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ASTIN BULLETIN

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EDITORIAL POLICY

ASTIN BULLETIN started in 1958 as a journal providing an outlet for actuarial studies in non-life insurance. Since then a well-established non-life methodology has resulted, which is also applicable to other fields of insurance. For that reason *ASTIN BULLETIN* will publish papers written from any quantitative point of view—whether actuarial, econometric, engineering, mathematical, statistical, etc.—attacking theoretical and applied problems in any field faced with elements of insurance and risk.

ASTIN BULLETIN appears twice a year, each issue consisting of about 80 pages.

Details concerning submission of manuscripts are given on the inside back cover.

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Members of ASTIN receive *ASTIN BULLETIN* free of charge. As a service of ASTIN to the newly founded section AFIR of IAA, members of AFIR also receive *ASTIN BULLETIN* free of charge.

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EDITORIAL AND ANNOUNCEMENTS

GUEST EDITORIAL

ACTUARIES PREPARE FOR 1993!

The Member States of the European Community (EC) have set themselves a target of 1 January 1993 for completing the single market in insurance within the EC. Many still regard this as an unrealistic target, but substantial progress has been made, and continues to be made, in putting the various parts of the programme together.

Efforts to bring about a more liberal European-wide insurance market began in 1956 when the Organization of European Economic Co-operation commissioned a report from Professor Campagne, Chairman of the Verzekeringskamer, the insurance supervisory authority of the Netherlands, on whether it was possible to establish minimum standards of solvency for insurance firms. It was hoped to move towards an agreed standard of solvency, so that each country would be able to rely on supervision carried out in the other countries for the purposes of allowing insurance companies from those countries to carry on business.

After the EC was established by the Treaty of Rome in 1957, the same issue was taken up by the insurance supervisory authorities of the Community. However, it took until 1973 before the Non-Life Establishment Directive was finally agreed. This introduced the current EC solvency margin regime for non-life insurers and opened the way for insurance companies to set up branches in other EC countries, with only the branch assets and liabilities being supervised in that country. Responsibility for checking the overall solvency of the company rested with the supervisory authority of the head office country.

The Directive left unanswered the question of how the assets and the liabilities should be valued in arriving at the solvency margin. This continues to be a matter for debate.

In June 1988 the Council of Ministers adopted the Second Non-Life Directive, which provides for freedom of services for "large risks". This took effect in July 1990, since when it has been possible for an insurer based in one country of the EC to write policies directly on commercial risks throughout the EC. Full extension of this concept to personal lines business as well as to commercial risks is intended under the proposals in the Non-Life Framework Directive, which were published in September 1990 and are currently under discussion in a Working Party of the Council of Ministers. This is based on the principle of a single licence, whereby each company would be supervised only by the supervisor in the member state where the head office is situated, but would receive a licence to operate throughout the EC, either through establishment of branches or directly on a services basis. There would be no further layers of prudential supervision in the host Member States.

Much of the delay in agreeing on arrangements to provide full freedom of services has arisen because of concerns that additional protection was needed for policyholders, in the shape of minimum rules for technical reserves and for permissible assets.

Discussions are still proceeding on the draft Non-Life Framework Directive, but the proposal includes certain limitations on the proportion of the technical provisions which can be backed by different types of assets, outlaws any requirement by Member States to require insurers to invest in particular types of asset and requires "sufficient" technical provisions to be established, along the lines set out in another Directive, relating to the accounts of insurance undertakings.

The Accounts Directive, as agreed by the Council of Ministers in July 1991, although not yet ratified by the European Parliament under the co-operation procedure, sets out the types of technical provisions which should be established, in particular for unearned premiums, unexpired risks and outstanding claims. However, it is still not clear quite what is expected by the key sentence in Article 56;

> "the amount of technical provisions must at all times be such that an undertaking can meet any liabilities arising out of insurance contracts as far as can reasonably be foreseen".

What can reasonably be foreseen? Is this a charter for really cautious reserves? I am sure that was not really the intention, given the context that this directive is about reporting to shareholders. This new wording adds a further twist to the development of the concept of the adequacy of technical reserves in the EC.

Statistical methods are acceptable, although Member States may require prior approval to be given to the use of such methods. The provision must allow for claims IBNR and for claim settlement costs. Implicit discounting of provisions to take account of future investment income is not permitted (for example by not allowing for future inflation) but explicit discounting may be permitted by Member States for longer-tailed run-offs (where the average expected date for claim settlement is at least four years after the accounting date) and where the discounting is done on a recognized basis, using approved methodology and a prudent rate of interest.

Although the requirements in relation to discounting do not specifically mention actuaries, the approach required is essentially an actuarial one and could increase the demand for actuarial involvement in establishing non-life technical provisions, although some Member States may decide not to allow any discounting at all.

The Groupe Consultatif, which is the umbrella organization representing the fourteen associations of actuaries within the Member States of the EC, has lobbied actively for there to be more explicit mention of actuaries in the Accounts Directive and for the actuary to be defined as someone who is a member of one of the national associations. Unfortunately, this has not yet been successful, even in respect of the role of the actuary in life insurance. The position is even more unsatisfactory in relation to non-life insurance, where there is no explicit mention at all of actuarial involvement.

This does not mean, however, that the battle is lost. Recent developments in Canada and the United States to require actuarial certification of loss reserves should strengthen the hand of the actuarial profession in Europe in seeking to establish its special role in this field. Italy has already led the way by requiring the auditors of a non-life insurance company to obtain a certificate from an actuary on the adequacy of the technical reserves.

Of course, setting the technical reserves of a non-life insurer is not just a mathematical exercise. It requires a deep appreciation of the nature of the business, a thorough analysis of the available data, including a realistic assessment of their shortcomings, and a proper appreciation of the many uncertainties affecting the number, size and timing of future claim payments. The whole issue must be approached in a professional way and not just by the application of mechanical techniques or computer software packages. The Institute of Actuaries and the Faculty of Actuaries in the United Kingdom have recently issued a revised version of GN12, a Guidance Note on the production of actuarial reports on general insurance business, which helps to set out the framework under which an actuary should operate. It would be useful if agreement could be reached on an international basis as to a minimum set of professional requirements for an actuary producing a report on non-life business or certifying or giving an opinion on the technical reserves.

The business of insurance is becoming increasingly complex and specialist skills are needed to face the challenges which this brings. Actuaries have a great deal to offer to the managements of general insurance companies, not only in the field of loss reserving, but also in rating, experience analysis, profitability testing, designing and managing reinsurance programmes, assessing reinsurance security, investment strategy, asset/liability matching and overall financial control. Actuaries are beginning to devise models which will assist in corporate planning and in the overall financial management of the company. However, a key requirement in all of these areas is to be able to communicate well with management and to have a good appreciation of the underlying business environment.

In 1871 Cornelius Walford (an actuary himself) wrote, in the section of The Insurance Cyclopaedia discussing the term "actuary":

> "...it may seem superfluous to add that an actuary must be something more than a mathematician. That the must be a mathematician admits of no question; but with that qualification ever so largely developed, and nothing more than that, he never becomes an actuary in the sense here implied. The other qualifications are sound judgement and enlarged knowledge of business affairs - sagacity. The latter can only be obtained with and from experience; the judgement should be inherent."

There is still a lot to play for in the development of a single market in insurance in the EC. 1993 will only be the beginning. The opportunity is there, however, for actuaries to make a vitally important, professional contribution to the sound growth of the non-life insurance market, in an increasingly European environment.

CHRIS DAYKIN

THE 2nd AFIR INTERNATIONAL COLLOQUIUM

The 2nd AFIR International Colloquium was held in Brighton, England from April 16-19, 1991. The Colloquium was presented in much the same manner as the first one in Paris a year earlier, with some exceptions. The British organisers permitted the authors of papers more time to formally present their findings and thus needed to use concurrent sessions for topics judged to be away from the mainstream interest. The business meetings were organised with two invited lectures, six plenary sessions, and four sessions at which a total of nineteen concurrent meetings took place.

The guest lectures were given by Brian Quinn, an Executive Director of the Bank of England, who has special responsibility for banking supervision, and whose name has been in the newspapers subsequently because of his responsibility for the affairs of the Bank of Credit and Commerce International. His talk on some of the problems of banking supervision was prophetic.

The other guest lecture was given by Professor Michael Brennan of UCLA, who gave an extensive review of the economic fundamentals that underlie much of the work of modern financial economists, and hence of actuaries aspiring to contribute to AFIR.

Since it was impossible to attend all the sessions at which papers were presented, I have chosen to review those that I personally found to be of greatest interest. By following this route, I am unable to do justice to many of the valuable contributions, but they can be found in the full set of papers published in the four volumes of Colloquium proceedings. At the beginning of the first volume is a 20-page introductory review by David Wilkie, Chairman of the Scientific Committee.

Each session followed the same format. A designated "opener" set the stage for the subject area, providing some necessary background, outlining significant issues and developments, and providing a brief overview of the various papers. Authors of the papers were then given an opportunity to highlight their work and offer comments and results not in their published material. After questions from the floor, a designated "closer" summarised and opined on the various contributions from the authors and others.

The first plenary session covered banking and credit and included papers on the French banking system where many actuaries are employed. Of particular interest to me was the paper "Credit Risk Research: Private Placement Bonds and Commercial Mortgage Loans" by Gery Barry, who heads a Society of Actuaries research group studying this topic. Their research group has struggled with the issue of how to define a credit risk event and how to measure credit risk loss for a private transaction that tends to be renegotiated when the borrower experiences financial difficulty. European institutional investors have generally avoided credit risk, but US institutional investors seem prone to take it on and then live or die by managing it successfully or mismanaging it. The recent difficulties and failures of many US life insurance companies underscore the importance of understanding this subject better. A related risk for these types of investments, not addressed by the Society of Actuaries research group but highly topical in the US, is "illiquidity risk" that threatens the solvency of a financial institution suffering from a public crisis of confidence and thus exposed to a severe cash flow drain.

The second and third plenary sessions covered the topic of interest rate and yield curve models. Two papers caught my attention. Robert Reitano delivered a very understandable lecture on the risk of non-parallel shifts in the yield curve. He introduced the concept of "partial durations" that measure an asset's or liability's market value sensitivity to a shift in a single segment of the yield curve. Summing all partial durations gives the familiar "total" duration measure. Boulier and Sikorav tested whether the yield curve for French government bonds obeys the Ho and Lee model of interest rate dynamics. Although the Ho and Lee model performs reasonably well, the authors emphasise the need to go beyond in two areas: in the real world, short-term yields are more volatile than long-term yields and yield curve dynamics depend on more than what happens to the short-term interest rate. Also, for the originally-published form of the Ho and Lee model, there is the problem of negative interest rates occurring with significant probability.

One of the interesting concurrent sessions featured papers on non-linear models and chaos theory. Chaos theory has captured the imagination of many actuaries and indeed might have intrigued Albert Einstein by offering an alternative theoretical foundation for quantum mechanical phenomena. Many philosophical questions arise when the relationships between stochastic systems and non-linear deterministic systems are considered. For example: what errors, if any, will occur when a system that is actually non-linear, deterministic, and chaotic is modelled as if it were purely stochastic? The standard tests for detecting the presence of non-linear deterministic behaviour assume a very long data series. Maddocks, Nisbet, Nisbet and Blythe used such a test, based on the concept of "correlation dimension", and found evidence of deterministic behaviour in weekly data for the Financial Times All Share Index (1965-1989), the Dow Jones Index (1969-1986), the Standard and Poor Composite Index (1965-1986), and the Nikkei Index (1966-1989). There remains a question, however, as to whether their data transformations introduced spurious correlations, and a question as to whether the observed behaviour is chaotic (with a *strange* attractor present), not merely deterministic. We await their further results.

In my opinion, the real highlight of the Colloquium was the presentation of two invited papers by Andrew Smith: one on option pricing formulas and another on the use of martingales in actuarial work. In his presentation at the Colloquium, Dr Smith treated the audience to simple examples illustrating a few of the points raised in his papers. His second paper demonstrated how easily a number of key results found in actuarial science can be derived from basic properties of martingales. That paper should be studied carefully by all actuaries aspiring to be of "the third kind," according to Professor Bühlmann's taxonomy. Professor Neave's carefully prepared and thoughtfully presented introduction to the session on theoretical developments also deserves special mention. It set the stage perfectly for Smith's martingale paper and for the related paper "Generalised Arrow Pricing to Understanding Financial Markets" by Ami, Kast and Lapied.

Brighton does not have the same attractions to the visitor as Paris, but nevertheless the organisers had arranged a wide programme of visits and entertainments for accompanying persons during the day and for participants too in the evenings. These included visits to the attractive towns and villages of Sussex, two castles steeped in English history, a vineyard (yes, there is English wine of quite respectable quality), theatrical performances, and even, for those who prefer gambling to insurance, greyhound racing, at which the bookies seemed to be confounded by a large number of punters, under instructions from Hans Bühlmann, who all chose to put their money on the same dog; I cannot say whether their performance in this field was better or worse than that of investment managers.

There will not be an AFIR Colloquium in 1992 because the 24th International Congress of Actuaries in Montreal is accorded the spotlight for the year. The 3rd AFIR Colloquium will be held in Rome in 1993, an event we eagerly await after the outstanding achievements turned in at Paris and Brighton.

JAMES A. TILLEY

XXIII ASTIN COLLOQUIUM, STOCKHOLM, 1991

The 23rd ASTIN Colloquium was held from 30th June to 4th July 1991 in Stockholm ("beauty on water"). Over 200 actuaries from more than 20 countries attended. The Colloquium began on Sunday, 30th June, with the reception in Berns Congress Centre. The working sessions were started off on Monday morning with a presentation by Prof. Ragnar Norberg (University of Copenhagen) on "A Continuous Time Approach to the Prediction of the Total Outstanding Claims of a Non-Life Insurance Business — a Strategy for Solvency Control Based on the Break-Up Point of View". The rest of the morning and the afternoon were devoted to working sessions on topic number 3 "Modern Statistical Techniques". Bengt von Bahr and Arne Sandström opened these sessions by a brief survey of the papers on this topic. Then each author had 10 minutes to present his paper in greater detail. The remaining time was open for discussion, which was very animated. In the evening, the Colloquium participants were invited to a buffet dinner and a tour of the Stockholm City Hall, in which the annual Nobel Prize awards banquet is held.

An all-day excursion to the Gripsholm Castle was on the program for Tuesday, the very day on which midsummer weather set in Sweden. In two ships, the participants were taken directly from the centre of Stockholm to the outlet of Lake Mälaren. After having passed through Stockholm the journey continued through a majestic forest and lake countryside to the Gripsholm Castle, situated directly on the lake some 60 km inland. Lunch had been served on the ships, so that the afternoon was left open for a tour of the numerous rooms of the 16th Century Castle with its impressive portrait gallery of important figures from Swedish history. The lovely location of the Castle was also an invitation to linger on, which only made it all the more difficult to get onto the waiting busses for the trip back.

Wednesday was again devoted exclusively to working sessions. These were introduced with a presentation by Karl-Olof Hammarkvist, Managing Director of a large Swedish direct insurance and reinsurance company, concerning a "Professional View on Reinsurance". The subsequent working sessions dealt with Colloquium topics number 2 ("High Tech Reinsurance") and 1 ("The Use of Financial Theory in Insurance"). The "rapporteurs" were Björn Ajne, Malcolm Campbell and Björn Palmgren. The afternoon was concluded with the General Meeting, for which a special election to the ASTIN Committee was scheduled in addition to regular items on the agenda. Thomas Mack (Germany) and Ermanno Pitacco (Italy) were elected to the ASTIN Committee which now consists of 13 members. In the evening, the traditional Colloquium dinner was served on the "Operaterrassen" with a wonderful view of the old part of the town and the Royal Castle of Stockholm.

The topic of the last working session on Thursday morning was "Speakers' Corner". Here papers were presented and discussed which had only been submitted shortly before the beginning of the Colloquium and could therefore not be included in the Colloquium publication, as it had already been sent to the participants in advance.

So much for the formal "contours" of the Colloquium, which had been extremely well organized by the Swedish Society of Actuaries, headed by Alf Guldberg. The members of the organizing committee (Sven Astrand, Hans Ekhult, Bengt Langhed, Peter Lindström, Gunilla Lisén, Harry Wide) and the scientific committee (Björn Ajne, Bengt von Bahr, Anders Blommé, Malcolm Campbell, Björn Palmgren, Arne Sandström) deserve special thanks.

The following is a brief summary of the contents of all papers presented. These summaries are of course influenced by the capabilities of the authors of this report and may therefore be somewhat biased. The order of the summaries is also the order in which they were presented at the Colloquium.

THOMAS MACK, KLAUS-PETER MANGOLD

LIST AND SUMMARY OF PAPERS

Topic 3: Modern Statistical Techniques

Björn AJNE and Arne SANDSTRÖM: *New Standard Regulations Regarding Allocation of the Safety Reserve in Sweden.*

The authors explain in detail how the current Swedish safety reserve regulation came into being. For every line of business, there is a maximum tax-free value admitted for the safety reserve which consists of a percentage of premium income and a percentage of the claims reserve (due to the settlement risk), each of these being for own account. The percentages had been fixed in such a way that these total approximately 4.5 times the standard deviation of the loss ratio. This value was established on the basis of confidence interval considerations (99% security) calculated from statistics. Allowance was also made for the positive correlation of loss ratios for two consecutive years observed. The paper contains many illustrative graphical displays.

John BORREGAARD, Chresten DENGSOE, Joakim HERTIG, Niels JESPERSEN and Christian Roholte LARSEN: *Equalization Reserves: Reflections by a Danish Working Party.*

Since there is no fixed regulation in Denmark governing the level of equalization reserves, the Danish Association of Actuaries formed a relevant working party to deal with this question. In the present paper, members of this party outline the method of balanced limits for setting up an upper limit u and a lower limit l for the equalization reserve. If the annual claims amount x is less than l, the difference $l-x$ is transferred to the equalization reserve. If x is greater than u, then $x-u$ is transferred from the reserve. On the basis of three sample calculations for Windstorm, Fire and Motor insurance, the authors demonstrate this method on the basis of theoretical distributions with realistic parameters. Furthermore, they propose setting the reserve at a level at which the probability of the reserve being completely exhausted within 5 or 10 years is below a given value (for example 15%). The levels which result from this for the examples cited are also shown.

Arne SANDSTRÖM: *On Moment Corrections when Data are Grouped into Non-Equidistanced Intervals.*

For equidistantly grouped data Sheppard's correction for calculating the moments can be applied. The author proposes formulae for the non-equidistant case, which use only group frequencies and lengths and mid-points of intervals. For equidistantly grouped data, these formulae lead to higher values than Sheppard's formula. If the mean value per interval is also known, upper and lower limits can also be given for the higher moments. The author indicates that it is thus possible to demonstrate a superiority of this formula over Sheppard's correction.

Rolf LARSSON and Erik HEVRENG: *On the Estimation of the Time Development of the Risk Premium in Non-Life Insurance.*

The authors fit mathematical distribution models to empirical claims data (number of policies, amount of every individual claim for 11 years and 4 different lines of Property business). For the inflation-adjusted claim amount, they assume a lognormal distribution with a scale parameter depending linearly on the year of observation. A Poisson distribution is assumed for the claims number per year, whose mean value is modelled as a product of the number of policies and a logarithmic linear yearly trend. In numerous tables, the authors give a year-by-year comparison between the observed value and the fitted value (for claims number, claims amount and risk premium per policy) and place special emphasis on the examination of the significance of modelled trends. Since the authors originally only had access to the amounts of the individual claims grouped by classes of claims amount, they originally estimated the parameters by means of the *EM*-algorithm briefly described in an appendix. The resulting estimators are also included in the tables. The authors therefore regard the comparison of the results of both methods as being one of the main issues of their paper.

Erik ELVERS: *A Note on the Generalized Poisson Distribution.*

The author fits the Generalized Poisson Distribution introduced by Consul to the data observed in an Automobile Third Party Liability insurance portfolio. These data do not represent the number of accidents, but the number of casualties (injuries or deaths) per accident (over a period of several years and for several classes of motor vehicles). In almost every case, the chi-square test rejects the distribution hypothesis.

Jean LEMAIRE: *Negative Binomial or Poisson-Inverse Gaussian?*

On the basis of six data sets not related to insurance, the author compares the fit of the Poisson, the Negative Binomial and the mixed Poisson-Inverse Gaussian distributions. The parameters are estimated with both the maximum likelihood method and the method of moments. The best fits were obtained with the Negative Binomial distribution.

Ermanno PITACCO: *An Inference Model for Risks with Variable Claim Frequency Rate.*

The author generalizes the classical Poisson-Gamma model for the purpose of experience rating in Sickness insurance. For an insured life aged y and a policy term of several years, for the conditional claims number variable $X(y+h)$ a Poisson distribution with parameter $\theta t(y+h)$ is assumed, where $t(y) < t(y+1) < t(y+2) < \ldots$ If the risk parameter θ follows a Gamma distribution, one can procede with a calculation analogous to the classical case. From this, the author analyses the properties of an experience rating system on the assumption that the individual claims amounts are independent of both the claims number and the parameter θ.

Lourdes CENTENO and João Manuel ANDRADE E SILVA: *Generalized Linear Models under Constraints.*

After giving a brief description of the generalized linear models and the relevant estimation algorithm, the authors address the problem that in some applications the coefficients of the independent variables should satisfy certain linear constraints. In the authors' example which is based on Portuguese Automobile Third Party Liability insurance statistics, for example, this always occurs where no distinction is to be made between two variables or if one wishes to establish a certain fixed linear relationship. The authors offer two possible solutions: either reformulating the model or changing the algorithm. In the authors' opinion, the latter is easier as long as there are several step-by-step changes in the constraints, as is often the case in practice.

Pierre PETAUTON: *Une Estimation Naturelle des Paramètres Structuraux dans les Modèles de Crédibilité.*

The credibility estimator

$$\hat{X}_i = z_i \overline{X}_{i+} + (1-z_i)m$$

in the models of Bühlmann or Bühlmann and Straub includes the unknown parameters z_i and m, which must be estimated from the data X_{ij}. To this end, the author proposes a new approach. For a fixed year j, he eliminates the relevant observations X_{ij}, $i = 1, 2, \ldots$, and applies the above credibility formula to the remaining data, which means,

$$\hat{X}_i(j) = z_i \overline{X}_{i+}(j) + (1-z_i)m$$

where $\overline{X}_{i+}(j)$ is the mean of the data without year j. This can be applied to each year j of every risk i. The new approach is then to choose the parameters z_i and m in such a way that they minimize the weighted quadratic deviation

$$\sum_{i,j} g_{ij}(X_{ij} - \hat{X}_i(j))^2$$

where the weights g_{ij} should be in inverse proportion to Var (X_{ij}). This idea can be easily carried out in the simple Bühlmann model and, with a small exception, produces the usual estimators. In the Bühlmann-Straub model, the author must alter the expression to be minimized in order to arrive at explicit solutions.

Erhard KREMER: *Large Claims in Credibility.*

The author implements ideas from robust statistics in the credibility theory in order to cope with the disturbing influence of outliers (large claims or risks) which disrupt the homogeneity assumption. To this end he proposes replacing the individual loss experience contained in the credibility formula with robust estimators. This is done by applying so-called *M*- or *L*-estimators. For three examples, the so-called Huber credibility *M*-estimator, the trimmed and the winsorized credibility *L*-estimators he demonstrates robustness, which is not a property of the classical credibility estimator. The author does not directly truncate individual claims, but gives large claims less weight in the estimation or does not even take them into consideration at all. In the same manner, for estimating the structural parameters, he does not use the observations themselves, but rather robust statistics of the observations in order to obtain a kind of robust empirical credibility estimator. In the last chapter the author extends his approach to the regression credibility model as well.

Arne EYLAND: *Classification of Passenger Cars in a Multiplicative Rating Model using Recursive Credibility Estimation.*

The author develops an evolutionary regression credibility model, whose basic features are derived from a paper of SUNDT (1987). In this model, the non-observable parameter which characterizes each single risk may vary over time (time-heterogeneous model) and its risk premium is regressively dependent on observable technical variables. The multiplicative tariff used by a Norwegian insurance company, in which the tariff variable "car model" is determined by the technical parameters engine power, price and weight, serves as an example. The problem of having to invert the covariance matrix of the observations for determining the credibility estimator is reduced in its dimension by using linear sufficient statistics. Furthermore, the selection of a special recursive covariance structure in the model makes it possible to calculate the credibility estimator recursively. The author also treats the special case of risk parameters that are constant in time (time-homogeneous model) which he believes to be less realistic. In a third, so-called time-heterogeneous model for two portfolios, the author describes the situation in which the overall portfolio can be broken down into two sub-portfolios, each of these being time-heterogeneous and transition from one sub-portfolio to the other being possible. This would be useful in a case in which a technical variable ("price") is "suddenly" no longer observable (e.g. because a car model is no longer sold) and thus has to be subsumed under the unobservable risk parameter. The author purposely excludes the problem of estimating the (numerous) structural parameters and indicates that this will be investigated in a later publication.

Topic 2: High Tech Reinsurance

William S. JEWELL: *The Value of Information in Forecasting Excess Losses.*

The author analyses parameter estimation for rating an excess-of-loss treaty. Usually, the reinsurer has at his disposal only information on losses exceeding a certain limit. The author explains the problem resulting from this limited knowledge for the case of a Poisson distributed number of losses with shifted Pareto distributed loss amounts. By means of many impressive graphs he shows that the likelihood function of the two distribution parameters (the Poisson parameter and the Pareto shift parameter — the shape parameter being treated as known) has a ridge which allows only a very uncertain parameter estimation, if any at all. This situation can only be improved if at least the number of all losses from the ground up is available. The paper also addresses questions like the direct estimation of the excess-of-loss fair premium.

Erhard KREMER: *A (New) Nonparametric Method for XL-Rating.*

The author premises the fact that claims expectancy under an excess-of-loss reinsurance treaty can be represented as the product of the total claims expectancy of the insured portfolio and a tail probability

$$\int_P^\infty g(x)\,dx$$

above the priority P. In view of the infinite integration range, he proposes using the approximate procedure of GRAY and LEWIS (1971) for calculating this tail probability. This procedure uses the quotient of two determinants of the first k derivatives of g at the priority P. In order to avoid parametrically estimating the density g with all its derivatives from the data, the author recommends using the kernel estimator. For this, it is necessary to choose a specific kernel function, for which the author gives a concrete recommendation.

Erhard KREMER: *A Note on XL-Rating in Earthquake Insurance.*

Referring to a paper by MAKJANIC (1980), the author proposes use of the generalized exponential distribution for the magnitude frequency instead of the traditional exponential distribution. The author also offers a modification of Makjanic's parameter estimation procedure.

Patrik DAHL: *Some Reflexions on Contingent Premium Payment Plans.*

The author investigates reinstatement agreements in non-proportional reinsurance on the assumption that no partial claims in respect of the reinsurer's liability are possible and that both parties evaluate the treaty with an exponential utility function. Furthermore, the author makes allowance for the possibility that insurer and reinsurer may calculate with differing claims probabilities. Allowance is also to be made for the resulting cash flow as the premium of the reinstatement agreement depends on claims experience. In another part of the paper, the author argues quite generally for use of the Polya process as a model for the claims number.

Marc-Henri AMSLER: *Réassurance du Risque de Ruine.*

The author proposes for pension funds, for example, that a reinsurance treaty with a term of several years be applied, under which the reinsurer is only obliged to pay in the event of ruin and only for the amount of the deficit. Here the reinsurer's payment needs only to be a loan, since, in view of the positive security loading, the reinsured's contingency reserve will tend to become higher and higher. At expiry of the treaty, the reinsurer is obliged to pay if the reinsured's contingency reserve is lower than at the outset. The author calculates the resulting reinsurance premiums in two examples.

Thomas MACK: *Claims Reserving: The Direct Method and its Refinement by a Lag-Distribution.*

The author describes a simple procedure for claims reserving, which has some similarity to the chain ladder method. It is however also possible to show that the procedure represents the maximum likelihood estimator if a gamma distribution is assumed. The procedure can be used in a more realistic manner if it is only applied for the average claims amount per cell. In this case, the triangle of the number of payments must also be completed to make a square. To this end the author proposes to fit a (truncated) lag-distribution to the observed numbers of payments per year of occurrence. This proved to be more suitable than the chain ladder method in the practical example given by the author.

Gunnar BENKTANDER: *A Special Case of Variable Rates in Excess of Loss.*

In order to make a variable premium with a minimum of m and maximum of M produce the same average result as a constant premium E, m and M must be fixed in a specific way. The author investigates this question for excess of loss reinsurance, basing his investigation on a Poisson-Pareto distribution model. On the basis of the simulations conducted by Christer Möller, he demonstrates that the "accordion rule" $mM = E^2$ closely approximates the simulated results. The author also raises the question as to whether there are distributions for which this rule applies exactly. A paper partially responding to this question was distributed by Björn Sundt.

Mette RYTGAARD: *Variations on Typical Excess of Loss Covers.*

The author initially investigates an excess of loss cover which provides a reinstatement of the cover for an additional premium only. In this case, the calculation of the reinsurance premium becomes simpler if one directly examines the reinsurer's net payment after deduction of the claims-related additional premiums. This is also shown to be true for other agreements, such as, for the variable premium. Especially in the case of the variable premium however the net payment of the reinsurer might also be negative due to the multiplicative premium loading, for which reason the author proposes a form of the variable premium which avoids this disadvantage.

Topic 1: The Use of Financial Theory in Insurance

David SANDERS: *Risk Theory and Capital Allocation.*

The author uses the formulae of the classical ruin theory to derive a link between the premium loading, the variance of the total claims amount, the capital allocated to support the business and the probability of ruin under the assumption that the total claims amount has a normal distribution. Using the required rate of return on capital as a part of the premium loading he obtains a functional relationship between premium and capital which has a minimum point. The author analyses the minimum condition from the standpoint of its practical implications and gives some numerical examples.

Speaker's Corner

Siegfried KUON, Michael RADTKE and Axel REICH: *The Right Way to Switch from the Individual Risk Model to the Collective One.*

The overall loss $S_{\text{ind}} = X_1 + \ldots + X_n$ in the individual model of n independent and not necessarily identically distributed risks $X_i \geqslant 0$ is usually approximated by a collective model $S_{\text{coll}} = Z_1 + \ldots + Z_N$ with a claims number variable N and i.i.d. claims amounts $Z_j > 0$. Here the distribution of the Z_j is

$$p(Z_j \leqslant x) = \sum_{i=1}^{n} q_i p(X_i \leqslant x | X_i > 0) \Big/ \sum_{i=1}^{n} q_i$$

with $q_i = p(X_i > 0)$. If one chooses a Poisson distribution for N so that $E(S_{\text{coll}}) = E(S_{\text{ind}})$ applies, it is known that $\text{Var}(S_{\text{coll}}) > \text{Var}(S_{\text{ind}})$. The authors show that this is also the case if one chooses the negative binomial distribution or the binomial distribution for N (at least if not all $E(X_i)$ are equal).

Furthermore, it is shown that in the case of a growing portfolio, i.e. $n \to \infty$, $\text{Var}(S_{\text{coll}})/\text{Var}(S_{\text{ind}})$ does not converge toward 1 and that, in like manner, neither distribution functions nor percentile premiums converge toward one another either. Finally, it is shown that, on the one hand, by passing from Z_j to aZ_j with $a < 1$, the variances can be made equal, but, on the other hand, that the relevant stop-loss net premiums will still differ from one another significantly.

Björn PALMGREN: *Financial Risk in Insurance.*

The author gives a brief survey of a number of questions on the financial risk of an insurance company, particularly from the standpoint of the supervisory authority, e.g., "When is a mix of assets efficient or at least acceptable?", "How great is the influence of transaction costs?", etc. The author sees it as an important task of the supervisory authority to investigate the efficiency of capital investments and conduct sensitivity tests on company solvency.

Ermanno PITACCO: *Selection and Experience Rating in Health Insurance.*

This paper is a sequel to the paper by the same author discussed earlier under topic 3. The age-related parameters $t(y+h)$ mentioned in the earlier paper are lower under the influence of medical selection than without selection. This also influences the a posteriori expected value. The author gives a numerical example and also shows how a given selection assumption affects different experience rating system concepts.

Erhard KREMER: *The Total Claims Amount of Largest Claims Reinsurance Treaties Revisited.*

The author takes up the total claims amount of the reinsurer under the generalized largest claims reinsurance which he himself introduced in an earlier paper. The purpose of this paper is to provide a formula for the distribution density of the total claim where the individual claims are either exponentially or uniformly distributed. To this end, the author first investigates the special case in which exactly n claims are incurred. The resulting expressions are not simple and must still be mixed with the occurrence probabilities.

Rob KAAS, Marleen VANNESTE and Marc GOOVAERTS: *Maximizing Compound Poisson Stop-Loss Premiums Numerically with Given Mean and Variance.*

In contrast to earlier papers, the authors limit themselves here to total claims amounts with a compound Poisson distribution. Furthermore, they assume that the single claim amount is limited and arithmetic. They provide a numerical solution which is arrived at by means of the gradient method, which however shows that there are many local maxima. In addition to this, the maximum frequently deviates only very slightly from the value that is produced with a two-point distribution approximation for the single claim amount. The authors therefore feel that one should be satisfied with the procedure outlined in another paper by KAAS (1991) which uses two-point distributions.

