only depends on the tail probability of X at t, i.e. $S_X(t) = P(X > t)$.

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