Björn SUNDT: *On Some Extensions of Panjer's Class of Counting Distributions.*

The author generalizes the class R_1 of the claims number distributions with

$$p_n = (a+b/n)\,p_{n-1}$$

to class R_k with the recursion property

$$p_n = \sum_{i=1}^{k} (a_i+b_i/n)\,p_{n-i}.$$

Distributions from R_k are a result of convoluting distributions from R_1. The author shows that the Panjer algorithm can also be translated to distributions of class R_k. A formula for convoluting any two distributions of class R_1, which is known to include the binomial, Poisson and negative binomial distributions, is a by-product of this.

Björn AJNE: *A Note on the Additivity of Chain-Ladder Projections.*

The author investigates the problem of when the separate application of the chain ladder method to 2 run-off triangles produces the same projections as the single application to the sum of both triangles and he outlines a necessary and sufficient condition for this. He also conjectures as to when separate application produces lower reserves.

Jon HOLTAN: *Bonus Made Easy.*

To begin with, the author points out a few weaknesses of the traditional bonus-malus systems, for example, that the amount of a claim has no influence. In order to get around these weaknesses, he proposes introduction of a relatively high deductible as a substitute for the bonus-malus system. With a high deductible, a claim would have to be financed in advance by the insurer. In order to find the optimum deductible amount and mode of loan repayment, he applies a special loss function. The author sees a potential practical difficulty in the fact that, with this solution, the credit risk of the insured might be a factor to be considered.

Heikki BONSDORFF: *On the Convergence Rate of Markovian Bonus-Malus Systems.*

Under certain conditions, transition probabilities of a Markovian bonus-malus system converge toward stable transition probabilities. For such convergent Markov chains, the expression "convergence rate" is well-defined and can be calculated with the help of the eigenvalues of the transition matrix. The author calculates this convergence rate for the Dutch, Swiss and Finnish bonus-malus system, the results of which show that the latter converges most quickly.

XXIV ASTIN COLLOQUIUM

The 24th ASTIN Colloquium will be held in the United Kingdom

The intention is to base the colloquium at St. John's College, Cambridge, for the period 25th-29th July 1993.

Further details and the preliminary registration form will appear in the IAA Bulletin, issue April 1992.

ARTICLES

RISK THEORY WITH THE GAMMA PROCESS

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Abstract

The aggregate claims process is modelled by a process with independent, stationary and nonnegative increments. Such a process is either compound Poisson or else a process with an infinite number of claims in each time interval, for example a gamma process. It is shown how classical risk theory, and in particular ruin theory, can be adapted to this model. A detailed analysis is given for the gamma process, for which tabulated values of the probability of ruin are provided.

Keywords

Aggregate claims; compound Poisson process; gamma process; infinite divisibility; risk theory; ruin probability; simulation; stable distributions; inverse Gaussian distribution.

1. Introduction

In classical collective risk theory, the aggregate claims process is assumed to be compound Poisson (Panjer and Willmot, 1984). Here we shall examine a more general model for the aggregate claims process: processes with independent, stationary and nonnegative increments. Such a process is either compound Poisson or else a process with an infinite number of claims in any time interval. The most prominent process with this intriguing property is the gamma process.

Since the process under consideration is either a compound Poisson process or a limit of compound Poisson processes, its properties can be derived from the basic properties of the compound Poisson process. The general results are derived in Section 2 (for the aggregate claims process) and Section 6 (for the probability of ruin). The gamma process is examined in detail in Sections 3, 4 and 5 (for the aggregate claims process) and Sections 7 and 8 (for the probability of ruin).

2. PROCESSES WITH INDEPENDENT, STATIONARY AND NONNEGATIVE INCREMENTS

Let $Q(x)$ be a nonnegative and nonincreasing function of x, $x > 0$, with the properties:

$$Q(x) \to 0 \quad \text{as} \quad x \to \infty$$

and

$$\int_0^\infty Q(x)\,dx < \infty. \tag{2.1}$$

Condition (2.1) can also be written as

$$\int_0^\infty x[-dQ(x)] < \infty,$$

which, if $q(x) = -Q'(x)$ exists, becomes

$$\int_0^\infty xq(x)\,dx < \infty.$$

Such a function $Q(x)$ defines an aggregate claims process $\{S(t)\}_{t\geq 0}$ in the following way. For each $x > 0$, let $N(t; x)$ denote the number of claims with an amount greater than x that occur before time t; let $S(t; x)$ be the sum of these claims. We assume that $\{N(t; x)\}_{t\geq 0}$ is a Poisson process with parameter $Q(x)$ and that $\{S(x; t)\}_{t\geq 0}$ is a compound Poisson process with Poisson parameter $Q(x)$ and individual claim amount distribution

$$P(y; x) = \begin{cases} 0 & y \leq x \\ \dfrac{Q(x)-Q(y)}{Q(x)} & y > x. \end{cases} \tag{2.2}$$

The process $\{S(t)\}$ is defined as the limit of the compound Poisson processes $\{S(t; x)\}$ as x tends to 0.

We write

$$Q(0) = \lim_{x\to 0} Q(x).$$

We need to distinguish two cases: $Q(0) < \infty$, and $Q(0) = \infty$. In the first case, $\{S(t)\}$ is a compound Poisson process with Poisson parameter $Q(0)$ and individual claim amount distribution

$$P(y) = 1 - \frac{Q(y)}{Q(0)}, \qquad y \geq 0. \tag{2.3}$$

This is the classical model for collective risk theory. Conversely, every compound Poisson process, given by Poisson parameter λ and individual claim amount distribution $P(y)$, is of this type if we set

$$Q(y) = \lambda[1-P(y)], \qquad y \geq 0. \tag{2.4}$$

In the second case, $\{S(t)\}$ is the limit of compound Poisson processes, but is not a compound Poisson process itself, because the expected number of claims per unit time, $Q(0)$, is infinite. Indeed, with probability one, the number of claims in any time interval is infinite. Nevertheless, $S(t)$ is finite, as the majority of the claims are very small in some sense. In both cases, $Q(y)$ is the expected number of claims per unit time with an amount exceeding y.

Since $\{S(t)\}$ is the limit of $\{S(t;x)\}$ as x tends to 0, we can use well-known results for the compound Poisson process to obtain results for the process $\{S(t)\}$. For example, it follows from

$$\begin{aligned} E[S(t;x)] &= tQ(x) \int_0^\infty [1-P(y;x)]\,dy \\ &= txQ(x)+t\int_x^\infty Q(y)\,dy \end{aligned}$$

that

$$E[S(t)] = t\int_0^\infty Q(y)\,dy = t\int_0^\infty y[-dQ(y)]. \tag{2.5}$$

To get the Laplace transform, we start with

$$\begin{aligned} E[e^{-zS(t;x)}] &= \exp\left\{tQ(x)\left[\int_x^\infty e^{-zy}\,dP(y;x)-1\right]\right\} \\ &= \exp\left\{t\int_x^\infty [e^{-zy}-1][-dQ(y)]\right\}. \end{aligned}$$

Letting $x \to 0$, we obtain

$$E[e^{-zS(t)}] = \exp\left\{t\int_0^\infty [e^{-zy}-1][-dQ(y)]\right\}. \tag{2.6}$$

The process $\{S(t)\}$, defined by the function $Q(x)$, has independent, stationary and nonnegative increments, and $E[S(t)] < \infty$. The converse is true in the following sense. Every process $\{X(t)\}$ with these properties is of the form

$$X(t) = S(t)+bt,$$

where $\{S(t)\}$ is a process of the type presented above and b is a nonnegative constant. This is a consequence of the connection between processes with independent and stationary increments and infinitely divisible distributions, and the characterization of infinitely divisible distributions with nonnegative support (FELLER, 1971, p. 450, Theorem 2; p. 571, formula (4.7)).

3. THE GAMMA PROCESS

Assume that the function $Q(x)$ is differentiable and that $-Q'(x)$ is

$$q(x) = \frac{a}{x} e^{-bx}, \qquad x > 0, \tag{3.1}$$

where a and b are positive constants. Let $\{S(t)\}$ be the associated aggregate claims process. In a time interval of length t, the expected number of claims with an amount exceeding x is

$$tQ(x) = at \int_x^\infty \frac{e^{-by}}{y} dy.$$

Since $Q(0) = \infty$, there is an infinite number of claims in each time interval. By (2.5) the expected aggregate claims in a time interval of length t are

$$E[S(t)] = t \int_0^\infty yq(y)\, dy = at \int_0^\infty e^{-by}\, dy = \frac{at}{b}. \tag{3.2}$$

To obtain the distribution of $S(t)$, we compute its Laplace transform by (2.6):

$$\begin{aligned} E[e^{-zS(t)}] &= \exp\left\{ t \int_0^\infty [e^{-zy} - 1]\, q(y)\, dy \right\} \\ &= \exp\left\{ at \int_0^\infty \frac{e^{-(z+b)y} - e^{-by}}{y} dy \right\} \\ &= \left(\frac{b}{z+b} \right)^{at}. \end{aligned} \tag{3.3}$$

To verify the last step, consider the function

$$\varphi(z) = \int_0^\infty \frac{e^{-(z+b)y} - e^{-by}}{y} dy;$$

observe that $\varphi(0) = 0$ and $\varphi'(z) = -(z+b)^{-1}$. Formula (3.3) shows that the distribution of $S(t)$ is gamma, with shape parameter $\alpha_t = at$ and scale parameter $\beta_t = b$. Hence the process $\{S(t)\}$ is called a *gamma process*.

A gamma process with $a = b = 1$ is called a *standardized gamma process*. For an arbitrary gamma process with parameters a and b, we may set $t^* = at$ and $S^*(t^*) = bS(t)$. It follows from (3.3) that

(3.4) $$E[e^{-zS^*(t^*)}] = \left(\frac{1}{z+1}\right)^{t^*}.$$

Thus the transformed process $\{S^*(t^*)\}$ is a standardized gamma process.

The gamma process, given by (3.1), can be imbedded in a larger family of processes given by

(3.5) $$q(x) = ax^{\alpha-1}e^{-bx}, \qquad x > 0,$$

with $-1 < \alpha < \infty$. We note that

(3.6) $$\int_0^\infty yq(y)\,dy = a\int_0^\infty y^\alpha e^{-by}\,dy = \frac{a}{b^{\alpha+1}}\Gamma(\alpha+1)$$

is indeed finite.

For $\alpha > 0$,

(3.7) $$Q(0) = \int_0^\infty q(y)\,dy = \frac{a}{b^\alpha}\Gamma(\alpha)$$

is finite. Hence $\{S(t)\}$ is a compound Poisson process, with Poisson parameter λ given by (3.7) and claim amount density

(3.8) $$p(x) = \frac{q(x)}{\lambda} = \frac{b^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-bx}, \qquad x > 0,$$

which is a gamma density.

For $-1 < \alpha \le 0$, $Q(0) = \infty$. When $\alpha = 0$, we have the gamma process. To determine the probability density function $f(x, t)$ of $S(t)$ for $-1 < \alpha < 0$, we apply formula (2.6),

(3.9) $$E[e^{-zS(t)}] = e^{ta\varphi(z)},$$

with

(3.10) $$\varphi(z) = \int_0^\infty (e^{-zy}-1)\,y^{\alpha-1}e^{-by}\,dy.$$

From $\varphi(0) = 0$ and

(3.11) $$\varphi'(z) = -\int_0^\infty y^\alpha e^{-(z+b)y}\,dy = -\frac{\Gamma(\alpha+1)}{(z+b)^{\alpha+1}}$$

we obtain

(3.12) $$\varphi(z) = \frac{\Gamma(\alpha+1)}{\alpha}\left[\frac{1}{(z+b)^\alpha} - \frac{1}{b^\alpha}\right]$$

$$= \Gamma(\alpha)\,[(z+b)^{-\alpha} - b^{-\alpha}].$$

(Note that (3.12) is also valid for $\alpha > 0$; in this case it can derived by first expressing (3.10) as the difference of two convergent integrals.) For simplicity, assume $a = -1/\Gamma(\alpha)$ and $b = 1$. Write $\beta = -\alpha$. Then (3.9) becomes

$$E[e^{-zS(t)}] = \exp\{t[1-(1+z)^{\beta}]\}. \tag{3.13}$$

Recall the *stable distribution* of order β that is concentrated on the positive axis (FELLER, 1971, Sections XIII.6 and XIII.7). Let $g_\beta(x)$ denote its probability density function. Its Laplace transform is

$$\int_0^\infty e^{-zx} g_\beta(x)\,dx = e^{-z^\beta}.$$

Hence the Laplace transform of the function

$$t^{-1/\beta} g_\beta(t^{-1/\beta}x), \qquad x > 0,$$

is $\exp(-tz^\beta)$. Finally, it follows from (3.13) that the probability density function of $S(t)$ is

$$f(x, t) = e^{t-x} t^{-1/\beta} g_\beta(t^{-1/\beta}x), \qquad x > 0. \tag{3.14}$$

For $\beta = 1/2$, a closed form expression for the stable density is available,

$$g_{1/2}(x) = \frac{1}{2\sqrt{\pi}\, x^{3/2}} \exp\left(-\frac{1}{4x}\right), \qquad x > 0, \tag{3.15}$$

and (3.14) becomes

$$f(x, t) = \frac{t}{2\sqrt{\pi}\, x^{3/2}} \exp\left[-\frac{(2x-t)^2}{4x}\right], \qquad x > 0, \tag{3.16}$$

which is the probability density of the *inverse Gaussian distribution*. A review on the inverse Gaussian distribution can be found in FOLKS and CHHIKARA (1978); WILLMOT (1987) has applied the inverse Gaussian distribution in modelling the claim number distribution, and GENDRON and CRÉPEAU (1989) and WILLMOT (1990) have modelled the individual claim amount distribution with the inverse Gaussian distribution.

4. PARAMETER ESTIMATION FOR THE GAMMA PROCESS

Let $\{S(t)\}$ be a gamma process with (at time $t = 0$) unknown parameters a and b. We claim that, if we can observe the process for a time interval of (arbitrarily short) length h, $h > 0$, the value of a can be obtained as a limit: For $0 < x < 1$, we define the random variable

$$A(x) = -\frac{N(h; x)}{h \ln(x)}; \tag{4.1}$$

then

$$\lim_{x \to 0} A(x) = a. \tag{4.2}$$

(We remark that a similar situation exists for the diffusion process with *a priori* unknown but constant infinitesmal drift μ and variance σ^2: If the sample path for an arbitrarily small time interval is known, σ^2 can be calculated.)

To prove (4.2), we write (4.1) as

$$A(x) = \frac{\displaystyle\int_x^\infty \frac{e^{-by}}{y}\,dy}{\displaystyle\int_x^1 \frac{dy}{y}} \cdot \frac{N(h;x)}{ah\displaystyle\int_x^\infty \frac{e^{-by}}{y}\,dy} \cdot a.$$

Applying L'Hôpital's rule, we see that the first ratio tends to 1 as x tends to 0. The second ratio is $N(h;x)/[hQ(x)]$; by the strong law of large numbers it converges to 1 (with probability one) as x tends to 0.

In the following we assume that the value of a is known, but that b is unknown. If the aggregate claims process has been observed to time t, $S(t)$ is a sufficient statistic, i.e., any additional information about the sample path is irrelevant for the estimation of b (De Groot, 1975, p. 304, #5). To illustrate this, let us treat the unknown b as a random variable Θ with prior probability density function $u(\theta)$, $\theta > 0$. Then the posterior density of Θ at time t, given the value of $S(t)$, is

$$u(\theta;t) = \frac{\theta^{at}\, e^{-\theta S(t)}\, u(\theta)}{\displaystyle\int_0^\infty r^{at}\, e^{-rS(t)}\, u(r)\,dr}.$$

Let us now assume that $u(\theta)$ is gamma, say,

$$u(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \qquad \theta > 0,$$

with $\beta > 0$ and $\alpha > 1$. Then the posterior density is also gamma, with parameters

$$\alpha_t = \alpha + at$$

and

$$\beta_t = \beta + S(t).$$

At time $t = 0$, the expected aggregate claims per unit time are

$$E\left(\frac{a}{\Theta}\right) = a\int_0^\infty \frac{u(\theta)}{\theta}\,d\theta = a\frac{\beta}{\alpha-1}.$$

Hence, with $S(t)$ known, the conditional expectation of the aggregate claims per unit time is

$$
(4.3) \qquad a\frac{\beta_t}{\alpha_t - 1} = a\frac{\beta + S(t)}{\alpha + at - 1} = (1 - Z_t)\, a\frac{\beta}{\alpha - 1} + Z_t\frac{S(t)}{t},
$$

where $Z_t = at/(at + \alpha - 1)$. Formula (4.3) corresponds to the well-known result for exact credibility in the gamma/gamma model.

5. SIMULATION OF THE GAMMA PROCESS

We can simulate a compound Poisson process by simulating the times and amounts of the claims. This straightforward approach is not applicable to the gamma process, since there are infinitely many claims in each time interval. We now present a method for simulating the gamma process.

Let $\{S(t)\}$ be the gamma process with parameters a and b. To simulate a sample path, we use the following result. For time $\tau > 0$, the conditional distribution of the ratio $S(\tau/2)/S(\tau)$, given $S(\tau)$, is symmetric beta with

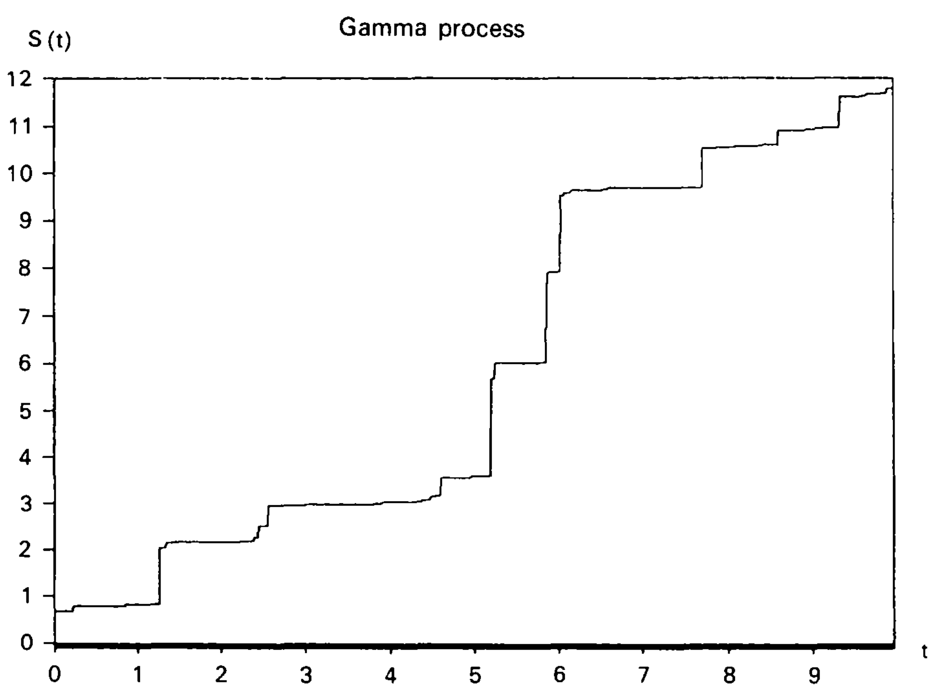

FIGURE 1.

parameter $a\tau/2$ (DE GROOT, 1975, p. 244, #5). Thus, if we want to simulate a sample path for $S(t)$, $0 \leq t \leq T$, we can proceed as follows. First we simulate a value for $S(T)$, whose distribution is gamma with shape parameter aT and scale parameter b. Then we obtain $S(T/2)$ by simulating a value for $S(T/2)/S(T)$, which has a symmetric beta distribution with parameter $aT/2$. Next, we obtain $S(T/4)$ and $S(3T/4)$ by simulating the values of $S(T/4)/S(T/2)$ and $[S(3T/4)-S(T/2)]$ / $[S(T)-S(T/2)]$, respectively, each of which has a symmetric beta distribution with parameter $aT/4$. Similarly, we can generate the values of $S(T/8)$, $S(3T/8)$, $S(5T/8)$, $S(7T/8)$, and so on.

We have simulated the standardized gamma process for various T. A sample path for $T = 10$ is shown in Figure 1.

6. RUIN THEORY

Let $\{S(t)\}$ be the aggregate claims process introduced in Section 2. In this section we present some ruin probability results for this process. In the next section, we specialize to the case that $\{S(t)\}$ is a gamma process.

Let the surplus of an insurance company at time t, $t \geq 0$, be

$$U(t) = u + ct - S(t). \tag{6.1}$$

Here u is a nonnegative number denoting the initial surplus and c is the rate at which the premiums are received. The relative security loading θ is defined by the equation

$$c = (1+\theta)\,E[S(1)] = (1+\theta)\int_0^\infty Q(x)\,dx. \tag{6.2}$$

We assume that $\theta > 0$. Let $\psi(u)$ denote the probability of ultimate ruin, i.e., the probability that the surplus becomes negative at some future time.

In view of formula (2.4), results for this model can be obtained via those for the compound Poisson model with the following recipe. We start with a formula for the case of the compound Poisson process with Poisson parameter λ and individual claim amount distribution $P(y)$. Then we substitute $Q(y)$ for $\lambda[1-P(y)]$ (or $q(y)$ for $\lambda p(y)$ if the derivatives exist) to obtain the corresponding formula for the more general model.

For example, in the compound Poisson model the probability of ruin satisfies the following defective renewal equation [e.g., BOWERS et al. (1986, p. 373, #12.11)]:

$$c\psi(u) = \lambda\int_0^u \psi(u-y)\,[1-P(y)]\,dy + \lambda\int_u^\infty [1-P(y)]\,dy, \qquad u \geq 0.$$

Substituting $Q(y)$ for $\lambda[1-P(y)]$, we get

$$c\psi(u) = \int_0^u \psi(u-y)\,Q(y)\,dy + \int_u^\infty Q(y)\,dy, \qquad u \geq 0. \tag{6.3}$$

For $u = 0$, this gives

$$\psi(0) = \frac{1}{c}\int_0^\infty Q(y)\,dy = \frac{1}{1+\theta}. \tag{6.4}$$

Let us now consider the maximal loss random variable

$$L = \max_{t\geq 0}\,\{S(t)-ct\}. \tag{6.5}$$

It is of interest since $1-\psi(u)$ is its distribution function. In the compound Poisson model, it is well known (BOWERS et al., 1986, Section 12.6) that L has a compound geometric distribution:

$$L = L_1+L_2+\ \dots\ +L_N. \tag{6.6}$$

Here $N, L_1, L_2, \dots$ are independent random variables, the L_i's are identically distributed with the probability density

$$h(x) = \frac{1-P(x)}{\int_0^\infty [1-P(y)]\,dy}, \qquad x > 0, \tag{6.7}$$

and N has a geometric distribution defined by

$$Pr(N = n) = \frac{\theta}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^n, \qquad n = 0, 1, 2, \dots. \tag{6.8}$$

If we multiply both numerator and denominator of (6.7) by λ, we see that (6.6) is valid for the general model, with

$$h(x) = \frac{Q(x)}{\int_0^\infty Q(y)\,dy}, \qquad x > 0. \tag{6.9}$$

These formulas can be used to determine numerical lower and upper bounds for the ruin probability; see Method 1 in DUFRESNE and GERBER (1989).

For the next result we assume that $p(x) = P'(x)$ and $q(x) = -Q'(x)$ exist. Let T denote the time of ruin. Put $X = U(T-)$, the surplus immediately before ruin, and $Y = |U(T)|$, the deficit at the time of ruin. We assume that $u = 0$. Given that ruin occurs, the joint probability density of X and Y in the compound Poisson case is

$$h(x, y) = \frac{p(x+y)}{\int_0^\infty [1-P(s)]\,ds}, \qquad x > 0, y > 0 \tag{6.10}$$

(DUFRESNE and GERBER, 1988). Thus, in the general model, the joint density of X and Y is

$$(6.11) \qquad h(x, y) = \frac{q(x+y)}{\int_0^\infty Q(s)\,ds}, \qquad x > 0, y > 0.$$

We note that both (6.10) and (6.11) are symmetric in x and y. The probability density of $Z = X + Y$ (the amount of the claim that cases ruin) is

$$(6.12) \qquad g(z) = \int_0^z h(x, z-x)\,dx = \frac{zq(z)}{\int_0^\infty Q(s)\,ds}, \qquad z > 0.$$

The conditional probability density of X given $Z = z$ (and $u = 0$) is

$$\frac{h(x, z-x)}{g(z)} = \frac{1}{z}, \qquad 0 < x < z.$$

This is the somewhat surprising result that the conditional distribution of X (given $Z = z$) is uniform between 0 and z.

We wish to remark that, if $Q(0) = \infty$, the notion of an individual claim amount distribution of the process $\{S(t)\}$ *per se* does not make sense. However, the conditional claim amount distribution, given certain information, may still exist. For example, (2.2) is the distribution of an individual claim amount given that it exceeds x. Likewise, $g(z)$ is the probability density function of the amount of the claim that causes ruin.

We now turn to Lundberg's asymptotic formula. The adjustment coefficient R is defined as the positive solution $r = R$ of the equation

$$(6.13) \qquad \int_0^\infty (e^{ry} - 1)\,[-dQ(y)] = cr.$$

(Note that some regularity conditions have to be imposed on $Q(y)$ to guarantee the existence of R.) It follows from (2.6) that, for all t,

$$(6.14) \qquad E[e^{R[S(t)-ct]}] = 1.$$

Lundberg's famous asymptotic formula states that

$$(6.15) \qquad \psi(u) \sim Ce^{-Ru} \qquad \text{for } u \to \infty.$$

In the compound Poisson case,

$$C = \frac{\theta\lambda \int_0^\infty y dP(y)}{\lambda \int_0^\infty y e^{Ry} dP(y) - c} \tag{6.16}$$

(SEAL, formula (4.64)), which is translated as

$$C = \frac{-\theta \int_0^\infty y dQ(y)}{-\int_0^\infty y e^{Ry} dQ(y) - c}. \tag{6.17}$$

7. RUIN THEORY FOR THE GAMMA PROCESS

We now consider the special case that $\{S(t)\}$ is a gamma process. As we pointed out in Section 3, any gamma process can be transformed into a standardized gamma process. Thus we assume that, for $x > 0$,

$$q(x) = \frac{e^{-x}}{x} \tag{7.1}$$

or

$$Q(x) = \int_x^\infty \frac{e^{-y}}{y} dy. \tag{7.2}$$

In ABRAMOWITZ and STEGUN (1964, p. 227), the exponential integral (7.2) is denoted as $E_1(x)$.

Since

$$\int_0^\infty Q(x)\,dx = \int_0^\infty xq(x)\,dx = \int_0^\infty e^{-x}\,dx = 1,$$

formula (6.2) becomes

$$1+\theta = c. \tag{7.3}$$

By (6.9) the common probability density function of the random variables $\{L_i\}$ is

$$h(x) = Q(x) = E_1(x), \qquad x > 0, \tag{7.4}$$

and their distribution function is

$$H(x) = \int_0^x h(y)\,dy = 1-e^{-x}+xE_1(x), \qquad x \geq 0. \tag{7.5}$$

From (6.11) and (6.12) we obtain

$$h(x, y) = \frac{e^{-(x+y)}}{x+y} \tag{7.6}$$

and

$$g(z) = e^{-z}, \tag{7.7}$$

respectively. Formula (7.7) is especially interesting, as it says that (if $u = 0$) the amount of the claim that causes ruin is exponentially distributed.

Substituting (3.4) and (7.3) in (6.14) yields the equation

$$\frac{1}{1-r} = e^{r(1+\theta)}. \tag{7.8}$$

The adjustment coefficient R is the positive root of (7.8). It follows from (6.17) and (7.3) that the asymptotic constant C in Lundberg's formula is

$$C = \frac{\theta}{\dfrac{1}{1-R} - (1+\theta)} = \frac{\theta(1-R)}{R-\theta(1-R)}. \tag{7.9}$$

Remark: As pointed out in Section 3, the gamma process is the limit of a certain family of compound Poisson processes, each with a gamma claim amount distribution. For these WILLMOT (1988) has given an elegant method to evaluate the probability of ruin.

8. THE PROBABILITY OF RUIN FOR THE GAMMA PROCESS

As in the last section we assume that the aggregate claims process is the standardized gamma process. Since (7.5) gives an explicit expression for $H(x)$, we can apply the method of lower and upper bounds to calculate the probability of ruin (DUFRESNE and GERBER, 1989). We have calculated lower and upper bounds for $\psi(u)$ for different values of the initial surplus u $(0, 1, 2, \ldots, 20)$ and the relative security loading θ $(0.1, 0.2, 0.3, \ldots, 1.0)$, for intervals of discretisation with length 0.01 and 0.001. For $\theta = 0.5$ these bounds are displayed in Table 1. Thus the exact value of the probability of ruin is known with sufficient accuracy (4 decimals). Table 2 shows these values.

Illustration: Assume that the annual aggregate claims have an expectation $\mu = 100{,}000$ and a standard deviation $\sigma = 20{,}000$. The initial reserve is 48,000 and the annual premium (net of expenses) is 120,000. What is the probability of ultimate ruin?

TABLE 1

LOWER AND UPPER BOUNDS FOR THE PROBABILITY OF RUIN

$\theta = 0.5$

u	Lower bounds		Upper bounds	
0	0.666667	0.666667	0.666667	0.666667
1	0.321352	0.322741	0.323055	0.324488
2	0.175016	0.176268	0.176550	0.177839
3	0.096653	0.097604	0.097819	0.098798
4	0.053619	0.054288	0.054439	0.055129
5	0.029801	0.030250	0.030352	0.030817
6	0.016577	0.016870	0.016936	0.017240
7	0.009225	0.009412	0.009454	0.009649
8	0.005135	0.005252	0.005279	0.005401
9	0.002858	0.002931	0.002948	0.003024
10	0.001591	0.001636	0.001646	0.001693
11	0.000886	0.000913	0.000919	0.000948
12	0.000493	0.000510	0.000513	0.000531
13	0.000275	0.000284	0.000287	0.000297
14	0.000153	0.000159	0.000160	0.000166
15	0.000085	0.000089	0.000089	0.000093
16	0.000047	0.000049	0.000050	0.000052
17	0.000026	0.000028	0.000028	0.000029
18	0.000015	0.000015	0.000016	0.000016
19	0.000008	0.000009	0.000009	0.000009
20	0.000005	0.000005	0.000005	0.000005

0.001

0.01

length of the interval of discretisation

Solution: We assume that the premiums are received continuously and the aggregate claims process can be modelled by a gamma process with parameters a and b. Then $a/b = \mu = 100{,}000$ and $a/b^2 = \sigma^2 = (20{,}000)^2$. It follows that $b = \mu/\sigma^2 = 1/4{,}000$. In order to use Table 2 (which is for the standardized gamma process), we have to transform the initial reserve to $u = 48{,}000 \times b = 12$. The relative security loading $\theta = 0.2$ does not change. Looking up Table 2, we obtain the probability of ruin $\psi(12) = 0.018$.

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TABLE 2

THE PROBABILITY OF RUIN FOR THE STANDARDIZED GAMMA PROCESS

Relative security loading θ

u	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.9091	0.8333	0.7692	0.7143	0.6667	0.6250	0.5882	0.5556	0.5263	0.5000
1	0.7395	0.5736	0.4613	0.3816	0.3229	0.2782	0.2434	0.2155	0.1929	0.1743
2	0.6184	0.4165	0.2990	0.2253	0.1764	0.1424	0.1178	0.0994	0.0854	0.0743
3	0.5182	0.3038	0.1952	0.1344	0.0977	0.0741	0.0582	0.0470	0.0388	0.0327
4	0.4345	0.2219	0.1277	0.0805	0.0544	0.0388	0.0289	0.0224	0.0178	0.0145
5	0.3643	0.1621	0.0836	0.0482	0.0303	0.0204	0.0144	0.0107	0.0082	0.0065
6	0.3054	0.1185	0.0548	0.0289	0.0169	0.0107	0.0072	0.0051	0.0038	0.0029
7	0.2561	0.0866	0.0359	0.0173	0.0094	0.0056	0.0036	0.0025	0.0018	0.0013
8	0.2148	0.0632	0.0235	0.0104	0.0053	0.0030	0.0018	0.0012	0.0008	0.0006
9	0.1801	0.0462	0.0154	0.0062	0.0029	0.0016	0.0009	0.0006	0.0004	0.0003
10	0.1510	0.0338	0.0101	0.0037	0.0016	0.0008	0.0005	0.0003	0.0002	0.0001
11	0.1266	0.0247	0.0066	0.0022	0.0009	0.0004	0.0002	0.0001	0.0001	0.0001
12	0.1062	0.0180	0.0043	0.0013	0.0005	0.0002	0.0001	0.0001		
13	0.0890	0.0132	0.0028	0.0008	0.0003	0.0001	0.0001			
14	0.0746	0.0096	0.0019	0.0005	0.0002	0.0001				
15	0.0626	0.0070	0.0012	0.0003	0.0001					
16	0.0525	0.0051	0.0008	0.0002						
17	0.0440	0.0038	0.0005	0.0001						
18	0.0369	0.0027	0.0003	0.0001						
19	0.0309	0.0020	0.0002							
20	0.0259	0.0015	0.0001							

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EVALUATING COMPOUND GENERALIZED POISSON DISTRIBUTIONS RECURSIVELY

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ABSTRACT

In this paper we give a recursive scheme, involving Panjer's recursion, to compute the distribution of a compound sum of integer claims, when the number of summands follows a Generalized Poisson distribution. Also, an elegant derivation is given for some basic properties of this counting distribution.

1. THE GENERALIZED POISSON DISTRIBUTION

The Generalized Poisson distribution, see CONSUL (1989), is an integer-valued, non-negative distribution with two parameters θ and λ. A random variable N having this distribution with parameters θ and λ is also denoted as a GP (θ, λ) random variable. In the first section we repeat the mathematical properties of this distribution, giving a short and elegant derivation. The second section contains a recursive algorithm to compute the probabilities of a compound Generalized Poisson distribution. This algorithm is obtained by the well-known technique of differentiating the generating function and comparing coefficients of resulting power series. This function, however, is known only in an implicit form, so the process is not as trivial as usual.

An actuarial application of the Generalized Poisson distribution, linking it to the ruin model, can be found in GERBER (1990). Other chance mechanisms generating this distribution are described in CONSUL (1989). One of these is the Galton-Watson branching process, which is a model with many conceivable actuarial applications. In this process, the spreading of a certain disease is modeled as follows. Suppose M individuals are originally infected. Each of these infects L_i other individuals, $i = 1, \ldots, M$. These in turn infect L_{ij} new victims, $j = 1, \ldots, L_i$, and so on. Now if M is a Poisson (θ) distributed random variable, and the L_i, L_{ij}, ... are independent Poisson (λ) random variables, the total number N of people infected has a Generalized Poisson distribution with parameters θ and λ.

The parameters θ and λ are non-negative; the Poisson distribution is the special case with $\lambda = 0$. Assume $\lambda < 1$ to ensure that N remains finite with probability one.

Consider the total number of individuals B_i infected by the ith person, including this person himself, and define B_{ij} analogously for the jth person infected by $i, j = 1, \ldots, L_i$. Obviously B_i and B_{ij} are random variables with the same distribution. We can write B_i as:

$$(1) \qquad B_i = 1 + \sum_{j=1}^{L_i} B_{ij}.$$

Let B be distributed as B_i and B_{ij}. From relation (1), and using some well-known properties of compound Poisson (λ) distributions, we can directly derive expressions for the mean, variance and generating function of B. The mean can be computed as follows:

$$(2) \qquad E[B] = 1 + \lambda E[B] \Rightarrow E[B] = \frac{1}{1-\lambda}.$$

The variance and the second moment can be computed from:

$$(3) \qquad \operatorname{Var}[B] = \lambda E[B^2] \Rightarrow E[B^2] = \frac{1}{(1-\lambda)^3}.$$

If $G_B(u) = E[u^B]$ is the generating function of the B_i and B_{ij} random variables, it must satisfy the following relation:

$$(4) \qquad G_B(u) = uG_{L_i}(G_B(u)) = u\, e^{\lambda(G_B(u)-1)}.$$

Writing $t = t(u) = G_B(u)$, we obtain from (4):

$$(5) \qquad u = t\, e^{-\lambda(t-1)}.$$

The probabilities $P[B = i]$ are the coefficients of the power series representation of $t(u)$. To determine them from relation (4), we use a slightly simplified form of Relation 3.6.7 in ABRAMOWITZ and STEGUN (1965; Lagrange's expansion): if $u = f(t)$, $f(0) = 0$, $f'(0) \neq 0$, and g is any function infinitely differentiable, then

$$(6) \qquad g(t) = g(0) + \sum_{k=1}^{\infty} \frac{u^k}{k!} \left[\frac{d^{k-1}}{dt^{k-1}} \frac{g'(t)\, t^k}{f(t)^k} \right]_{t=0}.$$

The distribution of B is found by taking $g(t) = t$ and using $u = f(t)$ as in (5), resulting in the Borel distribution:

$$(7) \qquad P[B = i] = \frac{(i\lambda)^{i-1} e^{-i\lambda}}{i!}, \qquad i = 1, 2, \ldots.$$

Since a GP (θ, λ) random variable is a compound Poisson (θ) sum of Borel (λ) random variables, its generating function equals $e^{\theta(G_B(u)-1)}$, so the density of a GP (θ, λ) random variable N is found by taking $g(t) = e^{\theta(t-1)}$ in (6), leading to:

$$(8) \qquad P[N = n] = \frac{\theta(\theta+n\lambda)^{n-1} e^{-\theta-n\lambda}}{n!}, \qquad n = 0, 1, \ldots.$$

To compute mean and variance of N directly from (8) involves rather tricky mathematics. Using (2) and (3), however, it is trivial exercise; note that for $\lambda > 0$ the mean exceeds the variance:

$$E[N] = \frac{\theta}{1-\lambda}; \qquad \mathrm{Var}\,[N] = \frac{\theta}{(1-\lambda)^3}. \tag{9}$$

Being a compound Poisson (θ) sum (of Borel (λ) distributions), a GP (θ, λ) random variable is easily seen to be infinitely divisible, as for any $n = 1, 2, \dots$ it can be written as the convolution of n GP $\left(\frac{\theta}{n}, \lambda\right)$ variables.

2. A RECURSIVE ALGORITHM FOR THE PROBABILITIES OF A COMPOUND GP DISTRIBUTION

To actuaries the total of the incurred claims is more relevant than their number. If the costs associated with occurrence i, $i = 1, \dots, N$, are given by a random variable Z_i, then the total costs are given by the following compound Generalized Poisson (θ, λ) random variable:

$$S = \sum_{i=1}^{N} Z_i. \tag{10}$$

Here the GP (θ, λ) distributed counting variable N is assumed to be independent of all Z_i, and the sequence $Z_1, Z_2, \dots$ is i.i.d. We assume the Z_i to be integer-valued and positive. (By excluding zero-claims, we avoid problems later on, when we have to compute $P[S = 0]$ to start a recursion.)

Actuaries prefer to use counting distributions that are suitable for computations of quantities like probabilities of ruin and stop-loss premiums. Since PANJER (1981) actuaries are aware that there is a very efficient recursive algorithm to compute probabilities of S as in (10) if N is Binomial, Negative Binomial or Poisson. SUNDT and JEWELL (1981) derive similar recursions for a wider class of counting distributions. In this section we will derive recursion formulae expressing $P[S = s]$ in $P[S = j], j = 0, 1, \dots, s-1$ for the case of a Borel and a GP counting variable, too.

To this end, we will derive recursion relations for the coefficients of the generating function $G_S(u)$. Using the fact that a GP (θ, λ) distribution can be viewed as a compound Poisson (θ) sum of Borel (λ) distributions, we can rewrite S as follows:

$$S = \sum_{i=1}^{M} Y_i \qquad \text{where} \qquad Y_i = \sum_{j=1}^{B_i} Z_{ij}. \tag{11}$$

Here M is a Poisson (θ) random variable, B_i is a Borel (λ) random variable and Z_{ij} is an i.i.d. sequence of claim amounts. Each term Y_i has a compound Borel distribution.

If N has a GP (θ, λ) distribution, and S is as in (10), by (4) and (5) we have

$$(12)\quad G_S(u) = e^{\theta(G_B(G_Z(u))-1)} = e^{\theta(t-1)} \text{ with } t \text{ such that } t\,e^{-\lambda(t-1)} = G_Z(u).$$

This implicit description of the generating function of a compound Generalized Poisson distribution will enable us to derive relations between its probabilities. We do so in two steps. The first and most important step is to compute the coefficients of $G_B(G_Z(u))$, which amounts to computing the probability function of the compound Borel distributed Y_i random variables. The second step uses these coefficients to compute the coefficients of $G_S(u)$, simply by invoking Panjer's recursion formula.

Taking the derivative with respect to u of the logarithm of the second part of (12) provides us with the following relation:

$$(13)\quad \frac{d}{du}\log\left(t(u)\,e^{-\lambda(t(u)-1)}\right) = \frac{t'(u)}{t(u)} - \lambda t'(u) = \frac{d}{du}\log G_Z(u) = \frac{G_Z'(u)}{G_Z(u)}.$$

Rearranging leads to the following equality:

$$(14)\qquad t'(u) = \frac{t(u)}{1-\lambda t(u)}\,\frac{G_Z'(u)}{G_Z(u)}.$$

We introduce the following notation for the coefficients of the power series representations for the three factors appearing in (14):

$$(15)\quad t(u) = \sum_{n=1}^{\infty} \alpha_n u^n; \quad \frac{t(u)}{1-\lambda t(u)} = u\sum_{n=0}^{\infty} \beta_n u^n; \quad \frac{G_Z'(u)}{G_Z(u)} = \frac{1}{u}\sum_{n=0}^{\infty} r_n u^n.$$

Since the coefficients of u^n in (14) must be equal on both sides, we obtain the following relation:

$$(16)\qquad \alpha_{n+1}(n+1) = \sum_{j=0}^{n} \beta_j r_{n-j}.$$

The coefficients r_n depend on the known probability function of Z. We write $p_n = P[Z_i = n]$, $n = 1, 2, \ldots$; assume $p_1 > 0$. So we have

$$(17)\qquad G_Z(u) = \sum_{n=1}^{\infty} p_n u^n.$$

Using (17), rearranging the last equation of (15) and comparing coefficients of u^n leads to

$$\text{(18)} \qquad \frac{\sum_{n=0}^{\infty} (n+1)p_{n+1}u^n}{u\sum_{n=0}^{\infty} p_{n+1}u^n} = \frac{1}{u}\sum_{n=0}^{\infty} r_n u^n \Rightarrow$$

$$(n+1)p_{n+1} = \sum_{j=0}^{n} r_j p_{n+1-j}, \qquad n = 0, 1, \ldots .$$

Then the r_n can be determined as follows:

$$\text{(19)} \qquad r_n = \frac{1}{p_1}\left((n+1)p_{n+1} - \sum_{j=0}^{n-1} r_j p_{n+1-j}\right) \qquad n = 0, 1, \ldots .$$

The coefficients α_n are the probabilities of Y_i to be determined. The auxiliary coefficients β_n can be expressed in $\alpha_1, \ldots, \alpha_{n+1}$, using the same technique leading to (18). Indeed the middle equation of (15) gives the result

$$\text{(20)} \qquad \frac{u\sum_{k=0}^{\infty} \alpha_{k+1}u^k}{1-\lambda\sum_{k=1}^{\infty} \alpha_k u^k} = u\sum_{k=0}^{\infty} \beta_k u^k \Rightarrow$$

$$\beta_n = \sum_{k=1}^{n} \beta_{n-k}\lambda\alpha_k + \alpha_{n+1}, \qquad n = 0, 1, \ldots .$$

Using (20) and the fact that $r_0 = 1$, see (19), we may write (16) as follows:

$$\text{(21)} \qquad \alpha_{n+1}(n+1) = \sum_{j=0}^{n-1} r_{n-j}\beta_j + \sum_{k=1}^{n} \beta_{n-k}\lambda\alpha_k + \alpha_{n+1}.$$

The following expression for α_{n+1} is found

$$\text{(22)} \qquad \alpha_{n+1} = \frac{1}{n}\left(\sum_{j=0}^{n-1} r_{n-j}\beta_j + \lambda\sum_{k=1}^{n} \beta_{n-k}\alpha_k\right), \qquad n = 1, 2, \ldots .$$

The probabilities α_n can now be computed successively. Indeed, if the probabilities $\alpha_1, \ldots, \alpha_n$ and the auxiliary quantities $\beta_0, \ldots, \beta_{n-2}$ are known, one computes β_{n-1} using (20), and next α_{n+1} using (22). Since $P[Z_i = 0] = 0$, the starting value α_1 can be computed as follows:

$$\text{(23)} \qquad \alpha_1 = P\left[\sum_{i=1}^{B} Z_i = 1\right] = P[B = 1]\,P[Z_1 = 1] = p_1 e^{-\lambda}.$$

Note that by the requirement $p_1 > 0$ we have $\alpha_1 > 0$.

Having computed the coefficients $\alpha_1, \alpha_2, \dots$, which are the probabilities of the random variables Y_i, we can compute the probabilities of S simply by using Panjer's recursion formula for the Poisson (θ) case, starting from $P[S = 0] = P[N = 0] = e^{-\theta}$:

$$P[S = s] = \frac{\theta}{s} \sum_{j=1}^{s} j\alpha_j P[S = s-j], \qquad s = 1, 2, \dots. \tag{24}$$

Remark

Taking $p_1 = 1$, $p_j = 0$ otherwise, one gets $r_j = 0$ for $j \neq 0$. Then (22) and (20) lead to a recursion for the Borel (λ) distribution (7); combining it with (23) gives a recursion for the GP (θ, λ) distribution (8).

3. CONCLUSIONS

The Generalized Poisson distribution may be a useful model when the chance mechanism used in the first section is appropriate, or any of the other models in CONSUL (1989). It can be used as an alternative to the Negative Binomial distribution when the tails of the counting distribution are thicker than those of the Poisson. It is mathematically a more complex distribution than the counting distributions usually assumed by actuaries (Binomial, Poisson or Negative Binomial), but we think that using the lines of thought given in the first section, actuaries will be able to use this distribution in their practical work.

The possible objection that this counting distribution is not suitable for actuarial calculations, which mostly involve compound sums, is removed by the recursive algorithm given in Section 2.

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RECURSIVE CALCULATION OF SURVIVAL PROBABILITIES

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Abstract

In this paper we present an algorithm for the approximate calculation of finite time survival probabilities for the classical risk model. We also show how this algorithm can be applied to the calculation of infinite time survival probabilities. Numerical examples are given and the stability of the algorithms is discussed.

Keywords

Survival probability; finite time; infinite time; recursive calculations; numerical stability.

1. Introduction

The primary aim of this paper is the approximate calculation of the probablity of survival in continuous and finite time for a general classical risk process. We assume, without loss of generality, that the expected number of claims per unit time for this process is 1 and that the expected amount of a single claim is also 1. This process can be characterized as follows:

(1.1)
- — the number of claims occurring up to time t, denoted ${}_0N_t$, has a Poisson distribution with parameter t,
- — the amount of the i-th claim is ${}_0Y_i$, where $\{{}_0Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative random variables which are also independent of the claim number process, and whose first two moments exist,
- — the premium income per unit time is $1+\theta$, where θ is the premium loading factor. (We shall assume $\theta \geq 0$, but some of our later results require only that $(1+\theta) > 0$.)

(We use the subscript "0" where appropriate to indicate that we are dealing with our initial process.) For a given initial reserve $u(\geq 0)$ we denote by ${}_0\delta(u, t)$ the probability of survival in continuous time up to time t, so that

$${}_0\delta(u,t) = \Pr\left[u+(1+\theta)\tau - \sum_{i=1}^{{}_0N_\tau} {}_0Y_i \geq 0 \text{ for all } \tau,\ 0 < \tau \leq t\right]$$

Our approach to the calculation of ${}_0\delta(u, t)$ is to show that ${}_0\delta(u, t)$ can be approximated by the probability of survival in discrete and finite time for a particular risk process, and then to discuss the calculation of this latter probability. The particular risk process we use is a classical risk process characterized as follows:

(1.2)
- — the number of claims occurring up to time t, denoted N_t, has a Poisson distribution with parameter λt,
- — the amount of the i-th claim is Y_i where $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables which are independent of the claim number process,
- — the Y_i's are distributed on the non-negative integers,
- — the premium income per unit time is 1.

We introduce the following notation for this particular risk process:

$$b_k = \Pr[Y_i = k] \quad \text{for} \quad k = 0, 1, 2, \ldots$$

$$\delta(u, t) = \Pr\left[u+\tau - \sum_{i=1}^{N_\tau} Y_i \geq 0 \quad \text{for} \quad \tau = 1, 2, \ldots, t\right]$$

so that $\delta(u, t)$ denotes the probability of survival in discrete time up to time t for this particular risk process, given initial reserve u, which we always assume to be non-negative. With suitable choices for λ and the b_k's we can then argue that

$$(1.3) \qquad {}_0\delta(u, t) \simeq \delta(u\beta, (1+\theta)\beta t)$$

for some positive constant β.

Formula (1.3) can be justified by using a discretizing and re-scaling argument as follows:

STEP 1 Discretize the initial process:

Let $\{{}_1Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables distributed on the discrete points $0, 1/\beta, 2/\beta, \ldots$, for some $\beta > 0$, in such a way that the distribution of ${}_1Y_i$ approximates to that of ${}_0Y_i$.

Let $b_k = \Pr[{}_1Y_i = k/\beta]$ for $k = 0, 1, 2, \ldots$

Let $${}_1\delta(u, t) = \Pr\left[u+(1+\theta)\tau - \sum_{i=1}^{{}_0N_\tau} {}_1Y_i \geq 0 \text{ for all } \tau,\ 0 < \tau \leq t\right]$$

so that ${}_1\delta(u, t)$ is the probability of survival in continuous time before time t, given initial reserve $u(\geq 0)$, for the initial process but with ${}_0Y_i$ replaced by the discrete random variable ${}_1Y_i$.

Then, if ${}_1Y_i$ is a "good" approximation to ${}_0Y_i$,

$$ {}_1\delta(u,\ t) \simeq {}_0\delta(u,\ t). $$

STEP 2 Change the monetary unit:

Define ${}_2Y_i$ to be equal to $\beta\, Y_i$, so that

$$ \Pr\,[{}_2Y_i = k] = b_k \quad \text{for} \quad k = 0, 1, 2, \ldots $$

Denoting by ${}_2\delta(w,\ t)$ the probability

$$ \Pr\left[w + (1+\theta)\beta\tau - \sum_{i=1}^{{}_0N_\tau} {}_2Y_i \geq 0 \text{ for all } \tau,\ 0 < \tau \leq t \right] $$

it can be seen that

$$ {}_1\delta(u,\ t) = {}_2\delta(u\beta,\ t) $$

and hence

$$ {}_0\delta(u,\ t) \simeq {}_2\delta(u\beta,\ t). $$

STEP 3 Change the time unit:

Let ${}_3N_t$ be a Poisson process with parameter $\lambda = 1/[(1+\theta)\beta]$.

Let ${}_3\delta(w,\ t) = \Pr\left[w + \tau - \sum_{i=1}^{{}_3N_\tau} {}_2Y_i \geq 0 \text{ for all } \tau,\ 0 < \tau \leq t \right].$

Then it can be seen that

$$ {}_2\delta(w,\ t) = {}_3\delta(w,\ (1+\theta)\beta t) $$

and hence that

$$ {}_0\delta(u,\ t) \simeq {}_3\delta(u\beta,\ (1+\theta)\beta t). $$

Finally in our argument to justify (1.3), note that the risk process emerging from STEP 3 is the risk process characterized by (1.2) and that $\delta(u,\ t)$ is the discrete time probability of survival corresponding to ${}_3\delta(u,\ t)$. Intuitively, $\delta(u,\ t)$ should be a good approximation to ${}_3\delta(u,\ t)$ if, for a given t, the number of re-scaled time units, $(1+\theta)\beta t$, is large, so that there are frequent checks for survival in the discrete case.

For the remainder of this paper our theoretical work will be based on the risk process characterized by (1.2). We introduce the following notation for this process:

X_n denotes the aggregate claims from time $n-1$ to time n, so that

$$ X_n = \sum_{i=N_{n-1}+1}^{N_n} Y_i \quad \text{for} \quad n = 1, 2, \ldots \;(= 0 \text{ if } N_{n-1} = N_n) $$

g_k is the probability that X_n takes the value k, for $k = 0, 1, 2 \ldots$

m_k is the k-th moment about zero of an individual claim amount

Z_n is the accumulated surplus up to time n, given initial surplus $u \geq 0$, so that

$$Z_n = u+n - \sum_{i=1}^{n} X_i \quad \text{for} \quad n = 1, 2, \ldots$$

Note that since Y_i is distributed on the non-negative integers we can evaluate the g_k's using the recursive method of PANJER (1981). We shall assume for the remainder of this paper that the g_k's are known and that u is a non-negative integer. Note also that Z_{n+1} can take only the values

$$\begin{array}{lll} Z_n + 1 & (\text{if} \quad X_{n+1} = 0) & \\ Z_n & (\text{if} \quad X_{n+1} = 1) & \\ Z_n - 1 & (\text{if} \quad X_{n+1} = 2) & \text{etc.} \end{array}$$

2. THE METHOD OF DE VYLDER AND GOOVAERTS

DE VYLDER and GOOVAERTS (1988) present a very neat recursive algorithm for the approximate calculation of $_0\delta(u, t)$. Their method involves discretizing the risk process and then re-scaling it, in almost exactly the same way as we have described in our Section 1. In terms of the process characterized by (1.2), their algorithm is as follows:

$$\delta(w, 1) = \sum_{j=0}^{w+1} g_j \quad \text{for} \quad w = 0, 1, 2, \ldots \tag{2.1}$$

$$\delta(w, m) = \sum_{j=0}^{w+1} \delta(w+1-j, m-1) g_j \quad \text{for} \quad w = 0, 1, \ldots \quad \text{and} \quad m = 2, 3, \ldots \tag{2.2}$$

The rationale behind this algorithm is as follows:

— $\delta(w, 1)$ can be calculated directly from (2.1) since the g_k's are known,

— for $m \geq 2$, $\delta(w, m)$ can be calculated by conditioning on the surplus after 1 unit of time; with probability g_j this surplus is $(w+1-j)$ and the probability of survival over a further $(m-1)$ units of time is $\delta(w+1-j, m-1)$.

In terms of the calculations involved, formula (2.2) can perhaps be most easily appreciated by considering Figure 1. We suppose that we wish to calculate $\delta(u, t)$ for some given u and (positive integer) $t\,(> 1)$. We first calculate $\delta(w, 1)$ for $w = 0, 1, \ldots, u+t-1$ using (2.1). We then use (2.2) to calculate $\delta(w, 2)$ for $w = 0, 1, 2, \ldots, u+t-2$. In general, we calculate $\delta(w,\tau)$ for $w = 0, 1, 2, \ldots, u+t-\tau$ having first calculated $\delta(w, \tau-1)$ for

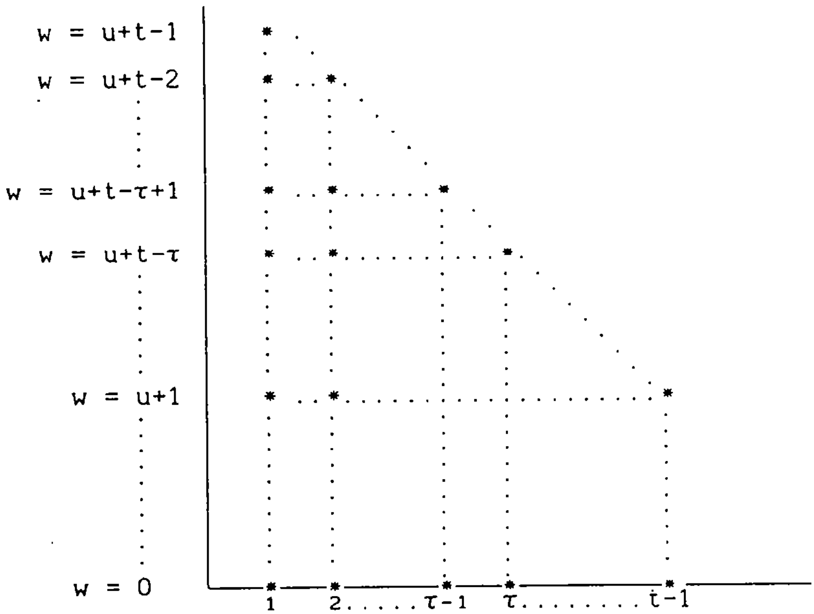

FIGURE 1. Combinations of w and τ for which values of $\delta(w, \tau)$ are required to calculate $\delta(u, t)$ using the method of De Vylder and Goovaerts.

$w = 0, 1, 2, \ldots, u+t-\tau+1$. It can be seen that to calculate $\delta(u, t)$ we have, at least in principle, to calculate $\delta(w, \tau)$ for all values of (w, τ) in the trapezoidal area given by $1 \leq \tau \leq t-1$ and $0 \leq w \leq u+t-\tau$.

There is one respect in which the above description represents a refinement of the algorithm presented by DE VYLDER and GOOVAERTS (1988). In their Section 7 they state that, " We can adopt any unit of money and any unit of time." However, re-scaling of the time unit results in a premium income per unit time which can be greater than 1; our re-scaling of the time unit, as described in our Section 1, results in a premium income per unit time which is equal to 1.

There are two respects in which the above description is a simplification of De Vylder and Goovaerts's algorithm. These are:

1. Truncation: DE VYLDER and GOOVAERTS (1988, Sections 4 and 5) point out that the algorithm as described above requires a lot of calculations to be carried out and hence requires a considerable amount of computer time. They propose, and use, a method for reducing the number of calculations required in such a way that the error resulting from this approximation can be bounded.
2. Averaging: DE VYLDER and GOOVAERTS (1988, formula (1)) point out that, in the notation of our Section 1,

$$\delta(u-1, t) \leq {}_3\delta(u, t) \leq \delta(u, t)$$

and, in their numerical example, they propose approximating ${}_3\delta(u, t)$ not by $\delta(u, t)$ but by $\bar{\delta}(u, t)$ where

$$\bar{\delta}(u, t) = \tfrac{1}{2}\{\delta(u-1, t)+\delta(u, t)\} \tag{2.3}$$

with $\delta(u-1, t)$ taken to be zero if u is zero.

A numerical example: Table 1 shows values of ${}_0\delta(u, t)$ for various combinations of u and t for the risk process with exponentially distributed individual claims and with two values for the premium loading factor θ, viz. 0.1 and 0.2. The key to Table 1 is as follows:

(1) denotes the exact value of ${}_0\delta(u, t)$ as given by WIKSTAD (1971);

(2) denotes the approximation to ${}_0\delta(u, t)$ given by DE VYLDER and GOOVAERTS (1988, Table 1);

(3) denotes the approximation to ${}_0\delta(u, t)$ given by (2.1) and (2.2) above.

TABLE 1 (See Section 2 for details)

(a) Premium loading factor $\theta = 0.1$

		$t = 1$	$t = 10$	$t = 100$
$u = 0$	(1)	0.5366	0.2146	0.1100
	(2)	0.3401	0.1562	0.0814
	(3)	0.5515	0.2239	0.1150
$u = 1$	(1)	0.7619	0.3874	0.2052
	(2)	0.4159	0.2322	0.1252
	(3)	0.7699	0.3953	0.2098
$u = 10$	(1)	0.9997	0.9681	0.7395
	(2)	0.9996	0.9663	0.7366
	(3)	0.9997	0.9687	0.7413

(b) Premium loading factor $\theta = 0.2$

		$t = 1$	$t = 10$	$t = 100$
$u = 0$	(1)	0.5490	0.2523	0.1717
	(2)	0.3498	0.1829	0.1262
	(3)	0.5636	0.2624	0.1789
$u = 1$	(1)	0.7695	0.4356	0.3040
	(2)	0.4212	0.2602	0.1847
	(3)	0.7772	0.4437	0.3094
$u = 10$	(1)	0.9997	0.9759	0.8601
	(2)	0.9996	0.9743	0.8573
	(3)	0.9997	0.9764	0.8615

The following points should be noted concerning Table 1:

(i) The same discretization has been used to calculate (2) and (3). This is the discretization given by De Vylder and Goovaerts (1988, Section 8); in particular the parameter β has been taken to be 20.

(ii) Both (2) and (3) have been calculated using the truncation proposed by De Vylder and Goovaerts (1988, Sections 4 and 5) with the same truncation parameter in each case.

(iii) The figures shown for (2) have been calculated using formula (2.3), i.e. by "averaging". The figures for (3) have not been calculated using (2.3). If (2.3) had been used to calculate the figures for (3) the effects would have been an improvement in the approximation to ${}_0\delta(u, t)$ for $u > 0$ (e.g. the approximation to ${}_0\delta(10, 10)$ with $\theta = 0.1$ would have been 0.9684) but a much worse approximation to ${}_0\delta(0, t)$ (e.g. for $\theta = 0.2$ the approximation to ${}_0\delta(0, 10)$ would have been 0.1312).

(iv) The important difference in the calculation of the values for (2) and (3) is the difference in the re-scaling of the time unit, as explained above.

(v) For all combinations of θ, u and t shown in Table 1, the approximation given by (3) is much closer to the exact value than is the approximatoin given by (2). We consider this to be a consequence of point (iv) above.

3. A NEW APPROACH TO THE CALCULATION OF $\delta(u, t)$

In this section we present an approach to the calculation of $\delta(u, t)$ different to that of Section 2. The starting point for this approach is formula (2.2). For

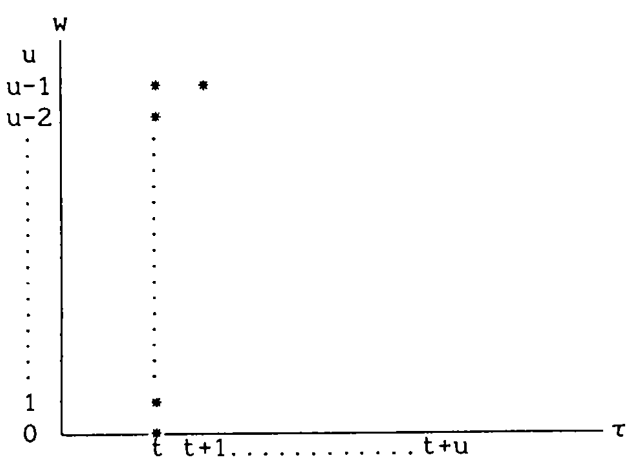

FIGURE 2. Combinations of w and τ for which values of $\delta(w, \tau)$ are required to apply formula (3.2).

$u > 1$ and $t > 0$, we can rewrite (2.2) as

$$\delta(u-1,\ t+1) = \sum_{i=0}^{u} g_i \delta(u-i,\ t) \tag{3.1}$$

To apply this approach, we do not need a formula corresponding to (2.1), but we have to consider the situation when $u = 1$. This is considered in detail in Section 4. On rearranging (3.1), we see that

$$\delta(u,\ t) = g_0^{-1}\left[\delta(u-1,\ t+1) - \sum_{i=1}^{u} g_i \delta(u-i,\ t)\right] \tag{3.2}$$

Figure 2 illustrates the survival probabilities required in order to calculate $\delta(u, t)$ from (3.2). By repeated application of this approach, we see that all values of $\delta(w, \tau)$ for $w = 0, 1, \ldots, u-1$ and $\tau = t, t+1, \ldots, t+u-w$ are required to calculate $\delta(u, t)$. Note that all values of $\delta(0, \tau)$, $\tau = t, t+1, \ldots, t+u$, are required, but these cannot be calculated from (3.2). These values are central to our algorithm and, for the moment, we assume that these values are known. A method for finding these values is considered in Section 4.

Figure 3 illustrates the combinations of w and τ for which values of $\delta(w, \tau)$ are required in order to calculate $\delta(u, t)$. The algorithm starts by calculating $\delta(1, t+u-1)$ from $\delta(0, t+u)$ and $\delta(0, t+u-1)$. Survival probabilities at time $t+u-2$ are then calculated, firstly $\delta(1, t+u-2)$, then $\delta(2, t+u-2)$. We next calculate survival probabilities at time $t+u-3$ and continue in this manner until we finally calculate survival probabilities at time t.

Calculation of $\delta(u, t)$ by this method requires that a total of $0.5u(u+3)$ survival probabilities must first be calculated. What is remarkable about this number is that it is independent of t. This contrasts with the algorithm discussed in Section 2, where the number of δ values required to calculate

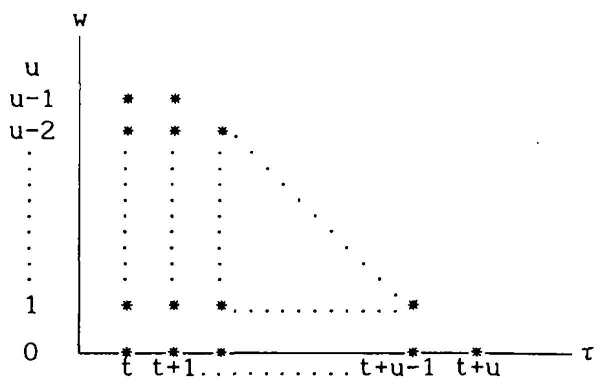

FIGURE 3. Combinations of w and τ for which values of $\delta(w, \tau)$ are required to calculate $\delta(u, t)$ by repeated application of formula (3.2).

$\delta(u, t)$ is $(t-1)(u+0.5t+1)$, which clearly depends on t. However, as we shall show in Section 4, the number of calculations required to produce a value for $\delta(0, t)$ does depend on t.

A further difference between this algorithm and that of Section 2 is that, using the approach of De Vylder and Goovaerts, the survival probabilities required to calculate $\delta(u, t)$ are all for time periods less than t. The new algorithm uses survival probabilities for time periods of at least t. This difference is not important if we are only interested in calculating the survival probability for one particular combination of u and t. De Vylder and Goovaerts' approach to calculting $\delta(u, t)$ also produces values of $\delta(u, j)$, for $j = 1, 2, \ldots, t-1$. Our new algorithm produces values of $\delta(j, t)$, for $j = 0, 1, \ldots, u-1$ (and the method of De Vylder and Goovaerts produces all the figures required to calculate these survival probabilities).

4. A FORMULA FOR $\delta(0, t)$

Let us first consider a survival probability that is slightly different to $\delta(u, t)$. Define

$$\delta^*(u, t) = \Pr\left[u+\tau - \sum_{i=1}^{N_\tau} Y_i \geq 1 \quad \text{for} \quad \tau = 1, 2, \ldots, t\right]$$

so that survival occurs only if the reserve level is strictly positive at each duration from 1 through to t, but the initial reserve level could be zero. When t is infinite, we shall write $\delta^*(u)$ rather than $\delta^*(u, \infty)$.

Let us consider $\delta^*(0, t+1)$, where $t > 0$. Since the initial reserve level is zero, survival under the definition of δ^* can occur only if there are no claims in the first unit of time. Hence

$$\begin{aligned}
\delta^*(0, t+1) &= g_0 \delta^*(1, t) \\
&= g_0 \Pr\left[1+\tau - \sum_{i=1}^{N_\tau} Y_i \geq 1 \quad \text{for} \quad \tau = 1, 2, \ldots, t\right] \\
&= g_0 \Pr\left[\tau - \sum_{i=1}^{N_\tau} Y_i \geq 0 \quad \text{for} \quad \tau = 1, 2, \ldots, t\right] \\
&= g_0 \delta(0, t)
\end{aligned}$$

We can use results given in GERBER (1980, pp. 19-22) for stochastic processes with exchangeable increments to find a formula for $\delta^*(0, t)$, and hence $\delta(0, t)$. We have that

$$\delta^*(0, t) = \sum_{j=1}^{t} \Pr[Z_n > 0, \quad \text{for} \quad n = 1, 2, \ldots, t-1 \quad \text{and} \quad Z_t = j]$$

where Z_n is as in Section 1 (with $u = 0$). Using the duality principle,

$$\begin{aligned}
&\Pr[Z_n > 0, \quad \text{for} \quad n = 1, 2, \ldots, t-1 \quad \text{and} \quad Z_t = j] \\
&\qquad = \Pr[Z_n < Z_t \quad \text{for} \quad n = 1, 2, \ldots, t-1 \quad \text{and} \\
&\qquad\quad Z_t = j]
\end{aligned}$$

and since the process $\{Z_n\}$ is skipfree upwards, we have that

$$\Pr[Z_n < Z_t \quad \text{for} \quad n = 1, 2, \ldots, t-1 \quad \text{and} \quad Z_t = j] = \frac{j}{t} \Pr[Z_t = j]$$

Thus, $\delta^*(0, t) = \sum_{j=1}^{t} \frac{j}{t} \Pr[Z_t = j]$ and $\delta(0, t) = g_0^{-1} \sum_{j=1}^{t+1} \frac{j}{t+1} \Pr[Z_{t+1} = j]$.

Define S_t to be aggregate claims up to time t, so that $S_t = \sum_{n=1}^{t} Y_n$. Let $F(j,$ $= \Pr[S_t \leq j]$ and let $f(j, t) = \Pr[S_t = j]$, for $j = 0, 1, 2, \ldots$. Since the initial surplus is zero, $Z_{t+1} = j \Rightarrow S_{t+1} = t+1-j$, so that

$$\delta(0, t) = g_0^{-1} \sum_{j=1}^{t+1} \frac{j}{t+1} f(t+1-j, t+1) \tag{4.1}$$

$$= g_0^{-1} \sum_{j=0}^{t} \frac{1}{t+1} F(j, t+1) \tag{4.2}$$

Note that since S_t has a compound Poisson distribution with individual claims distributed on the non-negative integers, $F(j, t)$ can be calculated using PANJER'S (1981) recursion formula.

It is interesting to note that the formula for $\delta^*(0, t)$ can also be expressed in terms of $F(j, t)$ as

$$\delta^*(0, t) = \frac{1}{t} \sum_{j=0}^{t-1} F(j, t) \tag{4.3}$$

This expression is clearly analogous to the well known formula for ${}_0\delta(0, t)$ for the general risk process specified by (1.1), as given in, e.g., SEAL (1978, p. 48).

5. SOME NUMERICAL EXAMPLES AND SOME COMMENTS ON NUMERICAL STABILITY

5.1. Numerical examples using the algorithm in Sections 3 and 4

Table 2 shows values of, and approximations to, ${}_0\delta(u, t)$ for a risk process with exponentially distributed individual claims and premium loading factor, θ, equal to 0.1. The key to Table 2 is as follows:

(1) denotes the exact value of ${}_0\delta(u, t)$, as given by SEAL (1978, Table 2.4),

(2) denotes the approximation to ${}_0\delta(u, t)$ calculated using the algorithm discussed in Sections 3 and 4, i.e. using (4.2) and (3.2), with the parameter $\beta = 20$,

(3) denotes the ratio of the value in (2) to the value in (1),

(4) as in (2) but with $\beta = 10$,

(5) denotes the ratio of the value in (4) to the value in (1).

The two sets of approximations to ${}_0\delta(u, t)$ shown in Table 2, i.e. (2) and (4), have been calculated using the method for discretizing the individual claim amount distribution given by DE VYLDER and GOOVAERTS (1988, Section 8). In the former case it is exactly the same discretization, in the latter case only the parameter β is different.

We make the following comments about Table 2:

(i) The approximations to ${}_0\delta(u, t)$ are always larger than the correct values. This is not surprising since we are using discrete time survival probabilities as approximations to continuous time survival probabilities. This is the problem that DE VYLDER and GOOVAERTS (1988) were trying to alleviate by "averaging". See Section 2.

(ii) The relative error in the approximation to ${}_0\delta(u, t)$ for $\beta = 10$ is consistently about twice the relative error for $\beta = 20$. We would expect the relative error for $\beta = 10$ to be larger since it involves a "coarser" discretization of the individual claim amount distribution and also involves "checking for survival" less frequently.

(iii) Where values of (u, t) are given in both Table 1 and Table 2, the approximations to ${}_0\delta(u, t)$ given by formulae (2.1) and (2.2) (i.e. values (3) in Table 1) can be compared with the approximations given by formulae (4.2) and (3.2) with $\beta = 20$, (i.e. values (2) in Table 2). (These values can reasonably be compared since they use precisely the same discretization of the individual claim amount distribution.) It can be seen that the

TABLE 2 (See Section 5 for details)

		$t = 1$	$t = 5$	$t = 10$	$t = 20$	$t = 40$
$u = 0$	(1)	0.5366	0.2804	0.2146	0.1682	0.1362
	(2)	0.5515	0.2921	0.2239	0.1757	0.1423
	(3)	1.0278	1.0417	1.0433	1.0446	1.0455
	(4)	0.5660	0.3036	0.2332	0.1831	0.1485
	(5)	1.0548	1.0827	1.0867	1.0886	1.0903
$u = 1$	(1)	0.7619	0.4881	0.3874	0.3094	0.2529
	(2)	0.7699	0.4971	0.3953	0.3160	0.2584
	(3)	1.0105	1.0184	1.0204	1.0213	1.0217
	(4)	0.7775	0.5059	0.4030	0.3224	0.2638
	(5)	1.0205	1.0365	1.0403	1.0420	1.0431

TABLE 2 (See Section 5 for details)

		$t = 1$	$t = 5$	$t = 10$	$t = 20$	$t = 40$
$u = 2$	(1)	0.8803	0.6456	0.5309	0.4327	0.3574
	(2)	0.8844	0.6522	0.5373	0.4383	0.3623
	(3)	1.0047	1.0102	1.0121	1.0129	1.0137
	(4)	0.8883	0.6587	0.5435	0.4439	0.3670
	(5)	1.0091	1.0203	1.0237	1.0259	1.0269
$u = 3$	(1)	0.9409	0.7605	0.6469	0.5388	0.4503
	(2)	0.9429	0.7652	0.6520	0.5436	0.4546
	(3)	1.0021	1.0062	1.0079	1.0089	1.0095
	(4)	0.9449	0.7698	0.6569	0.5483	0.4588
	(5)	1.0043	1.0122	1.0155	1.0176	1.0189
$u = 4$	(1)	0.9712	0.8416	0.7386	0.6289	0.5325
	(2)	0.9722	0.8449	0.7425	0.6329	0.5363
	(3)	1.0010	1.0039	1.0053	1.0064	1.0071
	(4)	0.9732	0.8481	0.7464	0.6369	0.5399
	(5)	1.0021	1.0077	1.0106	1.0127	1.0139
$u = 5$	(1)	0.9862	0.8973	0.8094	0.7044	0.6046
	(2)	0.9867	0.8996	0.8125	0.7078	0.6079
	(3)	1.0005	1.0026	1.0038	1.0048	1.0055
	(4)	0.9871	0.9017	0.8154	0.7110	0.6111
	(5)	1.0009	1.0049	1.0074	1.0094	1.0108
$u = 6$	(1)	0.9934	0.9346	0.8631	0.7668	0.6674
	(2)	0.9937	0.9361	0.8654	0.7696	0.6703
	(3)	1.0003	1.0016	1.0027	1.0037	1.0043
	(4)	0.9939	0.9375	0.8675	0.7722	0.6730
	(5)	1.0005	1.0031	1.0051	1.0070	1.0084
$u = 7$	(1)	0.9969	0.9591	0.9031	0.8179	0.7219
	(2)	0.9970	0.9600	0.9047	0.8201	0.7243
	(3)	1.0001	1.0009	1.0018	1.0027	1.0033
	(4)	0.9971	0.9609	0.9063	0.8222	0.7267
	(5)	1.0002	1.0019	1.0035	1.0053	1.0066
$u = 8$	(1)	0.9986	0.9747	0.9322	0.8590	0.7687
	(2)	0.9986	0.9753	0.9334	0.8608	0.7708
	(3)	1.0000	1.0006	1.0013	1.0021	1.0027
	(4)	0.9987	0.9759	0.9346	0.8625	0.7728
	(5)	1.0001	1.0012	1.0026	1.0041	1.0053
$u = 9$	(1)	0.9993	0.9846	0.9532	0.8919	0.8087
	(2)	0.9994	0.9850	0.9541	0.8933	0.8105
	(3)	1.0001	1.0004	1.0009	1.0016	1.0022
	(4)	0.9994	0.9854	0.9549	0.8947	0.8122
	(5)	1.0001	1.0008	1.0018	1.0031	1.0043
$u = 10$	(1)	0.9997	0.9908	0.9681	0.9179	0.8427
	(2)	0.9997	0.9910	0.9687	0.9190	0.8442
	(3)	1.0000	1.0002	1.0006	1.0012	1.0018
	(4)	0.9997	0.9912	0.9693	0.9200	0.8456
	(5)	1.0000	1.0004	1.0012	1.0023	1.0034

two sets of values are identical up to the fourth significant figure and hence it appears as though the two algorithms are "as accurate as each other".

5.2. Some comments on numerical stability

The algorithms specified by formulae (2.1) and (2.2) and by formulae (4.2) and (3.2) involve a considerable number of numerical operations. In such situations the numerical stability of an algorithm must be of concern. (An algorithm is numerically unstable if small errors in individual numerical operations, as a result of machine rounding for example, can combine to give uncontrollably large errors in the final results. See, for example, CONTE and DE BOOR (1980)).

DE VYLDER and GOOVAERTS (1988, Section 5) demonstrate that the algorithm specified by formulae (2.1) and (2.2) *is* numerically stable. However, the algorithm specified by formulae (4.2) and (3.2) does not appear to be stable. The authors have experienced difficulties (e.g. calculated δ values outside the range zero to one) when using formulae (4.2) and (3.2) to approximate ${}_0\delta(u, t)$ for values of u greater than about 30 with individual claim amounts having an exponential distribution (with mean 1). These difficulties seem to occur:

(i) independently of the value of t, and,

(ii) independently of the value of β.

This last observation may be a little surprising since reducing the value of β reduces the number of numerical operations required to approximate ${}_0\delta(u, t)$ for given values of u and t.

We can prove the following result concerning the error in the calculation of $\delta(u, t)$ using formulae (4.2) and (3.2). Instead of $\delta(u, t)$, let us assume that $\hat{\delta}(u, t)$ has been calculated, due to rounding errors, and that $\hat{\delta}(u, t)$ satisfies (3.2). We define $\varepsilon(u, t)$ to be the error in the calculation of $\delta(u, t)$, so that

$$\varepsilon(u, t) = \delta(u, t) - \hat{\delta}(u, t)$$

and, for given u and t, ε to be the modulus of the maximum error in the calculation of $\delta(0, \tau)$ for $\tau = t, t+1, \ldots, t+u$, so that

$$\varepsilon = \max_{t \leq \tau \leq t+u} |\varepsilon(0, \tau)|$$

Then we can show that

(5.1)
$$|\varepsilon(w, \tau)| \leq \varepsilon(2g_0^{-1})^w \quad \text{for} \quad \tau = t, t+1, \ldots, t+u, \quad \text{and} \quad w = 0, 1, 2, \ldots, t+u-\tau.$$

Proof: The proof is by induction, working back from $\tau = t+u$ to $\tau = t$. Note first that (5.1) holds for $\tau = t+u$ since the only possible value for w in this case is 0 and

$$|\varepsilon(0, t+u)| \leq \varepsilon = \varepsilon(2g_0^{-1})^0$$

by definition of ε. Now assume (5.1) holds for $\tau = \tau^*+1$, for some τ^*, so that

$$(5.2) \qquad |\varepsilon(w, \tau^*+1)| \leq \varepsilon(2g_0^{-1})^w \quad \textit{for} \quad w = 0, 1, \ldots, t+u-\tau^*-1$$

We have to show that

$$(5.3) \qquad |\varepsilon(w, \tau^*)| \leq \varepsilon(2g_0^{-1})^w \quad \textit{for} \quad w = 0, 1, \ldots, t+u-\tau^*$$

to complete the induction. We shall prove (5.3) by induction on w. Note that (5.3) holds for $w = 0$ by definition of ε. Suppose (5.3) holds for $w \leq w^*$ for some w^*, where $0 \leq w^* < t+u-\tau^*$. From (3.2) the basic equation satisfied by $\varepsilon(w^*+1, \tau^*)$ is

$$\varepsilon(w^*+1, \tau^*) = g_0^{-1}\left\{\varepsilon(w^*, \tau^*+1) - \sum_{i=1}^{w^*+1} g_i\varepsilon(w^*+1-i, \tau^*)\right\}$$

from which we have

$$\begin{aligned}
|\varepsilon(w^*+1, \tau^*)| &\leq g_0^{-1}\left\{|\varepsilon(w^*, \tau^*+1)| - \sum_{i=1}^{w^*+1} g_i|\varepsilon(w^*+1-i, \tau^*)|\right\} \\
&\leq g_0^{-1}\left\{\varepsilon(2g_0^{-1})^{w^*} + \sum_{i=1}^{w^*+1} g_i\varepsilon(2g_0^{-1})^{w^*+1-i}\right\} \\
&\leq g_0^{-1}\{\varepsilon(2g_0^{-1})^{w^*} + \varepsilon(2g_0^{-1})^{w^*}\} \\
&= \varepsilon(2g_0^{-1})^{w^*+1}
\end{aligned}$$

using (5.2) and (5.3). Hence, by induction, (5.3) holds for $w = w^*+1$ and hence, also by induction, (5.1) holds for $\tau = \tau^*$.

This result is somewhat unsatisfactory since it gives only an upper bound for $|\varepsilon(u, t)|$ rather than more detailed information about how this error behaves, and also because for large values of u it may very well be greater than 1. Note that for values of β used in this paper g_0 is close to 1. For example, in Table 2 with $\beta = 20$ the value of g_0 is 0.95663, but the maximum value of w is 200 so that ε will need to be very small indeed for the upper bound in (5.1) to be less than 1!

However, the result does have some interesting features:

(a) The upper bound for $|\varepsilon(u, t)|$ is explicitly a function of u, not of t (although ε itself will be a function of t). See remark (i) earlier in this section.

(b) Suppose we wish to approximate ${}_0\delta(u, t)$ for some given u and t using formulae (4.2) and (3.2). Suppose further that we do this twice using different values for β, one twice the value of the other, say β and $\hat{\beta} = 2\beta$. Then, in an obvious notation,

$$g_0 = e^{-1/(1+\theta)\beta}; \ \hat{g}_0 = e^{-1/2(1+\theta)\beta}$$

The value of ${}_0\delta(u, t)$ is approximated by $\delta(u\beta, (1+\theta)\beta t)$ and $\delta(2u\beta, 2(1+\theta)\beta t)$ in each case and the upper bounds given by (5.1) for the errors will be

$$\begin{aligned} |\varepsilon(u\beta, (1+\theta)\beta t)| &\leq \varepsilon(2e^{1/(1+\theta)\beta})^{u\beta} \\ &= \varepsilon\, 2^{u\beta}\, e^{u/(1+\theta)} \\ |\varepsilon(2u\beta, 2(1+\theta)\beta t)| &\leq \hat{\varepsilon}(2e^{1/2(1+\theta)\beta})^{2u\beta} \\ &= \hat{\varepsilon}\, 2^{2u\beta}\, e^{u/(1+\theta)} \end{aligned}$$

so that one component of the upper bound is independent of β. See remark (ii) earlier in this section.

5.3. A pragmatic solution to the problem of instability

We can deal with the problem of numerical instability resulting from the use of formulae (4.2) and (3.2), at least superficially, by constraining the results to behave properly. Consider formula (4.2) first. We know that

$$0 < \delta(0, t+1) \leq \delta(0, t) \leq 1$$

for any $t \geq 0$. Let $\hat{\delta}(0, t)$ be the value calculated using (4.2). Rather than use this value in formula (3.2) we can use $\delta'(0, t)$ where

$$\delta'(0, t) = \min\{1, \max(0, \min(\hat{\delta}(0, t), \delta'(0, t-1)))\} \quad \text{for} \quad t \geq 1 \tag{5.4}$$

In our numerical examples, we did not experience stability problems in the calculation of $\delta(0, t)$.

We can adjust (3.2) in a similar fashion. For $u \geq 1$ the constraints on $\delta(u, t)$ are

$$0 \leq \max\{\delta(u, t+1), \delta(u-1, t)\} \leq \delta(u, t) \leq 1$$

Let $\delta'(,)$ denote the (constrained) value of $\delta(,)$ actually used and, for given u and t, let $\hat{\delta}(u, t)$ be the "value" of $\delta(u, t)$ calculated using (3.2) with $\delta'(u-1, t+1)$ and $\delta'(u-1, t)$ appearing on the right hand side. Then

$$\delta'(u, t) = \min\{1, \max(\hat{\delta}(u, t), \delta'(u, t+1), \delta'(u-1, t))\} \tag{5.5}$$

(At this stage the reader could be forgiven for thinking that we are treating the symptoms of instability rather than the disease itself!)

Table 3 shows values of, and approximations to, ${}_0\delta(u, t)$ for larger values of u and t than those in Table 2. The premium loading factor θ is 0.1 and, as in our previous Tables, individual claim amounts are exponentially distributed.

TABLE 3 (See Section 5 for details)

		$t = 50$	$t = 100$	$t = 150$
$u = 0$	(1)	0.1284	0.1100	0.1028
	(2)	0.1399	0.1200	0.1121
	(3)	1.0896	1.0909	1.0905
$u = 11$	(1)	0.8467	0.7724	0.7361
	(2)	0.8493	0.7753	0.7390
	(3)	1.0031	1.0038	1.0039
$u = 22$	(1)	0.9844	0.9562	0.9352
	(2)	0.9847	0.9568	0.9359
	(3)	1.0003	1.0006	1.0007
$u = 33$	(1)	0.9990	0.9937	0.9870
	(2)	0.9993	0.9940	0.9875
	(3)	1.0003	1.0003	1.0005
$u = 44$	(1)	1.0000	0.9993	0.9979
	(2)	1.0000	1.0000	1.0000
	(3)	1.0000	1.0007	1.0021
$u = 55$	(1)	1.0000	0.9999	0.9997
	(2)	1.0000	1.0000	1.0000
	(3)	1.0000	1.0001	1.0003

The key to Table 3 is as follows:

(1) denotes the exact value of ${}_0\delta(u, t)$ given by SEAL (1978, Table 2.4),

(2) denotes the approximation ${}_0\delta(u, t)$ calculated using formulae (4.2) and (3.2) together with the adjustments given by (5.4) and (5.5),

(3) denotes the ratio (2)/(1).

The values in (2) have been calculated using $\beta = 10$ and the same discretization of the individual claim amount distribution as in our previous examples.

We make the following comments about Table 3:

(i) The relative errors follow the same general pattern as those in Table 2, i.e. increasing with t and decreasing with u, although the pattern is somewhat less regular than it was in Table 2.

(ii) The magnitudes of the relative errors are consistent with those for $\beta = 10$ in Table 2; in particular, introducing the constraints given by (5.4) and (5.5) has not made our approximations to ${}_0\delta(u, t)$ noticeably less accurate.

6. A TRUNCATION PROCEDURE

In their paper DE VYLDER and GOOVAERTS (1988, Section 5) show how the number of calculations, and hence the amount of computer time, involved in the calculation of $\delta(u, t)$ using formulae (2.1) and (2.2) can be reduced in such a way that the resulting error is bounded. This truncation procedure requires the specification of a parameter which, in their numerical example, De Vylder and Goovaerts take to be $\frac{1}{3} \times 10^{-6}$. (We have used this truncation procedure with the same parameter value for the calculation of the values of $\delta(u, t)$ in Table 1.) Even with the help of this truncation procedure, we have found that, typically for very small values of u, the algorithm specified by formulae (2.1) and (2.2) can sometimes take more computer time to calculate $\delta(u, t)$ than the algorithm specified by (4.2) and (3.2). Even so, the calculation of $\delta(u, t)$ using (4.2) and (3.2), and in particular the calculation of values of $\delta(0, t)$ using (4.2), can require a considerable amount of computer time. However, it is possible to limit the number of calculations involved in the calculation of $\delta(0, t)$ in such a way that the resulting error is bounded, as we show below.

Recall that Y_i denotes the amount of the i-th individual claim and that b_k denotes $P[Y_i = k]$ for $k = 0, 1, 2, \ldots$. We introduce the following notation:

$$B(k) = P[Y_i \leq k] \quad \text{for} \quad k = 0, 1, \ldots$$
$$B^{*n}(k) = P[Y_1 + Y_2 + \ldots + Y_n \leq k] \quad \text{for} \quad k = 0, 1, \ldots$$

Suppose ε, $0 < \varepsilon < 1$, is given. We define k_0 to be the smallest integer such that

$$B(k_0) > 1 - \varepsilon$$

The random variables $\{Y_{i,\varepsilon}\}_{i=1}^{\infty}$ are defined as follows:

$$\begin{aligned} Y_{i,\varepsilon} &= Y_i && \text{if} \quad Y_i \leq k_0 \\ &= \infty && \text{if} \quad Y_i > k_0 \end{aligned}$$

We define

$$\begin{aligned} b_\varepsilon(k) = P[Y_{i,\varepsilon} = k] &= b_k && \text{for} \quad 0 \leq k \leq k_0 \\ &= 0 && \text{for} \quad k_0 < k < \infty \\ &= \sum_{j=k_0+1}^{\infty} b_j && \text{for} \quad k = \infty \end{aligned}$$

$$B_\varepsilon(k) = P[Y_{i,\varepsilon} \leq k]$$
$$B_\varepsilon^{*n}(k) = P[Y_{1,\varepsilon} + Y_{2,\varepsilon} + \ldots + Y_{n,\varepsilon} \leq k]$$

It is an elementary exercise to show that

$$B^{*n}(k) - n\varepsilon \leq B_\varepsilon^{*n}(k) \leq B^{*n}(k) \quad \text{for} \quad k = 0, 1, 2, \ldots \quad \text{and} \quad n = 0, 1, 2, \ldots \tag{6.1}$$

Recall that $F(j, t)$ is the probability that the aggregate claims up to time t do not exceed j. Define $F_\varepsilon(j, t)$ to be the corresponding distribution function with individual claim sizes given by $Y_{i,\varepsilon}$ rather than Y_i, and $\delta_\varepsilon(0, t)$ to be the appropriate survival probability for this process. Then

$$F(j, t) - \lambda t \varepsilon \leq F_\varepsilon(j, t) \leq F(j, t) \tag{6.2}$$

$$\delta(0, t) - \lambda(t+1)\varepsilon g_0^{-1} \leq \delta_\varepsilon(0, t) \leq \delta(0, t) \tag{6.3}$$

for $t = 1, 2, \ldots$, and $j = 0, 1, 2, \ldots$. Formula (6.2) follows from (6.1) and from noting that

$$F(j, t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} B^{*n}(j)$$

with a corresponding formula for $F_\varepsilon(j, t)$. Formula (6.3) follows from (6.2) and (4.2).

The calculation of $\delta(0, t)$ and $\delta_\varepsilon(0, t)$ require the calculation of $F(j, t+1)$ and $F_\varepsilon(j, t+1)$ respectively, for $j = 0, 1, 2, \ldots, t$, and these latter calculations are carried out using PANJER'S (1981) recursion formula. There can be a considerable saving of computer time in using $\delta_\varepsilon(0, t)$ as an approximation to $\delta(0, t)$ since $F_\varepsilon(j, t+1)$ may be based on a risk process with considerably fewer possible values for an individual claim.

7. CALCULATION OF INFINITE TIME SURVIVAL PROBABILITIES

7.1. A recursive formula for the infinite time survival probability

In this section we shall assume that the mean of an individual claim amount, denoted m_1, is equal to β, i.e. that the discretisation of the initial individual claim amount in Section 1 has been done in such a way as to preserve the value of the mean claim amount. This condition is satisfied by the discretisation used in all the numerical examples in this paper. See DE VYLDER and GOOVAERTS (1988, Section 7).

The rationale underlying (2.2) can also be applied to infinite time giving

$$\delta(u-1) = \sum_{i=0}^{u} g_i \delta(u-i), \quad \text{where} \quad \delta(u) = \lim_{t\to\infty} \delta(u, t) \tag{7.1}$$

This is simply the infinite time version of (3.1), which can be rearranged to give

$$\delta(u) = g_0^{-1} \left[\delta(u-1) - \sum_{i=1}^{u} g_i \delta(u-i) \right] \tag{7.2}$$

We could apply formula (3.2) if we could calculate values of $\delta(0, t)$. We can apply (7.2) if we can calculate the value of $\delta(0)$. To do this, we consider the limit as $t \to \infty$ of formula (4.2), using ideas given in GERBER (1979, p. 113).

We have that

$$g_0\delta(0,t) = \frac{1}{t+1}\sum_{j=0}^{t} F(j,t+1)$$

$$= \frac{1}{t+1}\sum_{j=0}^{t} [1-(1-\mathrm{F}(j,t+1))]$$

$$= 1 - \frac{1}{t+1}\sum_{j=0}^{t} [1-F(j,t+1)]$$

$$= 1 - \frac{1}{t+1}\sum_{j=0}^{\infty} [1-F(j,t+1)] + \frac{1}{t+1}\sum_{j=t+1}^{\infty} [1-F(j,t+1)] \tag{7.3}$$

The summation in the second term on the right hand side of (7.3) is just the mean of the distribution of S_{t+1}. As S_{t+1} has a compound Poisson distribution with Poisson parameter $(t+1)/[(1+\theta)m_1]$, this term reduces to $1/(1+\theta)$.

Hence,

$$g_0\delta(0,t) = \frac{\theta}{1+\theta} + \frac{1}{t+1}\sum_{j=t+1}^{\infty} [1-F(j,t+1)] \tag{7.4}$$

Finally, consider $1-F(j,t+1) = \Pr[S_{t+1} \geq j+1]$. Now S_{t+1} has mean $(t+1)/(1+\theta)$ and variance $(t+1)m_2/[(1+\theta)m_1]$. We can apply Chebychev's inequality as follows:

$$\Pr(S_{t+1} \geq j+1) = \Pr\left[S_{t+1} - \frac{t+1}{1+\theta} \geq j+1 - \frac{t+1}{1+\theta}\right]$$

$$\leq \Pr\left[\left|S_{t+1} - \frac{t+1}{1+\theta}\right| \geq j+1 - \frac{t+1}{1+\theta}\right]$$

$$\leq \frac{V(S_{t+1})}{\left(j+1 - \dfrac{t+1}{1+\theta}\right)^2}$$

Then, $\displaystyle\sum_{j=t+1}^{\infty} [1-F(j,t+1)] \leq V(S_{t+1}) \sum_{j=t+1}^{\infty}\left(j+1 - \frac{t+1}{1+\theta}\right)^{-2}$.

Consider the sum

$$S = \frac{1}{(t+2-\alpha)^2} + \frac{1}{(t+3-\alpha)^2} + \frac{1}{(t+4-\alpha)^2} + \dots \quad \text{where } \alpha = (t+1)/(1+\theta)$$

$$< \frac{1}{(t+2-\alpha)\,(t+1-\alpha)} + \frac{1}{(t+3-\alpha)\,(t+2-\alpha)} + \dots \quad \text{provided that } \theta > 0$$

$$= \left(\frac{1}{t+1-\alpha} - \frac{1}{t+2-\alpha}\right) + \left(\frac{1}{t+2-\alpha} - \frac{1}{t+3-\alpha}\right) + \dots$$

$$= \frac{1}{t+1-\alpha} = \frac{1+\theta}{\theta(t+1)}$$

Hence, $\displaystyle\frac{1}{t+1}\sum_{j=t+1}^{\infty}[1-F(j,\ t+1)] \le \frac{1}{t+1}\, V(S_{t+1})\,\frac{1+\theta}{\theta(t+1)} = \frac{1}{t+1}\,\frac{m_2}{\theta m_1}$

so that $\displaystyle g_0\delta(0,\ t) \le \frac{\theta}{1+\theta} + \frac{1}{t+1}\,\frac{m_2}{\theta m_1}$.

Finally, as $g_0\delta(0,\ t) \ge \dfrac{\theta}{1+\theta}$ by (7.4), we see that by letting $t \to \infty$ we have

$$\delta(0) = \frac{\theta}{g_0(1+\theta)} \tag{7.5}$$

Again it is interesting to compare results for our discrete time process with those for the general risk process as specified by (1.1). We note that $\delta^*(0) = \theta/(1+\theta)$, which is the same as the ultimate survival probability in continuous time from initial reserve 0 in the general risk process.

Formulae (7.1) and (7.5) correspond to equations (33) and (37) in a paper by DUFRESNE (1988), but he does not consider their numerical application. An earlier reference, also given by Dufresne, is GIEZENDANNER, STRAUB and WETTENSCHWILER (1972). An alternative method of finding $\delta(0)$ which does not require equation (4.2) is given in his paper.

We can now apply (7.1) in a recursive manner to calculate survival probabilities starting from

$$\delta(1) = g_0^{-1}(1-g_1)\,\delta(0)$$

We can use calculated values of $\delta(\beta u)$ to approximate to

$$_0\delta(u) = \lim_{t\to\infty}\ _0\delta(u,\ t).$$

7.2. A numerical illustration

Table 4 shows values of, and approximations to, ${}_0\delta(u)$. As in Tables 2 and 3, individual claims are exponentially distributed and the loading factor, θ, equals 0.1. The discretization of the exponential distribution is as before. The key to Table 4 is as follows:

(1) denotes the exact value of ${}_0\delta(u)$ (see, e.g., SEAL (1978, p. 60)),

(2) denotes the approximation to ${}_0\delta(u)$ calculated using formulae (7.2) and (7.5), with $\beta = 20$,

(3) denotes the ratio of the value in (2) to the value in (1),

(4) is as (2), but with $\beta = 40$,

(5) denotes the ratio of the value in (4) to the value in (1),

(6) is as (2), but with $\beta = 100$,

(7) denotes the ratio of the value in (6) to the value in (1).

We make the following comments about Table 4:

(i) The pattern of results is similar to that in Table 2. The approximate values are always greater than the exact values, and as the value of β increases, the relative error in the approximation decreases.

(ii) The authors experienced problems in calculating values of $\delta(u, t)$ for values of u greater than about 30. There were no such problems in calculating values of $\delta(40)$ and $\delta(80)$. However, for larger values of u, the same numerical problems as in Section 5.2 exist.

7.3. Numerical stability

As in Section 5.3, we can adopt a pragmatic approach and constrain the calculated values of $\delta(u)$ to behave properly. The calculation of $\delta(0)$ does not pose any problems. For $u \geq 1$, we constrain the function $\delta(u)$ to be such that

$$0 \leq \delta(u-1) \leq \delta(u) \leq 1$$

TABLE 4 (See Section 7 for details)

u	(1)	(2)	(3)	(4)	(5)	(6)	(7)
0	0.0909	0.0950	1.0451	0.0930	1.0231	0.0917	1.0088
2	0.2420	0.2454	1.0140	0.2438	1.0074	0.2427	1.0029
4	0.3681	0.3709	1.0076	0.3695	1.0038	0.3686	1.0014
6	0.4731	0.4754	1.0049	0.4743	1.0025	0.4736	1.0010
8	0.5607	0.5626	1.0034	0.5617	1.0018	0.5611	1.0007
10	0.6337	0.6353	1.0025	0.6346	1.0014	0.6341	1.0006
20	0.8524	0.8531	1.0008	0.8528	1.0005	0.8526	1.0002
40	0.9760	0.9761	1.0001	0.9761	1.0001	0.9761	1.0001
80	0.9994	0.9994	1.0000	0.9994	1.0000	0.9994	1.0000

Let $\delta'(u)$ denote the constrained value of $\delta(u)$ actually used and, for given u, let $\hat{\delta}(u)$ be the "value" of $\delta(u)$ calculated using (7.2) with δ' appearing on the right hand side. Then

$$\delta'(u) = \min\{1, \max(\delta'(u-1), \hat{\delta}(u))\} \tag{7.6}$$

We can calculate approximate values of ${}_0\delta(u)$ using formulae (7.2) and (7.5), together with the adjustment given by (7.6). We have not produced a table of results because the exact and approximate values (with $\beta = 20$ and with $\beta = 40$) are both 1 to four decimal places for $u \geq 110$.

8. SOME COMMENTS ON THE DEFINITION OF SURVIVAL

Our aim in this paper has been to show how to approximate the continuous time probability of survival ${}_0\delta(u, t)$ by the discrete time probability of survival $\delta(u\beta, (1+\theta)\beta t)$. Formulae (4.2) and (3.2) are exact for $\delta(u\beta, (1+\theta)\beta t)$. However, if we regard the latter as an approximation to ${}_0\delta(u, t)$ we find that, being a discrete time approximation to a continuous time probability of survival, it will tend to overstate ${}_0\delta(u, t)$, as noted in comment (i) in Section 5.1.

If, in addition, the claim amounts have a continuous distribution, as is the case in all the numerical examples considered in this paper, there is a further reason why $\delta(u\beta, (1+\theta)\beta t)$ may overstate the value of ${}_0\delta(u, t)$. This is that for survival to occur according to the former, the surplus need only stay above the value -1 (but could be zero at any time), whereas for survival to occur according to the latter, the surplus must never go below zero, no matter by how little.

For the risk process characterized by (1.2) we defined in Section 4 the survival probability $\delta^*(u, t)$ for $u \geq 0$ and $1 \leq t \leq \infty$ as follows:

$$\delta^*(u, t) = \Pr\left[u+\tau - \sum_{i=1}^{N_\tau} Y_i \geq 1 \quad \text{for} \quad \tau = 1, 2, \ldots, t\right]$$

This differs from $\delta(u, t)$ in that for survival it requires the surplus to be strictly greater than zero after time zero. For finite t, $\delta^*(0, t)$ can be calculated from formula (4.3). For t equal to infinity, $\delta^*(0)$ is equal to $\theta/(1+\theta)$, as explained in Section 7. For u greater than zero it is clear that:

$$\delta^*(u, t) = \delta(u-1, t)$$

It could be argued, for the reason given in the second paragraph in this section, that $\delta^*(u\beta, (1+\theta)\beta t)$ is a more logical approximation than $\delta(u\beta, (1+\theta)\beta t)$ to ${}_0\delta(u, t)$, although, depending to some extent on the discretization of the claim amount distribution, it may tend to understate ${}_0\delta(u, t)$.

Table 5 shows the results of approximating ${}_0\delta(u, t)$ by $\delta^*(u\beta, (1+\theta)\beta t)$ for the risk process with exponentially distributed individual claims, premium loading factor equal to 0.1 and parameter β equal to 20. The key to Table 5 is as follows:

TABLE 5 (See Section 8 for details)

		$t = 10$	$t = 20$	$t = 40$	$t = \infty$
$u = 0$	(1)	0.2146	0.1682	0.1362	0.0909
	(2)	0.2146	0.1682	0.1362	0.0909
	(3)	1.0000	1.0000	1.0000	1.0000
$u = 5$	(1)	0.8094	0.7044	0.6046	0.4230
	(2)	0.8094	0.7043	0.6045	0.4229
	(3)	1.0000	0.9999	0.9998	0.9998
$u = 10$	(1)	0.9681	0.9179	0.8427	0.6337
	(2)	0.9681	0.9178	0.8426	0.6337
	(3)	1.0000	0.9999	0.9999	1.0000

(1) denotes the exact value of ${}_0\delta(u, t)$ as given by SEAL (1978);

(2) denotes the value of $\delta^*(u\beta, (1+\theta)\beta t)$ calculated using the methods of Section 4 ($u = 0$ and $t < \infty$), of Sections 3 and 4 ($u > 0$ and $t < \infty$) or of Section 7 ($t = \infty$) as appropriate;

(3) denotes the ratio (2)/(1).

The approximations to ${}_0\delta(u, t)$ in Table 5 can be compared with the approximations (for $\beta = 20$) in Tables 2 and 4. It can be seen that the approximations in Table 5 are very much better than those in Tables 2 and 4. One explanation for this may be that two "errors" in the approximation of ${}_0\delta(u, t)$ by $\delta^*(u\beta, (1+\theta)\beta t)$, i.e.

(a) understating ${}_0\delta(u, t)$ by redefining survival/ruin, and

(b) overstating ${}_0\delta(u, t)$ by using a discrete time approximation to a continuous time survival probablity,

are working in opposite directions and cancelling each other out.

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PREMIUM CALCULATION IMPLICATIONS OF REINSURANCE WITHOUT ARBITRAGE

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Abstract

Constraints imposed on premium calculation principles are studied under one aspect of competitive market theory: the impossibility of systematic arbitrage. Principles based on second moments or utility theory are shown to lead to arbitrage possibilities, while some other principles do not.

Keywords

Premium calculation; arbitrage; utility theory; capital asset pricing model.

1. INTRODUCTION

Insurers are in business to make a profit, and risk theory has shown that a profit margin is in fact required by prudent insurers. How to build into different insureds' premiums margins which rightly reflect relative riskiness is the topic of Premium Calculation Principles. For instance, the expected value principle adds a constant percentage load to each contract, while the standard deviation and variance principles incorporate loads proportional to the second central moment, or its square root.

In this paper it is hypothesized that at any one time there is a market premium calculation principle operant, and that market forces require its use by all insurers and reinsurers, wittingly or not. It is further assumed that a market is available for any risk priced according to this principle. One aspect of competitive market theory, namely that arbitrage profit possibilities are quickly extinguished by market competition, will then be used to place constraints on what this market premium calculation principle could be. Essentially, principles will be ruled out by showing how a portfolio of direct, assumed, and ceded policies could be assembled to create an arbitrage profit if that principle were the market principle. A class of premium principles consistent with no arbitrage will be identified.

Results using the theory of arbitrage free markets to price financial assets were given by Harrison and Kreps (1979). Merton (1973) showed how options can be priced through this approach. Generally the theory is carried out assuming that transaction costs will have minimal effects. This will also be assumed below, and so the results are strictly applicable only under this assumption. However, some consideration as to the possible impacts of non-negligible transactions costs are addressed.

2. CONSTRAINTS ON PREMIUM PRINCIPLES

2.1. Empirical constraint

First, an empirical observed constraint will be employed: a premium calculation principle should produce a higher load, relative to expected losses, for an excess of loss cover than for a primary cover on the same risks. This constraint thus automatically rules out an expected value load. Further observational evidence on relative premiums for excess of loss covers by layer will be proposed below as tests of remaining principles.

2.2. Arbitrage constraints

The first constraint imposed by arbitrage considerations is additivity for independent risks. This is illustrated by an example. Loss experience for a group of 100 truck drivers who band together to buy insurance will be relatively more predictable than for one of the truckers alone, i.e., probable deviations from expected results will be smaller. Because of the greater uncertainty, an insurer may want to give a single trucker a proportionally larger load than the 100 together.

If this happens as a general market practice, however, a reinsurance arbitrage possibility is created. Reinsurers could assume the liability on the single truckers for lower risk premium than the insurers charged, guaranteeing those insurers a risk free profit, and then cede groups of truckers for a still lower premium that is nonetheless higher than the market would charge for such a group, thereby achieving a risk free profit themselves and an above market premium for their retrocessional markets.

In general, the possibility of such packaging of exposures shows that a market without arbitrage must charge additive premiums for independent risks. This constraint rules out the standard deviation load as a market pricing principle. The standard deviation of risk experience would be 10 times as great for the 100 truckers than for one alone, giving them one-tenth the proportional load of a single trucker. Thus 90% of the load for individual truckers in this example would be available for arbitrage profits.

The profit available from any such reinsurance arbitrage would be reduced by transaction costs. However, given the automatic facilities available in the reinsurance market, such costs are likely to be small compared to the 90% of profit available to the cedents at no risk. The market could in fact sustain a charge to the small risk equal to their share of these transaction costs, in addition to a load proportional by risk size to the large risks' load. It is doubtful, however, that this could produce a standard deviation based load.

The second constraint is additivity for non-independent risks. Again reasoning by example, consider a retired couple who own two mobile homes in the same trailer park in Oklahoma and who want to purchase homeowners insurance. When the wind comes sweeping down the plain, both homes stand a chance of being damaged. An insurer may thus feel exposed to more than twice the dollar variability in results insuring both than insuring just one, and may thus want more than twice the single home premium for the two.

But the market cannot charge a two trailer surcharge, because the couple could just buy separate policies. Alternatively, the insurer could cede them separately to two reinsurers. Either alternative illustrates the requirement that market premiums be additive for non-independent risks, and thus rules out the variance principle. Otherwise, de-packaging of exposures could create arbitrage profits. The de-packaging transaction cost to the original insured could be quite small, in that two policies could probably be obtained in one visit to a broker.

The additivity requirement can also be illustrated in the realm of excess reinsurance. Layering a risk reduces the variance, as the sum of the variances of the layers is less than the variance of the whole. The covariances are positive and they disappear in the layering process. If there is a price benefit to this layering, it must be passed on to the original insureds. Otherwise, if the total price is greater than the sum of the layer prices, arbitrage possibilities are created. For instance, since we are assuming that markets exist, the insureds could buy primary and excess coverage separately, and get the price benefit for minimal transaction costs.

3. UTILITY PRINCIPLES

The above constraints together also rule out premiums calculated as the certainty equivalent from a utility function, as will be shown next.

If $u(s)$ is the utility of the current surplus, the certainty equivalent of a portfolio of risks with uncertain losses X is that p which gives $u(s) = E[u(s+p-X)]$. That is, it is the constant amount which makes one with utility function u indifferent between taking both the premium and the portfolio of risk or taking neither.

A popular example is exponential utility, e.g., $u(s) = 1-\exp(-s/a)$. It is not difficult to show that $p = a \ln E[\exp(X/a)]$. It follows readily from this that the certainty equivalent of a portfolio of independent risks is the sum of the certainty equivalents of the risks in the portfolio. BORCH (1968) showed that additivity for independent risks holds only for the linear and exponential utility functions. Thus additivity for independent risks rules out any others.

For correlated risks X and Y, however, $E[\exp(X/a)\exp(Y/a)]$ is not the same as $E[\exp(X/a)]\,E[\exp(Y/a)]$, due to covariance, and so additivity will fail. Thus additivity for non-independent risks rules out exponential utility. Linear utility is a special case of the expected value principle, and so ruled out by empirical constraints.

4. POSSIBLE PREMIUM PRINCIPLES

4.1. Introduction

Two principles that can sometimes meet the above constraints are: 1) expected value principle applied to an adjusted probability distribution and 2) a load proportional to the covariance of the risk with a selected "target" variable. Since both operations are additive regardless of independence, the additivity constraints are always satisfied. That higher percentage loads for higher layers can sometimes hold as well is shown below.

4.2. Adjusted distribution principles

Consider for example a line of business with the (shifted) Pareto severity distribution $1-(1+x/b)^{-2}$. (For the sake of argument, assume that this distribution incorporates both process and parameter risk, if that distinction is of concern.) The expected claim size is given by b, and the claim size limited to x is $b(1-(1+x/b)^{-1})$. The premium calculation principle to be used is to replace b by $1.1\,b$ in the distribution function, and then compute the expected value of a loss under the adjusted distribution. This will be done for two covers: primary coverage up to the limit $10\,b$, and excess coverage above this limit.

Note that the original severity gives an expected loss of $10\,b/11$ and $b/11$ for these layers, respectively. Under the adjusted distribution, these become $110b/111$ and $12.1\,b/111$, respectively, for a total of $122.1\,b/111$, which is a 10% load overall. This breaks down as a 9% load for the primary layer and a 20% load for the excess. Although in this example, the charge was the mean from the adjusted distribution, a constant times this mean could be used as well.

This is an example of a scale transformation of a distribution. In general, if $f(x)$ is a density function and $a > 0$, $g(x) = af(ax)$ is a scale transformation. It (g) is also a distribution function, i.e., positive and integrates to unity, which can be seen by the change of variable $y = ax$. The distribution functions are related by $G(x) = F(ax)$. A scale transformation is particularly easy to implement if the distribution has a scale parameter, like b in the example above. Transforming the scale of the severity distribution produces the same scale change on aggregate losses, and this is essentially the only way to do so.

Replacing the distribution by any other distribution will satisfy the additivity constraints. A scale transformation is probably the most elementary approach to finding a revised distribution. The above example shows that this can result in higher loads for excess layers, which is the empirical constraint. A more intricate transformation is the combined scale-power transformation $g(x) = ac(ax)^{c-1}f((ax)^c)$, i.e., $G(x) = F((ax)^c)$. This transformation changes an exponential distribution into a Weibull, for example, or a Pareto into a Burr.

4.3. Covariance principles

For the covariance case, let G, the price of gold, be the target variable, and let the premium for a loss variable X be $aE(X)+b\operatorname{Cov}(X, G)$. Because the covariance of two variables X and Y with a fixed auxiliary variable is additive whether or not X and Y are independent, this satisfies the additivity constraints. Does it satisfy the empirical constraint? Presumedly G is highly correlated with the inflation rate, as are the excess losses, while the primary losses are probably less so. Thus the loading factor for excess could exceed that for primary under this principle.

The price of gold may not be a reasonable target variable. CAPM theory suggests using the gains on the stock market. A more general approach is given by ANG and LAI (1987), who argue from capital market and insurance market considerations that a reasonable target variable might be the difference between total market insured losses and total investment gains on all publicly traded instruments in the economy. As they show, this overcomes some of the problems insurance practitioners have had with CAPM, and it quite possibly could give a higher percentage load to excesss losses, but this would have to be verified. However, both this and CAPM theory assume either a quadratic utility function or that risk preferences can be captured with just two moments, both which are questionable.

If by some chance the target variable turned out to be the losses on a particular insurance portfolio, then the covariance pricing principle applied to that portfolio would be the variance principle. Changing the target variable for every contract so that the variance principle would always be applied would not satisfy the arbitrage constraints, however.

4.4. Covariance principle results from an adjusted distribution

Another method of adjusting a distribution is to multiply the density $f(x)$ by a non-negative function $h(x)$ such that $f(x)\,h(x)$ integrates to unity. Quite a range of such functions could be used, for as long as the integral is finite, it can be made to be unity by applying a factor to h.

As an example, let $h(x) = 1+b(E(Y|x)-E(Y))$ for some target variable Y, where b is small enough for h to be positive. Then $E(h(X)) = 1+b(EE(Y|X)-E(Y)) = 1$, which shows that $f(x)\,h(x)$ is a density function, and $aE(Xh(X)) = aE(X)+ab(E(XY)-E(X)\,E(Y)) = aE(x)+ab\ \mathrm{Cov}\ (X, Y)$, which shows that the adjusted distribution principle for this h is the covariance principle with target Y.

4.5. Other principles

Can every principle which satisfies additivity be expressed as an adjusted distribution principle? That is, does an additive premium principle induce a distribution function on loss random variables so that the price for any coverage can be expressed as the expected value of the losses for that coverage under that distribution?

It would seem that this could be approximated to any desired degree of accuracy, according to the following reasoning. For any $m > 0$, consider the coverage C_m which pays a small amount d just in the event that losses are at least m. First, it would seem that any coverage could be approximated by a linear combination of these coverages, i.e., as $\Sigma a_i C_i$. For instance, full coverage up to some limit M would be approximately $\sum_{i=1}^{M/d} C_{id}$. This pays d if losses are at least d, another d if they are at least $2d$, etc. This approximation gets better with smaller d. If there is no upper limit, the sum can go to infinity. If there are only partial payments, the coefficients a_i would be less than 1.

The price of the coverage C_m will be seen to induce a probability distribution that by additivity will in turn generate the prices for all coverages. The price of C_m given a distribution F would be $d(1-F(m))$. Thus the price function induces the distribution $F(m) = 1 - \text{Price}\,(C_m)/d$.

By additivity, the price of any layer of full coverage, i.e., a_i's all equal 1, would be the sum of these terms $d(1-F(i))$, which would be the expected value under the induced distribution F. If the coverage is not 100% in the layer, i.e., a_i's < 1, the price would have to reduce by the same percentage as the coverage, because by additivity the full coverage price would have to be the sum of the prices of the reduced coverage layer and its complementary layer. Thus the price would again be computed as the expected value under the induced distribution.

The result is that the only premium calculation principles that preserve additivity are those generated by transformed distributions. This is similar to the results of HARRISON and KREPS (1979), and later HARRISON and PLISKA (1981), who showed that in an arbitrage free market, pricing of financial instruments should take place according to the expectation under a risk adjusted probability distribution. It is also closely related to the results of DELBAEN and HAEZENDONCK (1989). They however apparently allow the random variable being priced to enter into the probability adjustment, so that a variance load can result, which contrasts to the arbitrage free considerations above.

5. APPLICATION

5.1. Minimum rates on line

An empirical reinsurance market phenomenon is minimum rate on line. The rate on line is the premium divided by the coverage limit, and most reinsurers establish a minimum they will accept for this ratio. Although there are various ad hoc explanations for this practice, it would be interesting to see to what extent it could be explained as a form of risk load. The example below shows that this can be partially accomplished by an adjusted distribution risk load.

The above shifted Pareto distribution is similar in form but less heavy tailed than severity distributions commonly used in US casualty insurance. The adjustment below can be done with more heavy distributions with similar effect. For this distribution, the expected loss in the layer (u, v) is $b^2(v-u)/(b+u)(b+v)$. For pricing excess coverage above $1000b$, assume use of a charge of $1.25\ 10^{-6}$ times the expected value from a distribution with

$1-F(x) = (1+x/b)^{-.1}$ for $x > 1000b$. For this distribution, the layer expected value is $(b/.9)\,[(1+v/b)^{.9}-(1+u/b)^{.9}]$. Take the case where $b = 1000$, which has severity mean = 1000. The expected loss per first dollar claim and the corresponding charge from this rule are shown for various \$1 million excess layers:

Retention	Layer Expected	Layer Charge
1,000,000	.499251	.602821
11,000,000	.007574	.490740
21,000,000	.002164	.460965
31,000,000	.001008	.443690
41,000,000	.000581	.431624
51,000,000	.000377	.422405

Although an absolute flat charge per million of coverage is never reached, it is closely approximated by this rule. The charges are certainly dropping off much more slowly than the expected losses. This example shows that the kind of leveling of charges seen in minimum rates on line can be produced by adjusting distributions. The key is to have a low absolute value for the negative exponent in the pricing distribution function. The layer charges and the point at which leveling off occurs can be adjusted through the b parameter and the constant multiplier, here $1.25\ 10^{-6}$.

This approach to minimum rates on line will approximate such a minimum for risks of a given size, but larger or smaller risks will have larger or smaller rates. A true minimum rate on line applicable to all risks or treaties would seem to generate arbitrage possibilities. A reinsurer could retrocede two minimum rated risks for the price of one. In a competitive market, competition would reduce the minimums for smaller treaties to gain this retrocessional opportunity.

In a quasi-monopolistic market where these savings are not passed on to the original cedent or insured, spirals of retrocession could be generated, where A retrocedes to B who retrocedes to C who retrocedes to A, etc., with an arbitrage profit being taken at each step.

6. SUMMARY

A general advantage of changing the distribution is that it is easy to calculate charges, at least after the adjusted distribution has been established. A particular advantage of a covariance load is that in the form of CAPM it has somewhat of an economic justification. It is not clear that arbitrage theory itself could further specify the adjustments to the distribution, however. The best test is probably empirical, i.e., what sells; life actuaries have been at this enterprise for years, adjusting mortality tables in different ways depending on whether an annuity or insurance is being marketed. The time honored practice of fudging the table thus has stronger justification than might have been anticipated.

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WORKSHOP

A MIXED MODEL FOR LOSS RATIO ANALYSIS

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ABSTRACT

The model introduced may be treated as a mixed two-way analysis of variance with fixed company effects and random time effects. Further, the risk volumes are integrated into the model in such a way that the unexplained variance is inversely proportional to the risk volume of each company. The proposed model is used to analyze loss ratio data from the general insurance market in Kuwait. The maximum likelihood estimates of the structural parameters are obtained. These estimates are then used to compute the loss ratios and solvency margins for the four domestic insurance companies.

KEYWORDS

General insurance; lognormal distribution; restricted maximum likelihood estimators; risk volume; solvency margins.

1. INTRODUCTION

Loss ratios play a very important role in risk theory. For one thing, they are used in credibility analysis to predict future losses, which is pertinent to ratemaking, and for another, they are used to compute solvency margins.

The traditional approach is to assume that the loss ratios follow a beta distribution, whose two parameters are then estimated by the method of moments, e.g., see DE WIT and KASTELIJN (1980). This approach was criticized by RAMLAU-HANSEN (1982) who assumed the loss ratios to follow a lognormal distribution. The shape of the lognormal curve is appealing in this context and it has been applied before to model loss ratio data, see HUNTER (1980). The model introduced by RAMLAU-HANSEN may be treated as a two-way analysis of variance with random company and time effects. It may also be viewed as a parametric credibility model with seasonal random factor, see SUNDT (1979).

The assumption of random-effects for the companies under study is appropriate when they are considered to be a random sample of the companies in the population. However, if they constitute the whole population or if they are considered to represent themselves, they should be assumed to have fixed-effects. To deal with such a situation, we propose here a mixed two-way analysis of variance model with fixed company effects and random time effects.

RAMLAU-HANSEN also assumed that the unexplained variance is constant and independent of the risk volume of each company. However, since a small portfolio would usually mean large fluctuations in loss ratios, and therefore a large company would need a lower solvency margin than a small company, we adopt here the more realistic assumption that the unexplained variance is inversely proportional to the risk volume, e.g., earned premiums, of each company.

The mixed model is introduced in Section 2, while the maximum likelihood estimators of the parameters are developed in Section 3. Then, in Section 4, the proposed model is used to estimate the loss ratios and solvency margins for the four national insurance companies in Kuwait using data from the general insurance market.

Regarding notation, we use I_m to denote the $m \times m$ identity matrix, H' and $\operatorname{tr}(H)$ to denote the transpose and trace, respectively, of a matrix H, and $N_m(\mu, \Sigma)$ to denote the m-dimensional multivariate normal distribution with mean vector μ and covariance matrix Σ.

2. THE MODEL

Let X_{ij} and p_{ij} denote the loss ratio and the earned premiums, respectively, for company i in year j. Set $Y_{ij} = \ln X_{ij}$ and assume that

$$Y_{ij} = \alpha_i + \beta_j + e_{ij}, \qquad i = 1, \ldots, a, \quad j = 1, \ldots, b,$$

where the α_i are unknown fixed constants, while the β_j and e_{ij} are mutually independent normal random variables having zero means and variances θ_2 and θ_1/p_{ij}, respectively. Thus, the parameter space is given by

$$\Theta = \{\alpha_1, \ldots, \alpha_a, \ \theta_1, \theta_2 \colon \alpha_i \in \mathscr{R}, \ i = 1, \ldots, a, \theta_1 > 0, \theta_2 \geq 0\}.$$

Now, for company i in year j, the loss ratio is given by

$$L_{ij} = \exp\{\alpha_i + .5(\theta_2 + \theta_1/p_{ij})\}. \tag{1}$$

Further, the upper limit of the loss ratio, at the probability level $(1-\varepsilon)$, is given by

$$U_{ij} = \exp\{\alpha_i + \Phi^{-1}(1-\varepsilon)\sqrt{\theta_2 + \theta_1/p_{ij}}\}, \tag{2}$$

where Φ denotes the cumulative standard normal distribution function. Hence, the solvency margin is obtained from

$$S_{ij} = \max\{0, (U_{ij} + E_i - 100)\}, \tag{3}$$

where E_i denote the expense ratio for company i.

3. THE MAXIMUM LIKELIHOOD ESTIMATORS

Let $n = ab$ and write the model in matrix form as

$$Y = A\alpha + B\beta + e,$$

where $Y, A, \ldots, e$ are defined in the obvious manner. In particular,

$$A = \begin{pmatrix} 1_b & 0 & \ldots & 0 \\ 0 & 1_b & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & 1_b \end{pmatrix},$$

and

$$B = (I_b, \ldots, I_b)'.$$

Then, define the diagonal matrix

$$P = \operatorname{diag}(p_{11}, \ldots, p_{ab}).$$

Therefore, under the model assumptions, we have that

$$Y \sim N_n(A\alpha, \theta_I P^{-1} + \theta_2 BB').$$

To simplify the discussion in the sequel we consider the transformation $P^{1/2} Y$. Thus,

$$P^{1/2} Y \sim N_n(P^{1/2} A\alpha, \Sigma),$$

where

$$\Sigma = \theta_1 I_n + \theta_2 P^{1/2} BB' P^{1/2}.$$

We now give an explicit expression for Σ^{-1}. Let

$$p_{+j} = \sum_{i=1}^{a} p_{ij}, \qquad j = 1, \ldots, b,$$

and define

$$\rho_j = \theta_2/(\theta_1 + \theta_2 p_{+j}), \qquad j = 1, \ldots, b.$$

It is easily verified that

$$\Sigma^{-1} = (1/\theta_1)\,[I_n - P^{1/2} B \Lambda B' P^{1/2}],$$

where $\Lambda = \operatorname{diag}(\rho_1, \ldots, \rho_b)$.

Following HARVILLE (1977), the likelihood equation for α is given by

(4) $$A' P^{1/2} \Sigma^{-1} P^{1/2} A\alpha = A' P^{1/2} \Sigma^{-1} P^{1/2} Y.$$

Now, let

$$p_{i+} = \sum_{j=1}^{b} p_{ij}, \qquad i = 1, \ldots, a,$$

and define H to be the $a \times a$ matrix whose elements are given by

$$H_{rs} = p_{r+} - \sum_{j=1}^{b} \rho_j p_{rj}^2, \qquad r = s = 1, \ldots, a,$$

$$= - \sum_{j=1}^{b} \rho_i p_{rj} p_{sj}, \qquad r \neq s.$$

Further, define h to be the $a \times 1$ vector whose elements are given by

$$h_r = \sum_{j=1}^{b} p_{rj} \left[Y_{rj} - \rho_j \sum_{i=1}^{a} p_{ij} Y_{ij} \right], \qquad r = 1, \ldots, a.$$

It is easily shown that equation (4) reduces to

$$\alpha = H^{-1} h. \tag{5}$$

Since the maximum likelihood estimators of θ_1 and θ_2 take no account of the loss in degrees of freedom resulting from estimating α, the vector of fixed company effects, we consider the restricted maximum likelihood approach of PATTERSON and THOMPSON (1971) to estimate the variance components, see HARVILLE (1977) for more details. To this end, let $Z = Y - B\beta^*$, where

$$\beta^* = \theta_2 B' P^{1/2} \Sigma^{-1} P^{1/2} (Y - A\alpha),$$
$$= (\theta_2/\theta_1) [I_b - B' PBA] B' P (Y - A\alpha).$$

Consequently, $Z_{ij} = Y_{ij} - \beta_j^*$, where

$$\beta_j^* = \rho_j \sum_{i=1}^{a} p_{ij} (Y_{ij} - \alpha_i), \qquad j = 1, \ldots, b.$$

Furthermore, define

$$S = I_n - P^{1/2} A (A' PA)^{-1} A' P^{1/2}.$$

Thus, the restricted likelihood equation for θ_1 is given by

$$\theta_1 = Y' P^{1/2} S P^{1/2} Z/(n-a),$$
$$= [Y' PZ - Y' PA (A' PA)^{-1} A' PZ]/(n-a),$$
$$= \left[\sum_{i=1}^{a} \sum_{j=1}^{b} p_{ij} Y_{ij} Z_{ij} - \sum_{i=1}^{a} \left(\sum_{j=1}^{b} p_{ij} Y_{ij} \right) \times \right.$$
$$\left. \times \left(\sum_{j=1}^{b} p_{ij} Z_{ij} \right) \Big/ p_{i+} \right] \Big/ (n-a). \tag{6}$$

On the other hand, let

$$T = [I_b + (\theta_2/\theta_1) B' P^{1/2} S P^{1/2} B]^{-1},$$
$$= (\theta_1/\theta_2) \Lambda [I_b + B' PAH^{-1} A' PB\Lambda].$$

Then,

$$\operatorname{tr}(T) = (\theta_1/\theta_2)\left[\sum_{j=1}^{b} \rho_j + \operatorname{tr}(H^{-1}G)\right],$$

where G is the $a \times a$ matrix whose elements are given by

$$G_{rs} = \sum_{j=1}^{b} \rho_j^2 p_{rj} p_{sj}, \qquad r, s = 1, \ldots, a.$$

It can be shown that the restricted likelihood equation for θ_2 is given by

$$\theta_2 = \sum_{j=1}^{b} \beta_j^{*2}/[b - \operatorname{tr}(T)]. \tag{7}$$

The likelihood equations (5)-(7) must be solved simultaneously for $\hat{\alpha}$, $\hat{\theta}_1$, and $\hat{\theta}_2$. However, except in some special cases, e.g., $p_{ij} = p$, or $p_{ij} = p_i$, say, explicit solutions of the above likelihood equations do not exist. Nevertheless, the form of the equations suggests the following iterative procedure, due to HENDERSON (1973). Set $\theta = (\theta_1, \theta_2)'$ and let $\theta^{(k)}, k = 1, 2, \ldots,$ denote the value produced by the procedure on its k^{th} iteration. Furthermore, for any quantity f which is a function of θ, we use $f^{(k)}$ to denote the value of f at $\theta = \theta^{(k)}$. Hence, we start the iteration by substituting an initial guess $\theta^{(0)}$ into

$$\theta_1^{(k+1)} = \left[\sum_{i=1}^{a} \sum_{j=1}^{b} p_{ij} Y_{ij} Z_{ij}^{(k)} - \sum_{i=1}^{a} \left(\sum_{j=1}^{b} p_{ij} Y_{ij}\right) \times \right.$$
$$\left. \times \left(\sum_{j=1}^{b} p_{ij} Z_{ij}^{(k)}\right) \middle/ p_{i+} \right] \middle/ (n-a) \tag{8}$$

and

$$\theta_2^{(k+1)} = \sum_{j=1}^{b} [\beta_j^{*(k)}]^2/[b - \operatorname{tr}(T^{(k)})], \tag{9}$$

and continue the iteration until $\theta^{(k+1)}$ is sufficiently close to $\theta^{(k)}$ in some norm.

HARVILLE (1977) showed that, if the above iterative procedure is started with strictly positive values for the variance components, then at no point can the values for the variance components ever become negative. It should also be noticed that this procedure must not be started with a zero value for any variance component, since the value for that component would then continue to be zero throughout the iterative procedure.

If we happen to have any prior information about θ, then we could use it to formulate an initial guess for θ. Otherwise, we could use the usual ANOVA estimators, obtained from (6) and (7) assuming that $P = I_n$, as initial guesses.

To this end, let $Y_{i+} = \sum_{j=1}^{b} Y_{ij}$, $Y_{+j} = \sum_{i=1}^{a} Y_{ij}$, and $Y_{++} = \sum_{i=1}^{a} \sum_{j=1}^{b} Y_{ij}$.

Now, define

$$R_2 = (1/a) \sum_{j=1}^{b} Y_{+j}^2 - Y_{++}^2/n,$$

and

$$R_1 = \sum_{i=1}^{a} \sum_{j=1}^{b} Y_{ij}^2 - (1/b) \sum_{i=1}^{a} Y_{i+}^2 - R_2.$$

Hence, the initial estimates are given by

$$\theta_1^{(0)} = R_1/[(b-1)(a-1)], \tag{10}$$

and

$$\theta_2^{(0)} = R_2/(b-1) - \theta_1^{(0)}]/a. \tag{11}$$

However, equation (11) may produce negative estimates which do not belong in the parameter space Θ. The usual practice of fixing negative estimates of variance components at zero would not be useful here since, as indicated above, we should not start the iterative procedure with a zero value for θ_2. Instead, an initial guess in the interior of Θ should be used to start the iterative procedure.

Thus, the procedure of computing the maximum likelihood estimates of the parameters starts by obtaining initial estimates of the variance components, e.g. from (10) and (11). These estimates are then substituted into (5) to estimates α. This estimate of α along with the initial estimates of θ_1 and θ_2 are then substituted into (8) and (9) to obtain $\theta^{(1)}$. This iterative process is to be continued until we achieve convergence after m iterations, say, at which time we get $\hat{\theta} = \theta^{(m)}$ and $\hat{\alpha} = \alpha^{(m)} = \alpha(\theta^{(m)})$.

The estimated loss ratios and solvency margins, for company i in year j, are obtained by substituting $\hat{\alpha}_i$, $\hat{\theta}_1$, $\hat{\theta}_2$ and p_{ij} into (1)-(3).

4. AN APPLICATION

Four domestic companies are operating in the Kuwaiti general insurance market, namely, Kuwait, Ahlia, Gulf and Warba. Since Warba started operating in 1978, we limited our data to the period 1978-1986. The incurred loss ratios during this period are given in Table 1, along with the associated data on earned premiums in millions of Kuwaiti dinars.

It must be more realistic to assume that the four domestic companies have fixed effects. Thus, the mixed model seems to be more suitable for the analysis of this set of data.

TABLE 1
INCURRED LOSS RATIOS (LR) AND EARNED PREMIUMS (EP)

Company	Kuwait		Ahlia		Gulf		Warba	
Year	LR	EP	LR	EP	LR	EP	LR	EP
1978	63	12.5	65	8.6	68	5.0	48	2.7
1979	67	12.1	68	9.6	66	5.7	84	5.6
1980	63	13.6	73	10.3	63	7.2	55	6.5
1981	58	15.8	78	11.9	67	8.5	56	7.2
1982	63	18.3	73	12.4	72	9.1	62	7.6
1983	67	20.2	54	12.8	89	9.2	53	7.9
1984	71	21.4	73	13.8	68	10.0	59	8.9
1985	68	18.6	74	12.6	72	10.0	58	8.3
1986	69	16.3	67	11.2	76	10.1	77	8.7

The initial estimates, computed from (10) and (11), were $\hat{\theta}_1^{(0)} = 0.01505$ and $\hat{\theta}_2^{(0)} = -0.00059$. Since $\hat{\theta}_2^{(0)}$ is negative, several initial guesses, all of which are in the interior of Θ, were used to start the iterative procedure and, in all cases, the procedure converged to the following estimates: $\hat{\theta}_1 = 0.12146$ and $\hat{\theta}_2 = 0$. These estimates were computed from (8) and (9) so that they are correct to 5 decimals. Then, they were substituted into (5) to get

$$\hat{\alpha} = (4.184,\ 4.237,\ 4.271, 4.114).$$

The loss ratios and solvency margins were computed from (1)-(3) using the above estimates along with the earned premiums of 1986. Further, it was assumed that $\varepsilon = 0.001$ and $E_i = 30\%$. The results appear in Table 2.

TABLE 2
ESTIMATED LOSS RATIOS AND SOLVENCY MARGINS

Company	Loss Ratio	Solvency Margin
Kuwait	65.9	15.1
Ahlia	69.6	24.7
Gulf	72.0	29.6
Warba	61.6	17.3

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DISTRIBUTIONS DE PARETO: INTÉRÊTS ET LIMITES EN RÉASSURANCE

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ABSTRACT

Instead of determining for a fire insurance portfolio the loss distribution purely based on the claims experience, we try to determine it based on the sums insured.

KEYWORDS

Loss distribution based on sums insured; Pareto distribution.

INTRODUCTION

Au lieu d'estimer, sur la base unique des sinistres, leur loi de distribution, on va essayer de le faire sur la base du profil du portefeuille qui les génère.

En Assurances Décès, s'il n'y a pas de relation entre le capital garanti en cas de décès et l'âge de l'assuré, les sinistres sont distribués comme le sont les capitaux assurés.

En Assurance Dommages, ce n'est généralement pas le cas.

Par exemple, si les risques assurés d'une compagnie sont distribués sur une loi de Pareto tronquée au plein de souscription, que la distribution des taux de dommages vérifie certaines propriétés acceptables, alors les sinitres ne sont pas issus d'une distribution tronquée de Pareto.

CADRE DE NOTRE ÉTUDE

Les hypothèses utilisées sont celles implictement admises pas les réassureurs:

- Le portefeuille assuré comprend N risques $(R_i)_{i=1,\dots N}$ de valeurs assurées $(K_i)_{i=1\dots N}$.
- Le portefeuille est stable dans le temps sur les m dernières années (nombre de polices par tranche de capitaux assurés) et pour l'année $m+1$.
- Pendant chacune de ces dernières années, et pour l'année $m+1$, pour chacun des N risques:
  - Si $(si_j)_{j=1\dots ni}$ désignent les n_i sinistres sur le risque i, $p(n_i = o) = \exp(-p_i) = \exp(-p)$
  - $\forall t$ dans $[0, 1]$, $\forall i$, $\forall j$, $p[(si_j/K_i) \leq t] = T(t)$ ne dépend pas de i et de j.

- T est une fonction continue sur [0, 1], dérivable sur [0, 1] strictement croissante sur [0, 1]
- n_i suit une loi de Poisson de paramètre p.
- Les sinitres survenant sont indépendants les uns des autres.

N.B.: Les hypothèses sur la fonction de distribution T de taux de dommages et sur la fréquence p sont assez restrictives.

On pourrait, sans trop de difficultés, les rendre moins contraignantes en écrivant que les risques i souscrits se caractérisent ainsi : ils sont issus d'une communauté M de risques I pour lesquels nous avons :

1. $\forall_I$, n_I suit une loi de Poisson de paramètre p_I.
2. $\forall_I, \forall_j, \forall_t$, $p([s_{Ij}/K_I] \leq t) = T_I(t)$ ne dépend que de I.
3. Il n'y a pas de corrélation entre p_I et K_I, d'une part, entre T_I et K_I d'autre part, ou encore :

   $\forall (a, b)\ a < b$,

   $E_M(p_I/[K_I \in [a, b]]) = p$ ne dépend pas de $[a, b]$

   $E_M(T_I(t)/[K_I \in [a, b]]) = T(t)$ ne dépend pas de $[a, b]$

Alors, toutes les espérances écrites par la suite subsistent ; elles sont des espérances d'espérances conditionnelles.

Les contraintes ainsi exposées n'altèrent en rien la latitude d'acceptation de l'assureur ni son niveau d'acceptation, notamment lorsqu'il y a beaucoup de coassurance.

PROBLÈME

On cherche à déterminer la loi L_x du nombre de sinistres supérieurs à x.

Montrons, dans un premier temps, que L_x est une loi de Poisson dont il suffira de connaître la moyenne $\bar{L}_x$ (E. STRAUB, 1971).

- Pour un risque i,

  L_{ix} suit une loi de Poisson de paramètre $p\left[1-T\left[\min\left(1, \frac{x}{K_i}\right)\right]\right]$

  — C'est évident si $x > Ki$.

  — Si $x \leq Ki$, alors :

$$p[L_{ix} = n)] = \sum_{j=n}^{+\infty} \binom{j}{n} \exp(-p) \frac{p^j}{j!}\, T\left(\frac{x}{K_i}\right)^{j-n} \left[1-T\left(\frac{x}{K_i}\right)\right]^n$$

$$= \exp(-p) \frac{p^n}{n!} \left[1-T\left(\frac{x}{K_i}\right)\right]^n \sum_{j=n}^{\infty} \frac{p^{j-n}}{(j-n)!}\, T\left(\frac{x}{K_i}\right)^{j-n}$$

$$= \exp(-p)\exp\left[pT\left(\frac{x}{K_i}\right)\right]\left[p\left[1-T\left(\frac{x}{K_i}\right)\right]\right]^n\frac{1}{n!}$$

$$= \exp\left[-\left[p\left(1-T\left(\frac{x}{K_i}\right)\right)\right]\right]\left[p\left[1-T\left(\frac{x}{K_i}\right)\right]\right]^n\frac{1}{n!}$$

- Pour chaque risque i,

  L_{ix} suit une loi de Poisson de paramètre $p\left[1-T\left[\min\left(1,\frac{x}{K_i}\right)\right]\right]$

- $L_x = \sum_i L_{ix}$. Les L_{ix} étant indépendantes,

- L_x suit une loi de Poisson de paramètre

$$p\sum_1\left[1-T\left[\min\left(1,\frac{x}{K_i}\right)\right]\right] = \bar{L}_x = L(x)$$

L'APPROCHE DES RÉASSUREURS

Pour évaluer le coût d'une couverture en excédent de sinistres, l'assureur et le réassureur ont besoin de connaître $L(x)$, l'espérance mathématique du nombre de sinistres supérieurs à x. Pour cela, on va passer du cas discret évoqué précédemment au cas continu.

Ainsi, la prime pure requise pour une couverture A *xs* B est-elle:

$$P = AL(A+B) - \int_B^{A+B}(x-B)\frac{dL(x)}{dx}\,dx$$

- Distribution des capitaux assurés:

- On supposera que les risques sont distribués en montant, sur une loi de Pareto de paramètre α pour $K > c$, c étant fixé. On montrera, sur un exemple en annexe, que cette hypothèse est acceptable.

  De la connaissance de la valeur certaine N_c, nombre de risques supérieurs à c, on estime N_x:

$$\hat{N}_x = N_c\left(\frac{c}{x}\right)^{\alpha}$$

[Si, lors de l'estimation, $\hat{\alpha}$ est non biaisé, alors $\hat{N}_x$ surestime en moyenne N_x.

En effet, $\hat{N}_x = N_x\left(\frac{c}{x}\right)^{\varepsilon}$ avec $E(\varepsilon) = 0$.

$$E\left[\left(\frac{c}{x}\right)^{\varepsilon}\right] = E\left(\exp\left(\varepsilon \ln \frac{c}{x}\right)\right) \geq \exp\left[E\left(\varepsilon \ln\left(\frac{c}{x}\right)\right)\right] = 1\Big]$$

- Calcul de $L(x)$:

$$L(x) = \int_x^{+\infty} -\frac{dN_u}{du} p\left[1 - T\left(\frac{x}{u}\right)\right] du$$

$$= \int_x^{+\infty} \alpha N_c \frac{c^\alpha}{u^{\alpha+1}} p\left[1 - T\left(\frac{x}{u}\right)\right] du$$

$$= \int_x^{+\infty} N_c \frac{c^\alpha}{u^{\alpha+2}} pxt\left(\frac{x}{u}\right) du$$

après intégration par partie.

Par ailleurs $L'(x) = -\alpha N_c \dfrac{c\alpha}{x^{\alpha+1}} p[1 - T(1)] - \displaystyle\int_x^{+\infty} p\alpha N_c \frac{c^\alpha}{u^{\alpha+2}} t\left(\frac{x}{u}\right) du$

soit $L'(x) = -\dfrac{\alpha}{x} L(x)$ ou encore $L(x) = x^{-\alpha} a^\alpha L(a)$, $\forall a \geq c$

(t désigne la dérivée de T).

On rappelle ici que, si $F(x) = \displaystyle\int_c^x f(x, u)\, du$, et que la différentielle df de f existe, alors F' existe et $F'(x) = f(x, x) + \displaystyle\int_c^x \frac{\partial f}{\partial x}(x, u)\, du.$

- *Intérêt d'une telle formule :*
  T et p existent mais sont inconnus. On va revenir à l'expérience du passé et estimer :

$L(a)$ par $\hat{L}(a) = \dfrac{\text{nombre de sinistres survenus supérieurs à } a \text{ en } m \text{ années}}{m}$

$$= \frac{m_a}{m}$$

pour tout $a \geq c$.

On choisit a le plus faible possible pour minimiser $\dfrac{\sigma[L(y)]}{L(y)} = \dfrac{1}{\sqrt{L(y)}}$

On prend donc $a = c$.

En effet, l'erreur relative sur $L(x)$ est d'autant plus faible qu'est faible l'erreur relative sur $L(a)$.

($\sigma[L(y)]$ désigne l'écart type de Ly).

- *Tarification:*

— Fréquence de sinistres supérieurs à B:

$$\hat{f}_B = \hat{f}_c \left(\frac{c}{B}\right)^\alpha$$

— Coût moyen d'un sinistre $d'xs$.

Coût moyen d'un sinistre supérieur à B: pour $\alpha > 1$

$$c_m(B) = \int_B^{+\infty} (x-B)\, dF_\alpha(x)/[1-F_\alpha(B)]$$

$$= \int_B^{+\infty} [1-F_\alpha(x)]\, dx/[1-F_\alpha(B)]$$

$$= \int_B^{+\infty} \frac{x^{-\alpha}}{a^{-\alpha}}\, dx/[1-F_\alpha(B)] = \frac{B}{\alpha-1}$$

— Coût moyen d'un sinistre $d'xs$:

$$c_m(A, B) = c_m(B) - \frac{1-F_\alpha(A+B)}{1-F_\alpha(B)}\, c_m(A+B)$$

$$= \frac{B}{\alpha-1} - \frac{A+B}{\alpha-1}\left(\frac{B}{A+B}\right)^\alpha$$

— Estimation de la prime pure requise:

$$\hat{P} = \frac{m_c}{m}\left(\frac{c}{B}\right)^\alpha \frac{B}{\alpha-1}\left[1 - \left(\frac{B}{A+B}\right)^{\alpha-1}\right]$$

Cette formule reste vraie pour $0 < \alpha < 1$.

Il suffit d'écrire

$$[1-F_\alpha(B)]\, C_m(A, B) = \int_B^{A+B} (x-B)\, dF_\alpha(x) + A[1-F_\alpha(A+B)]$$

Pour $\alpha = 1$

$$P = \frac{m_c}{m}\frac{c}{B} B \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[1 - \left(\frac{B}{A+B}\right)^{\varepsilon}\right] = \frac{m_c}{m} c \ln\left(\frac{B+A}{B}\right)$$

Le grand avantage de cette formulation réside dans la disparition de p et de T.

Le réassureur ne peut estimer p car il ne connaît que les sinistres supérieurs à un certain seuil s, qu'on a supposé ici inférieur à c. Pour les mêmes raisons, il ne peut estimer T.

A fortiori, dans l'hypothèse d'hétérogénéité, on ne peut estimer p_i et T_i pour chaque i, d'autant plus qu'on ne connaît pas les fonctions de structures de (p) et (T).

CRITIQUE DE LA MÉTHODE

On a supposé que T existait, mais était inconnue (si T était connu, $L(x)$ le serait immédiatement.

Des fonctions T circulent chez les réassureurs, sans qu'ils sachent bien lesquelles utiliser.

La méthode précédemment décrite a l'énorme avantage de ne pas requérir la connaissance de T.

Malheureusement, elle présente un énorme défaut. $L(x)$ est calculé en écrivant $N_x = N_c\left(\frac{c}{x}\right)^{\alpha}$ pour tout x et même si x est supérieur à F, le plein de souscription.

Or $N_y = o$ pour $y > F$. Il nous faut donc, au mieux, utiliser une loi de Pareto tronquée au point F. On supposera donc que toute la densité de risques supérieurs à F, suivant l'hypothèse paretienne, est concentrée en F, après dégagement par exemple en Facultatives *proportionnelles*.

$$\text{Ainsi, } L(y) = \underbrace{\int_y^F \frac{-dN_u}{du} p\left[1 - T\left(\frac{y}{u}\right)\right] du}_{L_1(y)} + \underbrace{N_c\left(\frac{c}{F}\right)^{\alpha} p\left[1 - T\left(\frac{y}{F}\right)\right]}_{L_2(y)}$$

En intégrant par partie,

$$L_1(y) = -N_F p\left[1 - T\left(\frac{y}{F}\right)\right] + \int_y^F yN_u/u^2 pt\left(\frac{y}{u}\right) du$$

$$L_1'(y) = \frac{dN_u}{du}(y)p\left[1 - T\left(\frac{y}{y}\right)\right] + \int_y^F \frac{dN_u}{du}\frac{p}{u}\, t(y/u)\, du$$

$$= -\alpha \int_y^F p\frac{N_u}{u^2}\, t\left(\frac{y}{u}\right) du$$

soit $\frac{y}{\alpha} L_1'(y) = -L_1(y) - N_c \left(\frac{c}{F}\right)^\alpha p \left[1 - T\left(\frac{y}{F}\right)\right]$

En posant $L_1(y) = \phi(y) y^{-\alpha} Nc \left(\frac{c}{F}\right)^\alpha p$

$$L_1'(y) = \frac{\phi'(y)}{\phi(y)} L_1(y) - \frac{\alpha}{y} L_1(y)$$

soit $\phi'(y) = -\alpha y^{\alpha-1} \left[1 - T\left(\frac{y}{F}\right)\right]$

soit $\phi(y) = \phi(F) + \alpha \int_y^F u^{\alpha-1} \left[1 - T\left(\frac{u}{F}\right)\right] du$

La condition $\phi(F) = 0$ [car $L(F) = 0$ puisque $T(u) = 1 \Rightarrow u = 1$] nous indique que:

$$L_1(y) = \alpha \left[\int_y^F u^{\alpha-1} \left[1 - T\left(\frac{u}{F}\right)\right] du\right] y^{-\alpha} N_c \left(\frac{c}{F}\right)^\alpha p$$

et

$$L(y) = \left[\left[\int_y^F u^{\alpha-1} \left[1 - T\left(\frac{u}{F}\right)\right] du\right] \alpha y^{-\alpha} + \left[I - T\left(\frac{y}{F}\right)\right]\right] N_c \left(\frac{c}{F}\right)^\alpha p$$

Exemple: Prenons
- $\alpha = 2$ pour $x > 1$ M FRF.
- $t(u) = 1$ sur $[0, 1]$
- $p = 1$‰; 10.000 risques de valeurs assurées > 1 M FRF.
- $F = 10$ M FRF.

$$L_1(y) = -\int_y^F p \frac{d}{du}\left[10.000 \left(\left(\frac{10}{u}\right)^6\right)^2\right]\left(1 - \frac{y}{u}\right) du$$
$$= 10^{13}\left[\frac{1}{3y^2} - \frac{1}{F^2} + \frac{2y}{3F^3}\right]$$

On a supposé 10.000 risques > 1 M FRF. Or, on n'en dénombre ici que 9.900 = 10.000 × F_α (10 M FRF).

Supposons, au pire, que les 100 risques «manquants» se situent en F comme on l'a fait précédemment.

Alors $L(y)$, espérance du nombre de sinistres supérieurs à y, s'écrit:

$$L(y) = L_1(y)+[1-F_\alpha(F)]N_c p\left[1-T\left(\frac{y}{F}\right)\right]$$

Pour notre exemple, $L(y) = L_1(y)+0{,}1\left[1-\frac{y}{F}\right]$

TABLEAU RÉCAPITULATIF

x	$L(x)$ (1)	$L_1(x)$ avec troncation	$L(x)$ (2)	(2)/(1)
1.000.000	3,3	3,2	3,3	$\simeq$100 %
2.000.000	0,83	0,74	0,82	98,7 %
3.000.000	0,37	0,29	0,36	97,3 %
4.000.000	0,21	0,14	0,20	95,2 %
5.000.000	0,13	0,06	0,11	85 %
6.000.000	0,093	0,033	0,073	78 %
7.000.000	0,068	0,015	0,045	66 %
8.000.000	0,052	0,006	0,026	50 %
9.000.000	0,041	0,001	0,011	27 %
10.000.000	0,033	0	0	0 %

La prise en compte de la troncation (2) modifie donc la loi de Pareto des sinistres (1). Les sinistres ne sont alors plus distribués sur une loi de Pareto.

Intuitivement, cela s'explique par le fait qu'il n'y a pas de générations de risques supérieurs à F pour remplacer ceux qui disparaissent (ceux $< x$). Cette modification peut être déterminante.

CONCLUSION

A travers cet exemple, force est de constater que la connaissance d'un profil de portefeuille, d'une espérance de sinistres en nombre à un seuil suffisamment bas pour que l'expérience nous en donne un bon indicateur ne suffisent par à déterminer une espérance (en nombre de sinistres) de sinistres importants qui pourraient affecter lourdement le compte d'exploitation d'un assureur.

La connaissance de la loi de distribution des taux de dommages des riques en portefeuille s'avère nécessaire.

ANNEXE 1

Estimation de a

1^er cas

Les valeurs assurées des J risques K_1, K_2, ... K_J des risques supérieurs à c sont connues.

La vraisemblance $V(K_1, K_2, \dots K_J, \alpha)$ de notre échantillon s'écrit:

$$V = \prod_{i=1}^{J} \left[\alpha \left(\frac{c}{K_i}\right)^{\alpha} x \frac{1}{K_i}\right] = \alpha^J c^{\alpha J} \left[\prod_{i=1}^{J} K_i\right]^{-1-\alpha}$$

En maximisant le logarithme de V, $\left[\frac{\partial}{\partial \alpha} [\ln V((K_i), \hat{\alpha})] = 0\right]$,

nous obtenons: $\hat{\alpha} = \dfrac{J}{\sum_i \ln \left(\frac{K_i}{c}\right)}$

— On peut montrer que $\hat{\alpha}$ est asymptotiquement non biaisé (METTE RYTGAARD, 1989).

— $\hat{\alpha}$ est une fonction continue de $(K_1, \dots K_J)$.

2^e^ *cas*

On ne connaît, en général, que le nombre de polices par tranche de capital assuré, pour les risques supérieurs à c.

Tranche	Numéro de tranche	Nombre de risques
$A_0 - A_1$	1	n_1
---------	-----	-----
$A_i - A_{i+1}$	$i+1$	n_{i+1}
---------	-----	-----
$A\ m-1 - +\infty$	m	n_m
Σ		N

On va procéder par étapes pour estimer $\hat{\alpha}$ en utilisant la continuité de ϕ: $(K_1, \dots K_J) \rightarrow \hat{\alpha}(K_1, \dots K_J) = \phi(K_1, \dots K_J)$ en utilisant l'estimateur du Khi-deux minimum, par l'algorithme de NEWTON-RAPHSON.

Première étape

On répartit dans chacune des tranches les risques uniformément.

$$\begin{cases} \text{Pour } i < m, \text{ les } n_i \text{ risques valent } B_{ij} = A_{i-1} + \dfrac{j}{n_i}(A_i - A_{i-1}). \\ \text{Pour } i = m, \text{ les } n_m \text{ risques valent } A_{m-1}. \end{cases}$$

Pour $j = 1 \dots n_j$.

On en déduit $\hat{\alpha}_1$, puisqu'on s'est ramené au premier cas.

Deuxième étape

On calcule, sous l'hypothèse, $\alpha = \hat{\alpha}_1$, les fréquences théoriques de chacune des m tranches de capital assuré.

$$\text{Pour } j = 1 \ldots m,\ P_{th_i}(\hat{\alpha}_1) = F_{\hat{\alpha}_1}(A_i) - F_{\hat{\alpha}_1}(A_{i-1}) \quad \text{avec} \quad \mathrm{A_m} = +\infty$$

Troisième étape

On écrit $\hat{\alpha}_2 = \hat{\alpha}_1 + \varepsilon$.

$\hat{\alpha}_2$ doit être un meilleur estimateur que $\hat{\alpha}_1$. ε doit être «petit» compte tenu de la continuité de ϕ.

$$\text{On écrit alors } P_{th_i}(\hat{\alpha}_2) = P_{th_i}(\hat{\alpha}_1) + \varepsilon \frac{\partial P_{th_i}}{\partial \alpha}(\hat{\alpha}_1), \text{ au premier ordre.}$$

Quatrième étape

On cherche à minimiser $G(x)$, avec

$$G(\hat{\alpha}_2) = \sum_{i=1}^{m} \left[\left(P_{th_i}(\hat{\alpha}_2) - \frac{n_i}{N} \right)^2 \Big/ P_{th_i}(\hat{\alpha}_2) \right]$$

$$= A(\hat{\alpha}_1) + \varepsilon B(\hat{\alpha}_1), \text{ au premier ordre.}$$

$$\text{On en déduit: } \hat{\alpha}_2 = \hat{\alpha}_1 - \frac{A(\hat{\alpha}_1)}{B(\hat{\alpha}_1)}$$

Avec:

$$A(\hat{\alpha}_1) = G(\hat{\alpha}_1)$$

$$B(\hat{\alpha}_1) = \frac{\partial G}{\partial \alpha}(\hat{\alpha}_1) = \sum_{i=1}^{m} \frac{\partial P_{th_i}}{\partial \alpha}(\hat{\alpha}_1) \times \left[2 \frac{\left[P_{th_i}(\hat{\alpha}_1) - \frac{n_i}{N} \right]}{P_{th_i}(\hat{\alpha}_1)} - \frac{\left[P_{th_i}(\hat{\alpha}_1) - \frac{n_i}{N} \right]^2}{P_{th_i}(\hat{\alpha}_1)^2} \right]$$

$$\text{et} \quad P_{th_i}(\alpha) = \left(\frac{A_o}{A_{i-1}} \right)^{\alpha} - \left(\frac{A_o}{A_i} \right)^{\alpha}$$

$$\text{et} \quad \frac{\partial P_{th_i}}{\partial \alpha}(\alpha) = \ln \frac{A_o}{A_{i-1}}\ x \left(\frac{A_o}{A_{i-1}} \right)^{\alpha} - \ln \frac{A_o}{A_i}\ x \left(\frac{A_o}{A_i} \right)^{\alpha}$$

On réitère éventuellement le procédé en substituant α_2 à α_1 et ainsi de suite.

Cet algorithme a des chances de converger sous l'hypothèse d'une adéquation parfaite à une loi de Pareto de paramètre α_c.

En effet:

$$G(\alpha_c) = o$$

$$G'(\alpha_c) = o$$

$$G''(\alpha_c) = 2 \sum_{i=1}^{m} \left[\frac{\partial P_{th_i}}{\alpha_c} (\alpha_c) \right]^2 x \frac{1}{P_{th(\alpha c)}} > o$$

$$\alpha_{n+1} = \alpha_n - \frac{G(\alpha_n)}{G'(\alpha_n)} = \alpha_n - \frac{(\alpha_n - \alpha_c)^2}{2} \frac{[G''(\alpha_c) + o_1(\alpha_n - \alpha_c)]}{(\alpha_n - \alpha_c)[G''(\alpha_c) + o_2(\alpha_n - \alpha_c)]}$$

avec $o_1(x) \to o$ et $o_2(x) \to o$ lorsque $x \to o$

Nous avons donc:

$$\alpha_{n+1} - \alpha_c = (\alpha_n - \alpha_c) \left[1 - \frac{1}{2} + o_3(\alpha_n - \alpha_c) \right] \text{ avec } o_3(x) \to o \text{ lorsque } x \to o.$$

Puisque $1 - \frac{1}{2} = \frac{1}{2}$, dans un voisinage de α_c, $|\alpha_{n+1} - \alpha_c| < \frac{3}{4} |\alpha_n - \alpha_c|$ on en déduit que $\alpha_n \to \alpha_c$, lorsque $n \to +\infty$, si α_1 est suffisamment proche de α_c.

Application numérique

Elle se fera sur le portefeuille Incendie (Risque Simples) d'une compagnie d'assurance allemande. On ne considérera ici que les risques supérieurs à $c = 5$, l'unité monétaire étant ici occultée.

Tranche	Nombre de risques
1 - 5	2.520
5 - 10	700
10 - 20	514
20 - 50	517
50 - 100	284
100 - 200	200
200 - 500	203
500 - 1.000	115
> 1.000	289
Σ	5.342

Nombre d'itérations	$\hat{\alpha}$
1	0,415
2	0,392
3	0,410
4	0,350
5	0,380
6	0,399
7	0,421
8	0,403
9	0,441
10	0,420
11	0,401
12	0,429

Observons par ailleurs $G(\alpha)\,N$.

$$\begin{array}{lll} G(0{,}38)\,N = 30{,}16 & G(0{,}40)\,N = 10{,}07 & G(0{,}42\ )\,N = 13{,}21 \\ G(0{,}39)\,N = 17{,}15 & G(0{,}41)\,N = 8{,}79 & G(0{,}405)\,N = 8{,}71 \\ & & G(0{,}415)\,N = 10{,}29 \end{array}$$

Si l'on teste l'hypothèse d'adéquation de la distribution des capitaux assurés à une distribution de Pareto de paramètre $\alpha = 0{,}405$ par la loi du Khi-deux, nous avons $\kappa^2_{(9-1-1)}(U) = \kappa^2_7(U) = 95\,\%$ implique $U = 14{,}1$

Or $8{,}71 < 14{,}1$; on peut donc accepter l'hypothèse paretienne avec un seuil de tolérance de 5 %. Cette méthode du Khi-deux minimum nous donne même le seuil critique, si on le souhaite.

ANNEXE 2

Il n'est pas toujours évident de trouver un portefeuille, comme celui qu'on vient de décrire, qui se prête à un ajustement des capitaux assurés sur une distribution de Pareto.

La question, ici, est de savoir pourquoi ces lois sont souvent prises en référence.

On part d'un principe bien simple : l'espérance du nombre de sinistres à charge doit être proportionnelle au nombre de risques exposés.

Les lois de Pareto (malgré la réserve importante qu'on a émise précédemment) sont solution.

On a vu, en effet, que $-\dfrac{L'(x)}{L(x)} = -\dfrac{N'_x}{N_x} = \dfrac{\alpha}{x}$ soit encore $L(x) = k\,N_x$.

Montrons que ce sont les seules solutions de ce problème, pour toute loi $T(t)$.

On a $\quad L(x) = d\,N_x \quad$ soit $\quad \dfrac{L'(x)}{L(x)} = \dfrac{N'_x}{N_x}$

Comme $$L(x) = \int_x^{+\infty} -\frac{dN_u}{d_u}\left[1 - T\left(\frac{x}{u}\right)\right] d_u$$

$$= \int_x^{+\infty} N_u \frac{x}{u^2}\, t\left(\frac{x}{u}\right) du$$

$$L'(x) = \int_x^{+\infty} \frac{dN_u}{d_u}\, x \frac{1}{u}\, t\left(\frac{x}{u}\right) du,$$

nous avons

$$0 = L'(x) - L(x)\frac{N_x'}{N_x} = \int_x^{+\infty} t\left(\frac{x}{u}\right) x \frac{1}{u}\left[N_u' - \frac{x}{u} N_u \frac{N_x'}{N_x}\right] du$$

Cette égalité est vraie pour toute fonction t.

Nous avons donc $\frac{N_u'}{N_u} = \frac{x}{u}\frac{N_x'}{N_x} = \frac{k}{u}$

N_u suit donc bien une distribution de Pareto de paramètre $-k$.

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STATISTICAL ANALYSIS OF NATURAL EVENTS IN THE UNITED STATES

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ABSTRACT

A statistical analysis is performed on natural events which can produce important damages to insurers. The analysis is based on hurricanes which have been observed in the United States between 1954 et 1986.

At first, independence between the number and the amount of the losses is examined. Different distributions (Poisson and negative binomial for frequency and exponential, Pareto and lognormal for severity) are tested. Along classical tests as chi-square, Kolmogorov-Smirnov and non parametric tests, a test with weights on the upper tail of the distribution is used: the Anderson – Darling test.

Confidence intervals for the probability of occurrence of a claim and expected frequency for different potential levels of claims are derived.

The Poisson Log-normal model gives a very good fit to the data.

KEYWORDS

Catastrophe risk; fitting models; frequency; severity; XL treaties.

1. INTRODUCTION

The United States of America are regularly hit by different types of natural events. Hurricanes affect the east part of the United States, tornadoes the middle one. Hailstorms and winter freeze may take place all over the United States. Earthquakes are observed in some specific zones as California (for example 1906 and 1989 San Francisco quakes).

These events cause very important losses. On the average the insured losses represent 4% of the premium income in classes as fire and multiperils for homeowners, farmowners and commercial risks.

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A very important hurricane may induce a 8 billion US \$ insured loss which would represent 20% of the premium income of these classes for one year. This percentage is even higher for an insurance company located in hurricane prone zones (Texas, Florida, Georgia, ...).

Direct insurers and reinsurers (underwriting non proportional treaties) must estimate their exposure in order to define an adequate reinsurance coverage.

The topic of the study is to get some results on the loss amount and frequency distributions of these events. In order to do homogeneous analysis, the study has been realized on a sample of hurricanes affecting the United States.

ISO keeps in its data base all losses (natural events) since 1949 whose amount exceeds 1 million US \$ (5 millions US \$ after 1982). Three factors explain the evolution of the losses amount from 1949: inflation, the number of

TABLE 1

HURRICANES EXCEEDING 30 MILLIONS \$

Year	Frequency	First loss	Second loss	Other losses		
1986	0					
1985	5	39,7	582,0	439,9	47,1	83,9
1984	1	41,2				
1983	1	893,1				
1982	1	192,0				
1981	0					
1980	1	106,2				
1979	2	216,7	1243,4			
1978	0					
1977	0					
1976	1	52,8				
1975	1	351,6				
1974	1	36,2				
1973	0					
1972	1	431,5				
1971	1	57,3				
1970	1	1602,1				
1969	1	822,2				
1968	0					
1967	1	260,1				
1966	1	58,7				
1965	1	6299,9				
1964	3	814,9	137,2	203,8		
1963	0					
1962	0					
1961	2	1263,5	53,7			
1960	1	1313,0				
1959	2	118,4	167,8			
1958	1	70,1				
1957	1	503,7				
1956	1	64,8				
1955	2	529,9	87,8			
1954	3	2465,4	317,9	2753,9		

people having the coverage against hurricanes in their insurance policy, demographic evolution.

These three factors have been taken into account in the trending of the losses (FRIEDMAN, 1987) in order to get an homogeneous data base in 1987 US \$. Nevertheless as the indexation coefficients for the first years were close to 100 and those for the years 1954 to 1982 were lower than 30, the observed period of time has been shortened to 33 years (1954 to 1986). During these years 37 hurricanes have been observed (cost of each hurricane in 1987 US \$ exceeding 30 millions).

2. HYPOTHESIS

Consider N the random variable (r.v.) of the yearly loss frequency $N(x_0)$ the r.v. of the losses exceeding x_0, with x_0 fixed. Let X_i be the amount of the loss i and $\underline{X} = (X_1, \ldots, X_N)$ the r.v. of the yearly loss amounts; the distribution of each X_i is supposed continuous.

K observations years ($K = 33$) are available. They produce a realization $(n_k, \underline{x}^{(k)})_{k=1,\ldots,K}$ of a K-sample $(N_1, \underline{X}^{(1)}), \ldots, (N_K, \underline{X}^{(K)})$ of $(N, \underline{X})$.

Two hypothesis are made

($H1$) N and $(X_1, X_2, \ldots)$ are independent random variables

($H2$) $X_1, X_2, \ldots$ are i.i.d. random variables.

($H1$) may be partly checked looking at the 25 years for which at least one loss has been observed. The grouping of the first losses in three classes gives the following contingency table (into parenthesis theoretical frequencies in case of independence).

Yearly frequency \ Loss	$\leqslant 200$	$200 < \leqslant 1000$	> 1000	Total
1	9 (7,92)	6 (6,48)	3 (3,60)	18
2	1 (1,76)	2 (1,44)	1 (0,80)	4
3 and over	1 (1,32)	1 (1,08)	1 (0,60)	3
Total	11	9	5	25

Chi-square independence test gives an observed $x^2_{\text{obs}} = 1{,}23$ which for the significance level (P-value) is $\hat{\alpha} = P(x_4^2 > x^2_{\text{obs}}) = 0{,}87$. So ($H1$) is accepted.

Remark: A grouping of yearly frequencies in two classes in order to follow the Cochran criterion $\left[\frac{n_i \cdot n \cdot j}{n} \geqslant 1 \,\forall\, (i,j) \text{ and } \frac{n_i \cdot n \cdot j}{n} \geqslant 5 \text{ for at least } 80\% \text{ of } (i,j)\right]$ would lead to the same conclusion.

For ($H2$) independence of X_1 and X_2, and identical distribution of X_1, X_2 and X_3 are checked using three non parametric tests: Kendall, Spearman and Kruskal-Wallis (GIBBONS, 1974).

— Independence of X_1 and X_2

For the 7 years during which at least 2 losses have been observed, the Kendall tau statistic can be written as follows:

$$T = \frac{2}{7\,(7-1)} \sum_{\substack{1 \leqslant i < j \leqslant K \\ n_i,\, n_j \geqslant 2}} A_{ij} \qquad \text{with}$$

$$A_{ij} = \begin{cases} 1 & (X_1^{(j)} - X_1^{(i)})\,(X_2^{(j)} - X_2^{(i)}) > 0 \\ 0 \quad \text{if} & (X_1^{(j)} - X_1^{(i)})\,(X_2^{(j)} - X_2^{(i)}) = 0 \\ -1 & (X_1^{(j)} - X_1^{(i)})\,(X_2^{(j)} - X_2^{(i)}) < 0 \end{cases}$$

The critical region for Kendall test at level $\alpha = 0{,}20$ is $W_\alpha = \{|T| > 0{,}4286\}$. The observed tau being $T = -0{,}333$, independence between X_1 et X_2 can be assumed for any reasonable level.

Let R_k be the rank of $X_1^{(k)}$ among the 7 observations (ordered increasingly) for which $n_k \geqslant 2$ and S_k be the rank of $X_2^{(k)}$, the Spearman rho statistic is

$$R = \frac{\sum_k (R_k - \bar{R})\,(S_k - \bar{S})}{\sqrt{\sum_k (R_k - \bar{R})^2}\,\sqrt{\sum_k (S_k - \bar{S})^2}}$$

The critical region for Spearman test with a level α of 0,20 is $W_\alpha = \{|R| > 0{,}536\}$, observed rho is computed at $-0{,}464$ so the conclusion is the same as for Kendall test.

— Identical distribution of X_1, X_2 and X_3

Let F_i ($i = 1, 2, 3$) be the cumulative distribution function (c.d.f.) of X_i, only years when at least i losses have occured being selected: $\{X_i^{(k)}$: k with $n_k \geqslant i\}$.

The null hypothesis $F_1 = F_2 = F_3$ is tested against the alternative $\exists i, j : F_i \neq F_j$ by the Kruskal-Wallis test. Under the assumption that loss amounts $(X_i^{(k)})_{i \geqslant 3}$ are identically distributed, we have a m_1-sample of X_1 ($m_1 = 25$), a m_2-sample of X_2 ($m_2 = 7$) and a m_3-sample of X_3 ($m_3 = 5$). These samples are assumed to be independent.

Let $M = \sum_{i=1}^{3} m_i$, R_i the sum of ranks of the ith sample observations in the combined (increasingly) ordered configuration of the M observations with $\sum_{i=1}^{3} R_i = \frac{M(M+1)}{2}$: $R_1 = 482$, $R_2 = 128$, $R_3 = 93$. Under the null hypothesis $E(R_i) = \frac{m_i(M+1)}{2}$ $\forall i$ $[E(R_1) = 475,\ E(R_2) = 133,\ E(R_3) = 95]$, the Kruskal-Wallis statistic

$$KW = \frac{12}{M(M+1)} \sum_{i=1}^{3} \frac{1}{m_i}\left[R_i - \frac{m_i(M+1)}{2}\right]^2$$

is free (its distribution is independent of the common F_i distribution). Asymptotically ($m_i \to +\infty\ \forall i$) KW is chi-squared distributed with 2 degrees of freedom. This asymptotic distribution is used in practice when $m_i \geqslant 5\ \forall i$. Here the critical region for the Kruskal-Wallis test $\{KW > c\}$ has a significance level $\hat{\alpha} = P(x_2^2 > KW) = 0{,}97$ (the observed KW statistic having a value of 0,054).

Remark:

1. If the size of the third sample $m_3\,(= 5)$ seems too small to use the asymptotic distribution of KW, it is still possible to test $F_1 = F'$ against $F_1 \neq F'$ [F' being the c.d.f. of X_i $(i \geqslant 2)$] with a m_1-sample of F_1 $(m_1 = 25)$ and a m'-sample of F' $(m' = 12)$. In this case the Kruskal-Wallis test is the Mann-Whitney-Wilcoxon test and has a significance level $\hat{\alpha} = 0{,}82$.
2. Under the assumption of the X_i's independence the Kruskal-Wallis test may be used to check the hypothesis ($H1$): no effect of the yearly loss frequency upon their amount:

   Considering the yearly loss amounts $\{X_1^{(k)} : k$ with $n_k = 1\}$ for years when exactly one hurricane occurs, $\{X_i^{(k)} = k$ with $n_k = 2;\ i = 1, 2\}$ for years with two hurricanes and $\{X_i^{(k)} : k$ with $n_k \geqslant 3;\ i = 1, 2, \ldots, n_k\}$ for years with more than two hurricanes as independent samples with respective sizes $m_1 = 18$, $m_2 = 8$, $m_3 = 11$ of distributions G_1, G_2, G_3, the Kruskal-Wallis test of the null hypothesis $G_1 = G_2 = G_3$ gives a significance level $\hat{\alpha} = 0{,}89$ (observed $KW = 0{,}25$).

   Hereafter ($H1$) and ($H2$) will be assumed to be true. X will be the random variable parent of X_i and F_X its c.d.f. (assumed to be continuous.)

3. LOSS FREQUENCY

The realization $(n_1, \ldots, n_K)$ of the K-sample $(N_1, \ldots, N_K)$ from N is given in the following Table 2. Let $\bar{n} = \frac{1}{K}\sum_k n_k$ and $\hat{\sigma}_n^2 = \frac{1}{K}\sum_k (n_k - \bar{n})^2$.

Different distributions fitting the loss frequency are examined.

— Poisson distribution $\mathscr{P}(\lambda)$ $(\lambda > 0)$

with $P_\lambda(N = n) = e^{-\lambda}\lambda^n/n!$ $(n \in \mathbb{N})$, $E(N) = V(N) = \lambda$, $\hat{\lambda} = \bar{n}$ is the maximum likelihood estimator (M.L.E.) of λ. A confidence interval at a level of at least $(1-\alpha)$ for λ is $[\hat{\lambda}_i, \hat{\lambda}_s]$ with

$$\hat{\lambda}_i = \frac{1}{2K}\, x^2_{2\sum_k n_k}(\alpha/2) \quad \text{and} \quad \hat{\lambda}_s = \frac{1}{2K}\, x^2_{2\left(\sum_k n_k + 1\right)}(1-\alpha/2).$$

In these expressions $x^2(\alpha/2)$ and $x^2(1-\alpha/2)$ are the $\alpha/2$ and $(1-\alpha/2)$ fractiles of the chi-square distribution.

— Negative binomial $N\mathscr{B}(r, p)$ $(r > 0, p \in]0,1[)$

with $P_{r,p}(N = n) = \dfrac{\Gamma(r+n)}{\Gamma(r)\,n!}\, p^r (1-p)^n$ $(n \in \mathbb{N})$, $E(N) = \dfrac{r(1-p)}{p}$ and

$V(N) = \dfrac{r(1-p)}{p^2} > E(N)$; the estimation of (r, p) by the M.L.E. or by the moments requires that the condition $\hat{\sigma}_n^2 > \bar{n}$ is fulfilled.

From the frequencies by year of hurricanes, we have $\bar{n} = \dfrac{37}{33} = 1{,}12121$ and $\hat{\sigma}_n^2 = 1{,}0762$. So a fit by a negative binomial distribution is impossible.

TABLE 2

YEARLY FREQUENCY OF HURRICANES EXCEEDING 30 MILLIONS $

Yearly frequency i	Observed freq. v_i	Theoretical freq. $K\hat{p}_i$	$\frac{(v_i - K\hat{p}_i)^2}{K\hat{p}_i}$
0	8	10,75	0,703
1	18	12,06	2,926
2	4	6,76	1,127
3	2	2,53	0,111
4	0		
5	1 } 1	0,90	0,011
6 and over	0		
Total	33	33	$4{,}878 = \chi^2_{\text{obs}}$

The x^2 goodness-of-fit test to a Poisson distribution with $\hat{\lambda} = \bar{n}$, $\hat{p}_i = P_{\hat{\lambda}}(N = i)$ and $c = 5$ classes gives (see Table 2) a significance level $\hat{\alpha}$ fulfilling condition

$$P(x^2_{c-2} > x^2_{\text{obs}}) \leqslant \hat{\alpha} \leqslant P(x^2_{c-1} > x^2_{\text{obs}}),$$

belonging to the interval [0,18; 0,30].

So the fit of N to a Poisson distribution $\mathscr{P}(\lambda)$ is accepted with for λ:

M.L.E. $\hat{\lambda} = 1{,}12121$

Confidence interval at a level at least 0,98

[0,73736; 1,63005]

Remark:

1. The M.L.E. of λ obtained from grouped data (5 classes) is $\hat{\lambda} = 1{,}09866$, so to state precisely the chi-square test gives a significance level $\hat{\alpha} = P(x^2_3 > 4{,}866) = 0{,}18$.
2. The fit of a Poisson distribution to that kind of event frequency can be checked with the distribution (see Table 3) of the frequency by year of all the north atlantic hurricanes which approached the United States from 1899 to 1986 (meteorogical data, US Department of commerce):

$$\bar{n} = 1{,}7045, \qquad \hat{\sigma}^2_n = 1{,}8218, \qquad \hat{\alpha} \in [0{,}72;\ 0{,}84].$$

TABLE 3

YEARLY FREQUENCY OF ALL NORTH ATLANTIC HURRICANES

Yearly frequency i	Observed freq. v_i		Theoretical freq. $K\hat{p}_i$	$\frac{(v_i - K\hat{p}_i)^2}{K\hat{p}_i}$
0	16		16,00	0,000
1	28		27,28	0,019
2	23		23,25	0,003
3	14		13,21	0,047
4	3		5,63	1,229
5	2			
6	2	4	2,63	0,714
7 and over	0			
Total	88		88	$2{,}012 = \chi^2_{\text{obs}}$

4. LOSS AMOUNT

Loss amounts are assumed to be i.i.d. random variables. Let

$n = \sum_{k=1}^{K} n_k\ (= 37)$, a realization $(x_1, \ldots, x_n)$ of a n-sample $(X_1, \ldots, X_n)$ of X is obtained; all losses are over 30.

The aim of the following lines is to estimate the probability $P(X \geqslant x_0) = 1 - F_X(x_0)$ that a loss amount exceeds x_0 and to derive a confidence interval at a level $1-\alpha$ (= 0,98).

1. Non parametric estimation

Let $X_{(1)} \leqslant \ldots \leqslant X_{(n)}$ be the ordered sample corresponding to $(X_1, \ldots, X_n)$ and $(F_n^*(x))_{x \in \mathbb{R}^+}$ the empirical c.d.f.

Considering $X_{(0)} = 30$, $X_{(n+1)} = +\infty$, we have, for $k = 0, \ldots, n$,

$$F_n^*(x) = \frac{k}{n} \quad \text{if} \quad X_{(k)} < x \leqslant X_{(k+1)}.$$

The statistic $1 - F_n^*(x_0) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{[x_0, +\infty[}(X_i)$ is an unbiased consistent estimator of $1 - F_X(x_0)$.

Furthermore if $D_n(1-\alpha)$ is the $(1-\alpha)$ fractile $[D_n(0{,}98) = 0{,}244$ for $n = 37]$ of the Kolmogorov-Smirnov statistic $D_n = \sup_{x \in \mathbb{R}^+} |F_n^*(x) - F_X(x)|$ associated to the sample, if we let, for $x \in \mathbb{R}^+$,

$$I_n(x) = \max\,[1 - F_n^*(x) - D_n(1-\alpha), 0]$$
$$S_n(x) = \min\,[1 - F_n^*(x) + D_n(1-\alpha), 1]$$

the band $([I_n(x), S_n(x)])_{x \in \mathbb{R}^+}$ is a level $(1-\alpha)$ confidence band for $1 - F_X(x_0)$ meaning that

$$P[I_n(x) \leqslant 1 - F_X(x) \leqslant S_n(x)\ \forall x \in \mathbb{R}^+] = 1 - \alpha.$$

The table with the values of $1 - F_n^*(x)$, $I_n(x)$ and $S_n(x)$ for $k = 0, \ldots, n$ and $x_{(k)} < x \leqslant x_{(k+1)}$ is presented in Appendix 1.

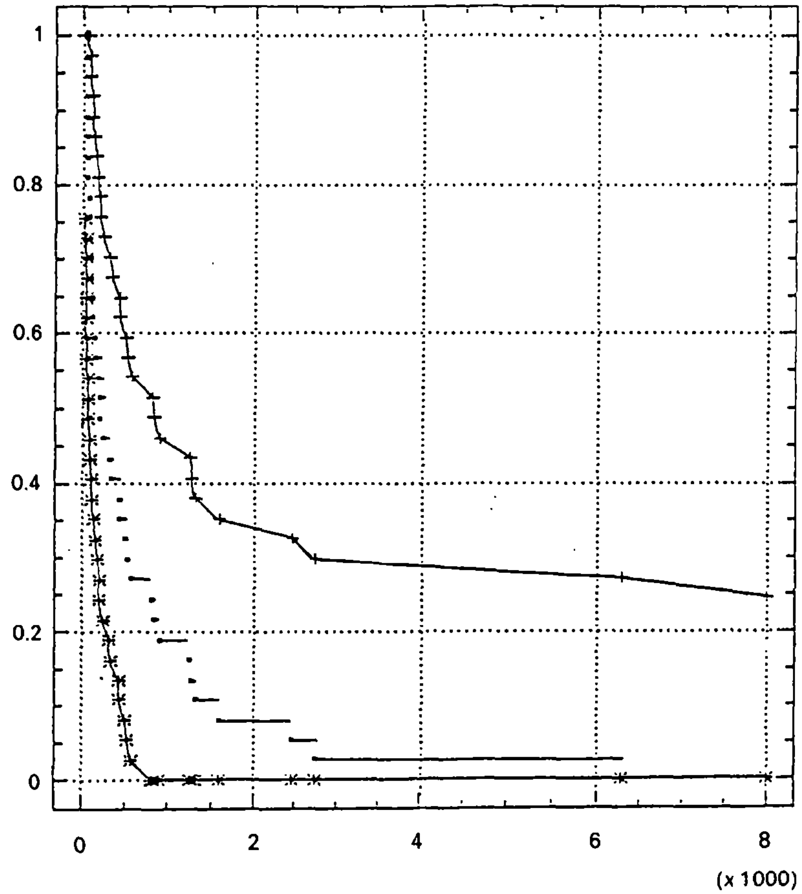
Joining with segments the points

$$\left(x_{(k)}, \min\left(\frac{n-k+1}{n} + D_n(1-\alpha), 1 \right) \right)_{k=0, \ldots, n+1}$$

for the superior envelope and

$$\left(x_{(k)}, \max\left(\frac{n-k}{n} - D_n(1-\alpha), 0 \right) \right)_{k=0, \ldots, n+1}$$

for the inferior envelope, a confidence band $(B_x)_{x \in \mathbb{R}^+}$ containing the first one $([I_n(x), S_n(x)]_{x \in \mathbb{R}^+}$ and graphically easier to draw is derived. Graph 1 shows the plot of $1 - F_n^*(x)$ and B_x for $30 \leqslant x \leqslant 8000$.



GRAPH 1. Plot of $1-F_n^*(x)$ and B_x.

2. Parametric family of distributions

A graphical approach and the value of significance levels of goodness-of-fit tests based on the empirical c.d.f. (D'AGOSTINO and STEPHENS, 1986) are used to test the fit of observations to a family $\mathcal{F} = \{F(x;\theta) : \theta \in \Theta\}$ of parametric distributions (θ varying in on open subset Θ of $\mathbb{R}^q$).

For the graphical procedure ($Q-Q$ plot) following results are applied: for $r = 1, \ldots, n$

$$E[F_X(X_{(r)})] = \frac{r}{n+1}\,; \quad V[F_X(X_{(r)})] = \frac{r(n-r+1)}{(n+1)^2(n+2)} = 0(1/n)\,.$$

For $n \geqslant 30$ (a generally accepted level) a realization of $F_X(X_{(r)})$ is very likely close to $\frac{r}{n+1}$. So it is possible to write $F_X(x_{(r)}) \simeq \frac{r}{\mathrm{n}+1}$ for $r = 1, \ldots, n$.

By an adequate transformation, depending of the examined family, the procedure is equivalent to estimate whether n points are roughly on a straight line.

Let $\hat{\theta}$ be the M.L.E. of θ in the hypothesis $F_X \in \mathcal{F}$, the goodness-of-fit test is based on Anderson-Darling statistic

$$\hat{A}_n^2 = n \int_0^{+\infty} \frac{[F_n^*(x) - F(x;\hat{\theta})]^2}{F(x;\hat{\theta})[1-F(x;\hat{\theta})]} \, dF(x,\hat{\theta})$$

$$= -n - \frac{1}{n} \sum_{r=1}^{n} (2r-1)\{\text{Log}\, F(X_{(r)};\hat{\theta}) + \text{Log}\,[1-F(X_{(n-r+1)};\hat{\theta})]\}.$$

This statistic gives one of the globally most powerful tests (D'AGOSTINO and STEPHENS, 1986). Moreover it is an adequate statistic of the here studied problem because of the weight $\left(\text{factor} \dfrac{1}{1-F(x;\hat{\theta})}\right)$ given to the tail of the distribution.

In order to compare with other tests, Kolmogorov-Smirnov statistic will be computed:

$$\hat{D}_n = \sup_{x \in \mathbb{R}^+} |F_n^*(x) - F(x,\hat{\theta})| = \max(\hat{D}_n^+, \hat{D}_n^-) \text{ with}$$

$$\hat{D}_n^+ = \max_{r=1,\ldots,n} \left[\frac{r}{n} - F(X_{(r)};\hat{\theta})\right] \text{ and } \hat{D}_n^- = \max_{r=1,\ldots,n} \left[F(X_{(r)};\hat{\theta}) - \frac{r-1}{n}\right].$$

Let $\hat{T}_n$ be one of these two test statistics and $\hat{T}_{n,x}$ its value for the realization $x = (x_1, \ldots, x_n)$ of $(X_1, \ldots, X_n)$. The distribution of $\hat{T}_n$ under the null hypothesis $H_0 : F_X \in \mathcal{F}$ depends generally only on n and the examined family. Thus a significance level $\hat{\alpha}(x) = P^{H_0}[\hat{T}_n > \hat{T}_{n,x}]$ may be computed from the table of this distribution.

Remark: It is not advisable to compare the fit of two families of distributions to the observations by a simple comparison of their $\hat{T}_{n,x}$. Indeed the same deviation has not the same likelihood to be reached under H_0. For example considering $\hat{D}_n$, for $n = 37$ and $\hat{D}_{n,x} = 0{,}165$: $P^{H_0}(\hat{D}_n > \hat{D}_{n,x}) = 0{,}24$ if $\mathcal{F} = \{F_0\}$ has only one distribution (fully specified), $P^{H_0}(\hat{D}_n > \hat{D}_{n,x}) = 0{,}08$ if $\mathcal{F}$ is the exponential distributions family, $P^{H_0}(\hat{D}_n > \hat{D}_{n,x}) = 0{,}15$ if $\mathcal{F}$ is the log-normal distributions family.

The histogram of the observations suggest to choose a dissymmetrical distribution. Successively exponential, Pareto and log-normal distributions will be tried: for these distributions there are statistical tables which give the goodness-of-fit significance levels $\hat{\alpha}(x)$.

2a. *Exponential distribution* $\varepsilon(\beta; 30)$

With $\beta > 0$, this distribution has the following density and c.d.f.

$$f_{\beta,30}(x) = \beta\, e^{-\beta(x-30)}\, \mathbb{1}_{[30,+\infty[}(x)$$
$$F_{\beta,30}(x) = 1 - e^{-\beta(x-30)}\ (x \geqslant 30) \quad \text{so} \quad -\text{Log}\,[1-F_{\beta,30}(x)] = \beta(x-30).$$

Let $(Y_1, \dots, Y_n)$ be a n-sample of $\{\varepsilon(\beta, 30) : \beta > 0\}$ and $(y_1, \dots, y_n)$ its realization.

* the n points $\left(y_{(r)} - 30,\ -\text{Log}\left(1 - \dfrac{r}{n+1}\right)\right)$ $r = 1, \dots, n$ are roughly on a straight line going through $(0,0)$ with a positive slope (the slope of an adjusted line on these points gives if necessary a graphical estimation of β).

* M.L.E. of β is $\hat\beta = \dfrac{n}{\sum_{i=1}^{n} (y_i - 30)}$, M.L.E. of $1 - F_{\beta,30}(x_0)$ is

  $1 - F_{\hat\beta,30}(x_0) = e^{-\hat\beta(x_0-30)}$ for $x_0 \geqslant 30$.

* a level $(1-\alpha)$ confidence interval with symmetric risks is

  for β: $\left[\hat\beta \dfrac{x^2_{2n}(\alpha/2)}{2n}\ ;\ \hat\beta \dfrac{x^2_{2n}(1-\alpha/2)}{2n}\right]$

  for $1 - F_{\beta,30}(x_0)$:

  $$\left(\exp\left\{\frac{-\hat\beta\, x^2_{2n}(1-\alpha/2)}{2n}(x_0-30)\right\};\ \exp\left\{\frac{-\hat\beta\, x^2_{2n}(\alpha/2)}{2n}(x_0-30)\right\}\right)$$

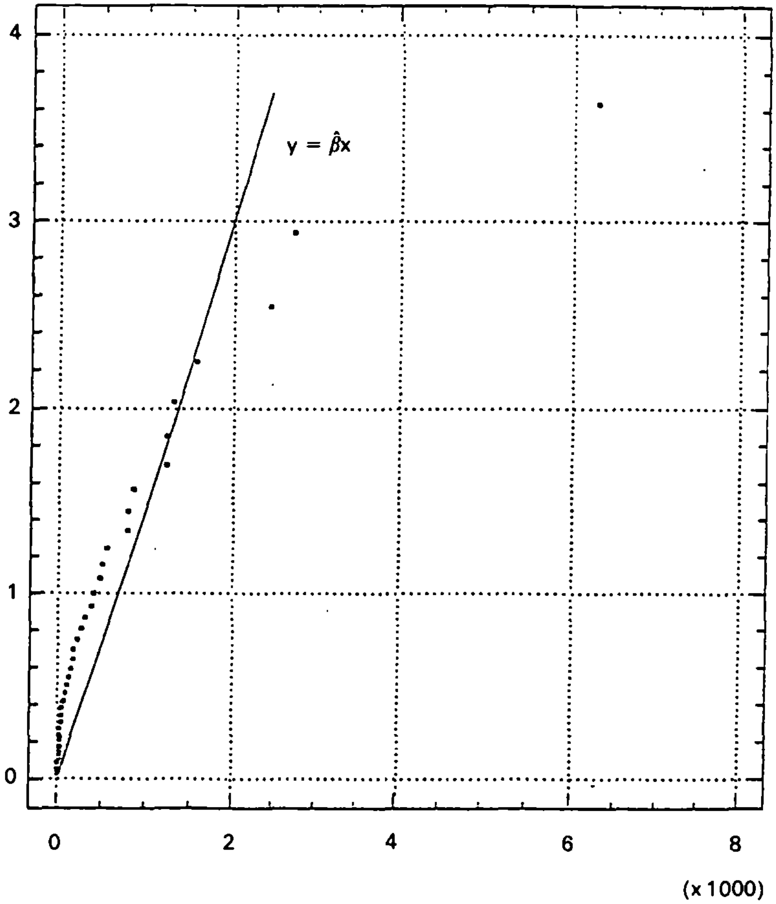
  as $\dfrac{2n\beta}{\hat\beta}$ is x^2_{2n} distributed (d.f. $2n$).

The graphical procedure applied to the 37-sample $(x_1, \dots, x_n)$ of X in graph 2 rejects in a first approach a fit to an exponential distribution: the tail of this distribution is too light to take into account the observed amounts of loss.

With $\hat\beta = \dfrac{1}{638{,}2} = 0{,}00157$, the significance levels of the goodness-of-fit tests

corroborate the lack of fit of the exponential distribution to the data:

$$\hat A^2_n = 5{,}98054 \qquad \hat\alpha(x) \ll 0{,}0025$$
$$\hat D_n = 0{,}2599 \qquad \hat\alpha(x) \ll 0.005.$$



GRAPH 2. Plot of the points $\left(x_{(r)} - 30, \ -\text{Log}\left(1 - \dfrac{r}{n+1}\right)\right)$ $r = 1, \dots, n$.

2b. *Pareto distribution* $P(\gamma; 30)$

With $\gamma > 0$ this distribution has the following density and c.d.f.

$$g_{\gamma,30}(x) = \frac{\gamma\, 30^{\gamma}}{x^{\gamma+1}} \mathbb{1}_{]30,+\infty[}(x),$$

$$G_{\gamma,30}(x) = 1 - \left(\frac{30}{x}\right)^{\gamma} \text{ so } -\text{Log}\,[1 - G_{\gamma,30}(x)] = \gamma \,\text{Log}\,\frac{x}{30} \ (x \geqslant 30).$$

Let $(Y_1, \dots, Y_n)$ be a n-sample of $\{P(\gamma; 30) : \gamma > 0\}$ and $(y_1, \dots, y_n)$ its realization.

* the n points $\left(\text{Log}\,\dfrac{y_{(r)}}{30},\ -\text{Log}\left(1-\dfrac{r}{n+1}\right)\right)$ $r = 1, \ldots, n$ are roughly on a straight line going through (0, 0) with a positive slope (the slope of an adjusted line on these points gives if necessary a graphical estimation of γ).

* M.L.E. of γ is $\hat{\gamma} = \dfrac{n}{\sum_{i=1}^{n} \text{Log}\,\dfrac{Y_i}{30}}$, M.L.E. of $1-G_{\gamma,30}(x_0)$ is

$$\widehat{1-G_{\gamma,30}(x_0)} = \left(\frac{30}{x_0}\right)^{\hat{\gamma}} \text{ for } x_0 \geqslant 30.$$

* a level $(1-\alpha)$ confidence interval with symmetric risks is

$$\text{for } \gamma: \left[\hat{\gamma}\,\frac{x^2_{2n}(\alpha/2)}{2n}\ ;\ \hat{\gamma}\,\frac{x^2_{2n}(1-\alpha/2)}{2n}\right]$$

$$\text{for } 1-G_{\gamma,30}(x_0): \left[\left(\frac{30}{x_0}\right)^{\hat{\gamma}\frac{x^2_{2n}(1-\alpha/2)}{2n}}\ ;\ \left(\frac{30}{x_0}\right)^{\hat{\gamma}\frac{x^2_{2n}(\alpha/2)}{2n}}\right]$$

as $\dfrac{2n\gamma}{\hat{\gamma}}$ is x^2_{2n} distributed.

Graph 3 shows that the n points $\left(\text{Log}\,\dfrac{x_{(r)}}{30},\ -\text{Log}\left(1-\dfrac{r}{n+1}\right)\right)$

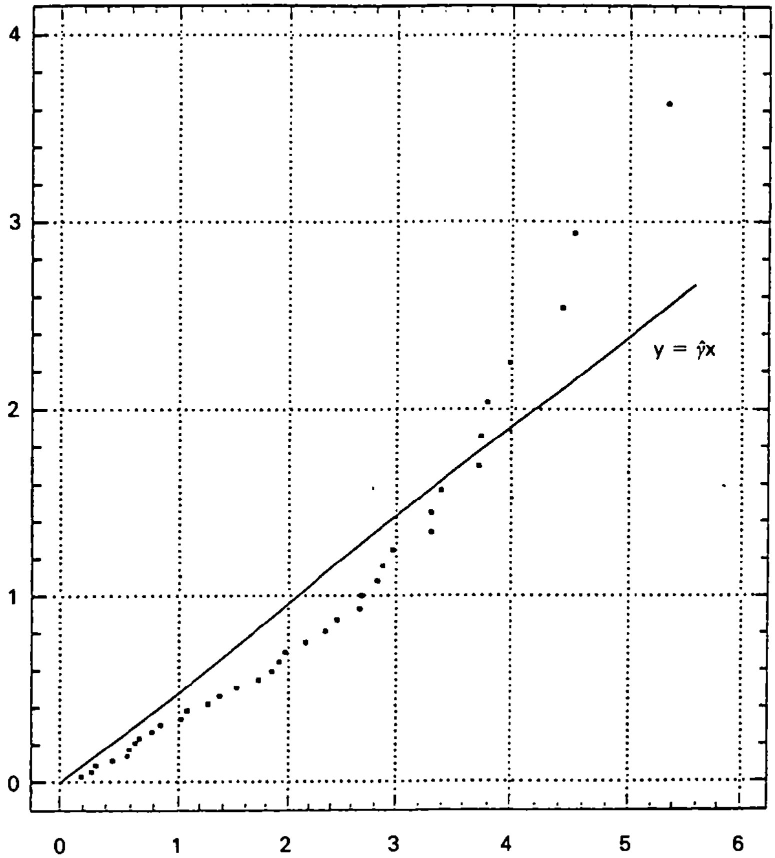
$r = 1, \ldots, n$ are not roughly on a straight line. Pareto distribution has a too heavy tail for the observed amounts of loss.

With $\hat{\gamma} = 0{,}465141$ the test statistics can be computed as follows

$$\hat{A}^2_n = 1{,}56365 \text{ with a significance level } \hat{\alpha}(x) = 0{,}025$$
$$\hat{D}_n = 0{,}14586 \text{ with a significance level } \hat{\alpha}(x) = 0{,}16\ .$$

Comparing the two significance levels demonstrates the interest of $\hat{A}^2_n$ relatively to $\hat{D}_n$. The fit to a Pareto distribution is rejected by $\hat{A}^2_n$ (tail of the distribution) though such a fit seems to be acceptable with $\hat{D}_n$, taking into account the small number of observations.

The fit to a Pareto distribution being rejected, the lower and upper limits of the confidence interval are not computed.



GRAPH 3. Plot of the points $\left(\operatorname{Log}\frac{x_{(r)}}{30},\ -\operatorname{Log}\left(1-\frac{r}{n+1}\right)\right)$ $r = 1, \ldots, n$.

2c. *Log-normal distribution* Log $N(\mu, \sigma; 30)$

With $\mu \in \mathbb{R}$ and $\sigma > 0$, a random variable Y is log-normally distributed if Log $(Y-30)$ is normally distributed $N(\mu, \sigma)$. Its density is

$$h_{\mu,\sigma,30}(x) = \frac{1}{\sqrt{2\pi}\,\sigma(x-30)} \exp\left\{-\frac{1}{2\sigma^2}[\operatorname{Log}(x-30)-\mu]^2\right\} \mathbb{1}_{]30,+\infty[}(x).$$

Let Φ be the c.d.f. of $N(0, 1)$, the c.d.f. of the log-normal distribution can be written $(x \geqslant 30)$

$$H_{\mu,\sigma,30}(x) = \Phi\left[\frac{\operatorname{Log}(x-30)-\mu}{\sigma}\right] \text{ therefore } \Phi^{-1}[H_{\mu,\sigma,30}(x)] = \frac{\operatorname{Log}(x-30)-\mu}{\sigma}.$$

Let $(Y_1, \ldots, Y_n)$ be a n-sample of $\{\text{Log}\, N(\mu, \sigma, 30) : \mu \in \mathbb{R}, \sigma > 0\}$

* The n points $\left(\text{Log}\,(y_{(r)} - 30),\ \Phi^{-1}\left(\frac{r}{n+1}\right)\right)$ $r = 1, \ldots, n$ are roughly on a straight line with a positive slope.

* M.L.E. of (μ, σ) is $\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \text{Log}\,(Y_i - 30)$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} [\text{Log}\,(Y_i - 30) - \hat{\mu}]^2}$$

M.L.E. of $1 - H_{\mu,\sigma,30}(x_0)$ is $\widehat{1 - H_{\mu,\sigma,30}}(x_0) = 1 - \Phi\left[\frac{\text{Log}\,(x_0 - 30) - \hat{\mu}}{\hat{\sigma}}\right]$

* the way to derive a confidence interval for $1 - H_{\mu,\sigma,30}(x_0)$ is explained later.

Graph 4 shows a very good fit of the log-normal distribution to the 37 observations. It is corroborated by the values of the test statistics computed with $\hat{\mu} = 5.19853$ and $\hat{\sigma} = 1.74297$:

$$\hat{A}_n^2 = 0.26265 \text{ with a significance level } \hat{\alpha}(x) = 0.70$$
$$\hat{D}_n = 0.07939 \text{ with a significance level } \hat{\alpha}(x) \gg 0.15.$$

The values of $\widehat{1 - H_{\mu,\sigma,30}}(x_0)$ for x_0 varying from 100 to 8000 are presented in Appendix 2 (column 1) and plotted in Graph 5.

2d. *Confidence interval for* $1 - \Phi\left[\frac{\text{Log}\,(x_0 - 30) - \mu}{\sigma}\right]$

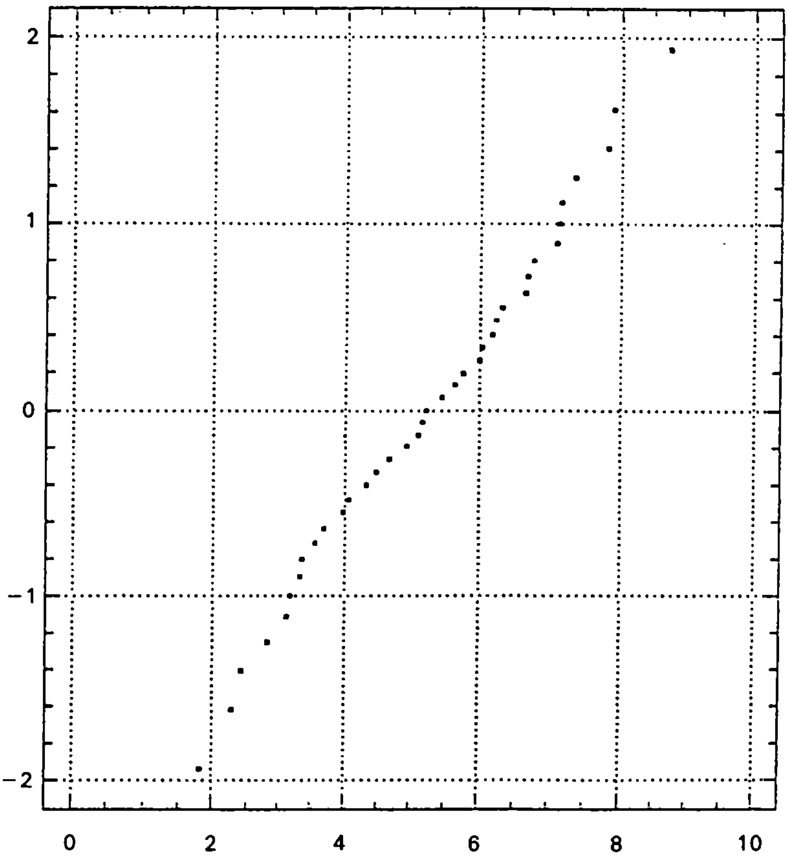
As the size of the sample ($n = 37$) is too small to use the confidence interval derived from the asymptotic normality of $(\hat{\mu}, \hat{\sigma})$ and the δ-method, the non-central Student distribution and its table (RESNIKOFF and LIEBERMAN, 1957) are to be used.

Let $Y_i = \text{Log}\,(X_i - 30)$ for $i = 1, \ldots, n$, $\overline{Y} = \frac{1}{n} \sum_i Y_i$ and

$S_Y^2 = \frac{1}{n-1} \sum_i (Y_i - \overline{Y})^2$, $(Y_1, \ldots, Y_n)$ is a n-sample of $N(\mu, \sigma)$.

So $\frac{\sqrt{n}}{\sigma}[\text{Log}\,(x_0 - 30) - \overline{Y}]$ is distributed as $N\left[\frac{\sqrt{n}}{\sigma}\{\text{Log}\,(x_0 - 30) - \mu\}, 1\right]$

and $\frac{(n-1)\, S_Y^2}{\sigma^2}$ *is* x_{n-1}^2 distributed. These two random variables being inde-



GRAPH 4. Plot of the points $\left(\text{Log}\,(x_{(r)}-30),\ \Phi^{-1}\left(\frac{r}{n+1}\right)\right)$ $r = 1, \ldots, n$.

pendent, the distribution of $\sqrt{n}\,\dfrac{[\text{Log}\,(x_0-30)-\bar{Y}]}{S_Y}$ is a $t_{n-1,\,\sqrt{n}\,[\text{Log}\,(x_0-30)-\mu]/\sigma}$ non-central Student distribution with $(n-1)$ degrees of freedom and centrality parameter $\sqrt{n}\,[\text{Log}\,(x_0-30)-\mu]/\sigma$.

In a more general way the γ-fractile $t_{\nu,\delta}(\gamma)$ of a Student distribution $t_{\nu,\delta}$ with ν d.f. and centrality parameter δ is, for fixed ν and γ, a strictly increasing continuous function of δ noted $C_{\nu,\gamma}$ with $P[t_{\nu,\delta} < C_{\nu,\gamma}(\delta)] = \gamma\ \forall\ \delta$.

Let $C_{\nu,\gamma}^{-1}(t)$ be its reciprocical function: for fixed $t \in \mathbb{R}$, $\delta = C_{\nu,\gamma}^{-1}(t)$ is the only solution of the equation, d being the unknown: $P[t_{\nu,d} < t] = \gamma$. From that it follows

$$\left[\frac{1}{\sqrt{n}}\, C^{-1}_{n-1,1-\alpha/2}\left(\frac{\sqrt{n}}{S_Y}\,[\mathrm{Log}\,(x_0-30)-\bar{Y}]\right),\right.$$
$$\left.\frac{1}{\sqrt{n}}\, C^{-1}_{n-1,\alpha/2}\left(\frac{\sqrt{n}}{S_Y}\,[\mathrm{Log}\,(x_0-30)-\bar{Y}]\right)\right]$$

is a level $(1-\alpha)$ confidence interval for $\dfrac{\mathrm{Log}\,(x_0-30)-\mu}{\sigma}$ with symmetric risks.

For $1-\Phi\left[\dfrac{\mathrm{Log}\,(x_0-30)-\mu}{\sigma}\right]$ the lower and upper limits of the confidence interval are

$$1-\Phi\left\{\frac{1}{\sqrt{n}}\, C^{-1}_{n-1,\alpha/2}\left(\frac{\sqrt{n}}{S_Y}\,[\mathrm{Log}\,(x_0-30)-\bar{Y}]\right)\right\} \text{ and}$$
$$1-\Phi\left\{\frac{1}{\sqrt{n}}\, C^{-1}_{n-1,1-\alpha/2}\left(\frac{\sqrt{n}}{S_Y}\,[\mathrm{Log}\,(x_0-30)-\bar{Y}]\right)\right\}.$$

From the fractiles of the Resnikoff-Lieberman table, it is possible to compute this interval for n = 37 and $1-\alpha$ = 0,98 (by linear interpolation and with a limited accuracy) only for $x_0 \geqslant 1500$. So it seems to be preferable to use the following approximation of fractile $t_{n-1,\delta}(\gamma)$ (VAN EEDEN, 1961):

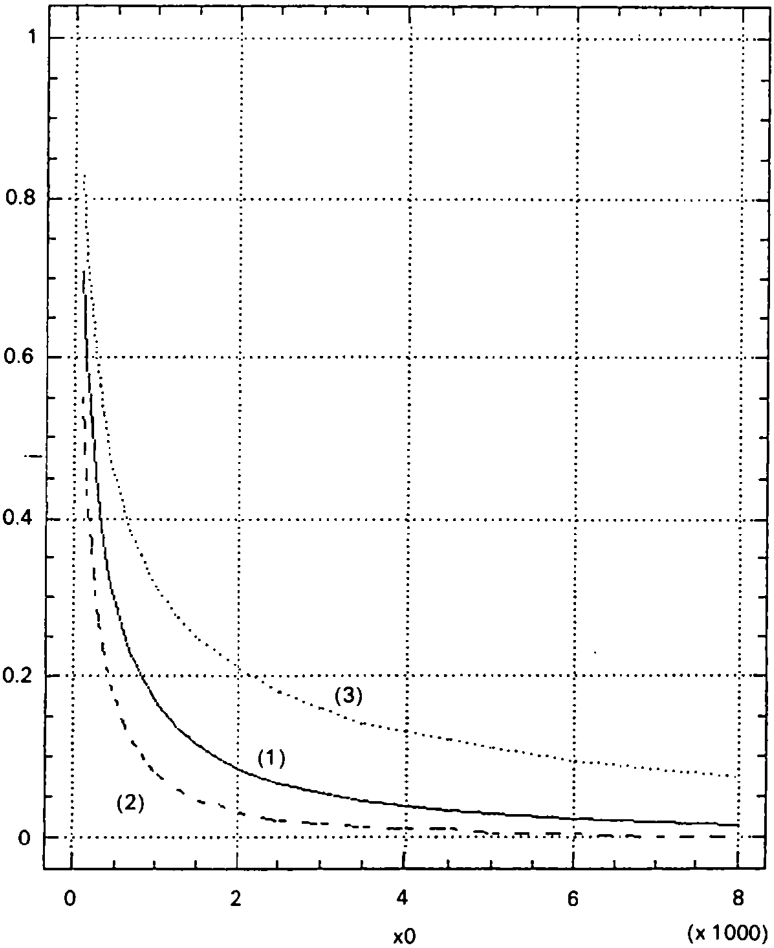
(1) $t_{n-1,\delta}(\gamma) \simeq t_{n-1}(\gamma)+h(\delta)$ with

$$h(\delta) = \delta + \frac{\delta}{4(n-1)}(1+2q^2+q\delta) + \frac{\delta}{96(n-1)^2}[3(4q^4+12q^2+1) + $$
$$+\, 6(q^3+4q)\delta-4(q^2-1)\delta^2-3q\delta^3]$$

and with $t_{n-1}(\gamma)$ and q being the γ-fractiles of the (central) Student distribution and of the normal distribution $N(0, 1)$.

Let $t_0 = \dfrac{\sqrt{n}}{s_y}\,[\mathrm{Log}\,(x_0-30)-\bar{y}]$, the approximation (1) provides $C^{-1}_{n-1,\gamma}(t_0)$ as solution of the equation (δ being the unknown): $t_{n-1}(\gamma)-t_0+h(\delta) = 0$. This equation can be numerically solved using the Newton-Raphson algorithm $\Big[$a starting value could be $\delta_0 = t_0-t_{n-1}(\gamma)$, obtained by neglecting the terms $\dfrac{1}{n-1}$ and $\dfrac{1}{(n-1)^2}$ in (1)$\Big]$.

Appendix 2 shows in columns 2 and 3 the lower and upper limits of the level 0,98 confidence interval for $1-H_{\mu,\sigma,30}(x_0)$. These limits are plotted in Graph 5.



GRAPH 5. Plot of $\widehat{1-H_{\mu,\sigma,30}(x_0)}$ (curve 1), lower limit (curve 2) and upper limit (curve 3) of the confidence interval for $1-H_{\mu,\sigma,30}(x_0)$. The log-normal case.

5. FREQUENCY OF LOSSES WITH AN AMOUNT $\geqslant X_0$

Let, for fixed $x_0 \geqslant 30$, $N(x_0)$ the r.v. of the yearly frequency of losses exceeding x_0. Using the same notations as before and considering that the r.v. N has a Poisson $\mathscr{P}(\lambda)$ distribution, under $(H1)$ and (H_2), the following results are obtained.

Theorem:

a) $N(x_0)$ is Poisson distributed with parameter $\lambda(x_0) = \lambda[1-F_X(x_0)]$.

b) If the distribution of X belongs to the family $\mathcal{F} = \{F(x;\theta) : \theta \in \Theta\}$, the M.L.E. of $\lambda(x_0)$ is $\widehat{\lambda(x_0)} = \hat{\lambda}[1-F(x_0;\hat{\theta})]$.

c) If $[\hat{\lambda}_i, \hat{\lambda}_s]$ and $[I(x_0), S(x_0)]$ are confidence intervals for λ and $1-F_X(x_0)$ at a level of at least $(1-\alpha/2)$, $[\hat{\lambda}_i I(x_0), \hat{\lambda}_s S(x_0)]$ is a confidence interval for $\lambda(x_0)$ at a level of at least $(1-\alpha)$.

Proof:

a) Direct calculation.

b) Because of the independence hypothesis and invariance of the M.L.E.

c) $P[\hat{\lambda}_i I(x_0) \leqslant \lambda(x_0) \leqslant \hat{\lambda}_s S(x_0)] \geqslant$
$$P[\hat{\lambda}_i \leqslant \lambda \leqslant \hat{\lambda}_s, I(x_0) \leqslant 1-F_X(x_0) \leqslant S(x_0)]$$

and the result with the Bonferroni inequality $P(A \cap B) \geqslant 1-P(A^c)-P(B^c)$ for any two events A and B. It is worthwhile to note that a direct use of the independence frequency-amount would give a level $\geqslant (1-\alpha/2)^2 = 1-\alpha+\dfrac{\alpha^2}{4}$ very close to $(1-\alpha)$

These results applied to the frequency and amounts of hurricanes give in the same way as for $1-F_X(x_0)$ but at a level $1-\alpha = 0.96$:

* In the non parametric case (Appendix 1)
  — an estimation of $\lambda(x_0)$: $\hat{\lambda}[1-F_n^*(x_0)]$ (Column 4)
  — a confidence band $([\hat{\lambda}_i I_n(x), \hat{\lambda}_s S_n(x)])_{x \in \mathbb{R}^+}$ for $\lambda(x_0)$ such as

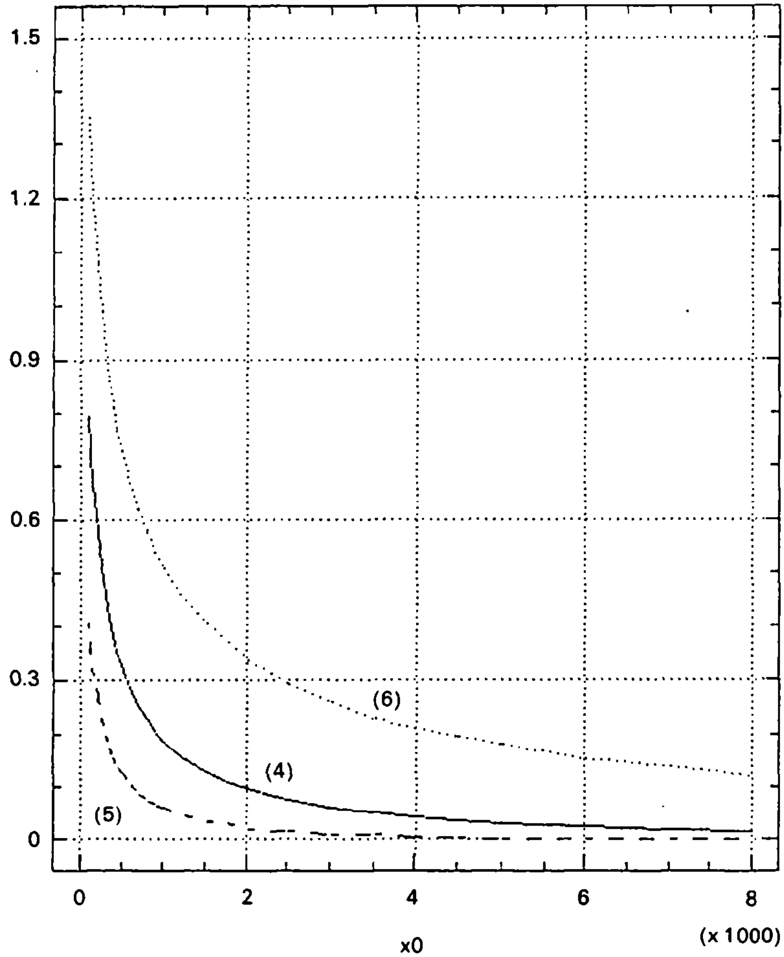
$$P[\hat{\lambda}_i I_n(x) \leqslant \lambda(x) \leqslant \hat{\lambda}_s S_n(x)\ \forall x \in \mathbb{R}^+] \geqslant 1-\alpha\,.$$

The values $\hat{\lambda}_i I_n(x)$ and $\hat{\lambda}_s S_n(x)$ are shown in Columns 5 and 6.

* In the log normal case (Appendix 2)
  — the M.L.E. $\widehat{\lambda(x_0)}$ of $\lambda(x_0)$ (Column 4)
  — the upper and lower limits of a confidence interval for $\lambda(x_0)$ (Columns 5 and 6).

Graph 6 shows a plot of these values.

In conclusion Table 4 shows for the values of x_0 for which observations are available, in order to judge of the goodness-of-fit: the M.L.E. $\widehat{\lambda(x_0)}$ derived from the model, the empirical mean $(\bar{n})$ and variance $(\hat{\sigma}_n^2)$ of the yearly frequency of losses exceeding x_0, the empirical distribution of the frequencies



GRAPH 6. Plot of $\widehat{\lambda(x_0)}$ (curve 4), lower limit (curve 5) and upper limit (curve 6) of the confidence interval for $\lambda(x_0)$. The log-normal case.

(columns obs.), to compare with the theoretical distribution derived from the Poisson log-normal model (column theor.).

Empirically the fit of the model seems very satisfactory.

CONCLUDING REMARKS

These results do not seem to be exclusive for hurricanes in the United States. So they could be used to modelize the frequency and amount distributions of natural events of any kind in the United States (for examples tornadoes) and even world wide.

TABLE 4

COMPARISON OF THE EMPIRICAL AND THEORETICAL FREQUENCIES OF $N(x_0)$

x_0	100		200		300		400		500	
$\widehat{\lambda(x_0)}$	0,793		0,577		0,459		0,382		0,327	
$\bar{n}$	0,758		0,606		0,515		0,454		0,394	
$\hat{\sigma}_n^2$	0,729		0,602		0,492		0,369		0,299	
Groups	obs.	theor.	obs.	theor.	obs.	theor.	obs.	theor.	obs.	theor.
0	15	15,47	18	18,53	19	20,85	20	22,52	21	23,80
1	13	11,72	11	10,69	12	9,57	11	8,60	11	7,78
2	3	4,44	3	3,08	1	2,20	2	1,64	1	1,27
3 and over	2	1,37	1	0,70	1 } 2	0,38 } 2,58	0 } 2	0,24 } 2,88	0 } 1	0,15 } 1,42
Σ	33	33	33	33	33	33	33	33	33	33

x_0	750		1000		1500		2000		2500	
$\widehat{\lambda(x_0)}$	0,240		0,188		0,129		0,096		0,075	
$\bar{n}$	0,303		0,212		0,121		0,091		0,061	
$\hat{\sigma}_n^2$	0,272		0,228		0,167		0,143		0,057	
Groups	obs.	theor.	obs.	theor.	obs.	theor.	obs.	theor.	obs.	theor.
0	24	25,96	27	27,34	30	29,01	31	29,98	31	30,62
1	8	6,23	5	5,14	2	3,74	1	2,88	2	2,30
2	1		1		1		1		0	
3 and over	0 } 1	} 0,91	0 } 6	} 0,52 } 5,66	0 } 3	} 0,25 } 3,99	0 } 2	} 0,14 } 3,02	0 } 2	} 0,08 } 2,38
Σ	33	33	33	33	33	33	33	33	33	33

APPENDIX 1

Values of:

$1-F_n^*(x)$	: Col. 1
$I_n(x)$	: Col. 2
$S_n(x)$	: Col. 3
$\hat{\lambda}[1-F_n^*(x)]$	: Col. 4
$\hat{\lambda}_i\, I_n(x)$	: Col. 5
$\hat{\lambda}_s\, S_n(x)$	: Col. 6

k	$x_{(k)} < x \leq x_{(k+1)}$		(1)	(2)	(3)	(4)	(5)	(6)
0	30,0	36,2	1,000	0,756	1,000	1,121	0,557	1,630
1	36,2	39,7	0,973	0,729	1,000	1,091	0,538	1,630
2	39,7	41,2	0,946	0,702	1,000	1,061	0,518	1,630
3	41,2	47,1	0,919	0,675	1,000	1,030	0,498	1,630
4	47,1	52,8	0,892	0,648	1,000	1,000	0,478	1,630
5	52,8	53,7	0,865	0,621	1,000	0,970	0,458	1,630
6	53,7	57,3	0,838	0,594	1,000	0,940	0,438	1,630
7	57,3	58,7	0,811	0,567	1,000	0,909	0,418	1,630
8	58,7	64,8	0,784	0,540	1,000	0,879	0,398	1,630
9	64,8	70,1	0,757	0,513	1,000	0,848	0,378	1,630
10	70,1	83,9	0,730	0,486	0,974	0,818	0,358	1,587
11	83,9	87,8	0,703	0,459	0,947	0,788	0,338	1,543
12	87,8	106,2	0,676	0,432	0,920	0,758	0,318	1,499
13	106,2	118,4	0,649	0,405	0,893	0,727	0,298	1,455
14	118,4	137,2	0,622	0,378	0,866	0,697	0,278	1,411
15	137,2	167,8	0,595	0,351	0,839	0,667	0,259	1,367
16	167,8	192,0	0,568	0,324	0,812	0,636	0,239	1,323
17	192,0	203,8	0,541	0,297	0,785	0,606	0,219	1,279
18	203,8	216,7	0,514	0,270	0,758	0,576	0,199	1,235
19	216,7	260,1	0,486	0,242	0,730	0,545	0,179	1,191
20	260,1	317,9	0,459	0,215	0,703	0,515	0,159	1,147
21	317,9	351,6	0.432	0,188	0,676	0,485	0,139	1,103
22	351,6	431,5	0,405	0,161	0,649	0,454	0,119	1,056
23	431,5	439,9	0,378	0,134	0,622	0,424	0,099	1,015
24	439,9	503,7	0,351	0,107	0,595	0,394	0,079	0,970
25	503,7	529,9	0,324	0,080	0,568	0,364	0,059	0,926
26	529,9	582,0	0,297	0,053	0,541	0,333	0,039	0,882
27	582,0	814,9	0,270	0,026	0,514	0,303	0,019	0,838
28	814,9	822,2	0,243	0,000	0,487	0,273	0,000	0,794
29	822,2	893,1	0,216	0,000	0,460	0,242	0,000	0,750
30	893,1	1243,4	0,189	0,000	0,433	0,212	0,000	0,706
31	1243,4	1263,5	0,162	0,000	0,406	0,182	0,000	0,662
32	1263,5	1313,0	0,135	0,000	0,379	0,152	0,000	0,618
33	1313,0	1602,1	0,108	0,000	0,352	0,121	0,000	0,574
34	1602,1	2465,4	0,081	0,000	0,325	0,091	0,000	0,530
35	2465,4	2753,9	0,054	0,000	0,298	0,061	0,000	0,486
36	2753,9	6299,9	0,027	0,000	0,271	0,030	0,000	0,442
37	6299,9	$+\infty$	0,000	0,000	0,244	0,000	0,000	0,398

APPENDIX 2
LOG NORMAL CASE

— Estimation of $1-F_X(x_0)$: Col. 1
— Lower and upper limits of the confidence interval for $1-F_X(x_0)$: Col. 2 and 3
— Estimation of $\lambda(X_0)$: Col. 4
— Lower and upper limits of the confidence interval for $\lambda(X_0)$: Col. 5 and 6.

x_0	(1)	(2)	(3)	(4)	(5)	(6)
100	0,707	0,55	0,83	0,793	0,41	1,35
150	0,593	0,44	0,73	0,665	0,32	1,19
200	0,514	0,36	0,66	0,577	0,27	1,08
250	0,455	0,31	0,61	0,511	0,23	0,99
300	0,409	0,27	0,565	0,459	0,20	0,92
350	0,372	0,24	0,53	0,417	0,175	0,86
400	0,341	0,21	0,50	0,382	0,16	0,81
450	0,315	0,19	0,47	0,353	0,14	0,77
500	0,292	0,17	0,45	0,327	0,13	0,73
600	0,255	0,14	0,41	0,286	0,105	0,67
700	0,226	0,12	0,38	0,254	0,09	0,62
800	0,203	0,105	0,355	0,228	0,08	0,58
900	0,184	0,09	0,33	0,206	0,07	0,54
1000	0,168	0,08	0,31	0,188	0,06	0,51
1250	0,137	0,06	0,28	0,153	0,045	0,45
1500	0,115	0,045	0,25	0,129	0,035	0,41
1750	0,098	0,04	0,23	0,110	0,03	0,375
2000	0,085	0,03	0,21	0,096	0,02	0,34
2500	0,067	0,02	0,18	0,075	0,015	0,295
3000	0,054	0,015	0,16	0,061	0,01	0,26
3500	0,045	0,01	0,14	0,051	0,01	0,23
4000	0,038	0,01	0,13	0,043	0,005	0,21
4500	0,033	0,01	0,12	0,037	0,005	0,195
5000	0,029	0,005	0,11	0,032	0,0	0,18
6000	0,022	0,005	0,095	0,025	0,0	0,155
7000	0,018	0,0	0,085	0,020	0,0	0,14
8000	0,015	0,0	0,075	0,017	0,0	0,12

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LETTER TO THE EDITORS

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W.M. Abbott
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BOOK REVIEW

Foundations of Casualty Actuarial Science: Published by Casualty Actuarial Society, One Penn Plaza, 250 West 34th Street, New York, NY 10119. 584 pages. $65. — (overseas $97.50).

From the Preface of *Foundations of Casualty Actuarial Science*:

> This landmark book is the first published, complete text containing the fundamentals of casualty actuarial science as practiced in North America. It is intended as an introduction to casualty actuarial concepts and practices. Its target audiences are members and students of the Casualty Actuarial Society, university and college students, plus insurance and general business professionals with a need for basic knowledge on these subjects.
>
> In designing the textbook, the Casualty Actuarial Society concluded that the readership would be best served by having each chapter written by an expert in the topic covered by the chapter. Therefore, each chapter is individually authored and the styles and organization vary somewhat. The chapters reflect the views of the individual authors and the content should not be considered as the official opinion of the Casualty Actuarial Society.

Those two paragraphs in the preface make it clear why reviewing the book *Foundations of Casualty Actuarial Science* is a difficult task for the reviewer as well as the reader. The ambitious specification of its aim, the wideness of its target audience, and the distribution of tasks among nine individual authors, set the book apart from the more usual one-author, one-topic, one-audience textbooks. A separate review of each chapter will be given.

Introduction, by Matthew Rodermund

In his introduction, the author recounts the history of the Casualty Actuarial Society from its beginnings in 1914. In his presentation, the history of CAS is inextricably interwoven with the development of Credibility Theory in North America. Thus the reader also finds a fascinating survey of the events leading to early applications of credibility theory, and its subsequent study and development. Special attention is given to the work of Albert H. Mowbray, Albert Whitney, Arthur Bailey, Laurence H. Longley-Cook, Allen L. Mayerson, Charles C. Hewitt; the contributions of several others are also mentioned including, of course, the work of Hans Bühlmann. Surprisingly, the work of William S. Jewell and Charles A. Hachemeister is not mentioned.

Matthew Rodermund takes a rather narrow view of casualty actuarial science, equating it, essentially, to the study of credibility theory. He is critical of risk theory which, in his words, still stands on the shoulders of credibility.

He is also very critical of "classical statistical theory", as opposed to the Bayesian discipline of credibility.

Ratemaking, by Charles L. McClenahan

The author of this chapter shows how one can perform a review of manual premium rates within the constraints of a given rating structure. Basic terminology is introduced and explained. Different approaches to the derivation of rates are presented, and special attention is given to the calculation of on-level premium (the level of current premium which is equivalent to a certain amount of statistical exposure). The necessity of projecting ultimate losses of immature accident years, and "trending" those to reflect the expected ultimate losses of future periods, is emphasised, and a technique for doing so is illustrated with a simple example. The effect of limits on severity trends is illustrated. The inclusion of loadings for expenses, profits and contingencies is discussed.

After overall rates have been determined, classification relativities must be found. A procedure for doing so is illustrated briefly with an example. Finally, any premium off-balance created by the classification relativities must be corrected for.

A worked-through example of a rate review for a fictitious auto insurance company is given as an appendix. This chapter also has a few pages of questions for discussion.

The author offers no model to explain the relationship between risk exposed and the generation of claims, or the difference in claim propensities between classes. Little guidance is given for the calculation of a class relativity, when the class exposure is small and data credibility is low.

For ratemaking at the overall level, this chapter contains much useful advice for a novice. Especially the emphasis on projecting and trending the ultimate losses of immature years, is timely (unfortunately, there are still insurance companies who base their rate decisions mainly on the loss ratio in last year's income statement).

Individual Risk Rating, by Margaret Wilkinson Tiller

This chapter discusses individual premium rating for large entities, or entities of special character (e.g. a Roller Skating Rink Risk Retention Group). Methods of individual risk rating are classified into Schedule Rating (adjusting a manual rate with discounts or loadings for observed risk factors), Experience Rating (adjusting next year's premium on the basis of previous years' loss experience), composite rating (experience rating using a composite exposure base for large, complex risks), and retrospective rating. Of each rating method, an example from real life is provided. The considerations necessary in designing an individual risk rating scheme, are mentioned and discussed.

Loss Reserving, by Ronald F. WISER

One of the major tasks of any practicing casualty actuary is the determination of loss reserves.

The author starts with giving an overview of accounting concepts and the place of loss reserves in corporate accounting. The tasks of a claims department are described lucidly. An actuarial model for loss development is set forward (I found that model hard to comprehend, but it is not used in what follows). The author then defines the necessary loss reserving terminology. Some questions of data availability and organisation are discussed; the estimation strategy must take the peculiarities of the data into account.

The author then offers a variety of angles from which to view loss development data in a preliminary, exploratory data analysis. Such an analysis is useful for detecting irregularities in respect of certain accident years. As far as I could see, the possibility of irregularities for calendar years is not mentioned. I also missed a formalised analysis of paid (or incurred) losses relative to the risk exposed; the amount of risk exposed is only verbally invoked as an explanatory variable.

The basis of the loss reserving method discussed in the next section is the chain-ladder method. The author explains how the raw, chain-ladder estimates can be adjusted judgementally, to dampen the effect of abnormal years. This method is applied both to paid loss development, and reserve development.

The Bornhuetter-Ferguson method is offered as a way of smoothing the estimated ultimate loss amounts, when data is sparse or very irregular.

The author then discusses the estimation of loss adjustment expenses, the incidence of which can follow a different pattern from paid or incurred losses. The necessity of comparing actual and predicted claims development is mentioned and discussed. Reserve discounting is mentioned only very briefly.

The author does not discuss the estimation of claims covered, but not incurred, or the idea of a premium deficiency provision. No way of assessing the uncertainty of the estimates is given.

Risk Classification, by Robert J. FINGER

Robert J. Finger discusses criteria for selecting rating variables, taking into account actuarial, operational, social and legal considerations and constraints. The need to classify risks, in order to prevent adverse selection, is thoroughly explained[1].

A detailed description of Motor Vehicle rating structures is given, while rating structures for other lines of business are only sketched. The author then presents a measure of efficiency of a rating structure, in a section which I found hard to comprehend. The estimation of class relativities is briefly discussed. The choice between an additive and a multiplicative rating structure is mentioned.

[1] While lecturing on the same topic, the reviewer was recently asked by a student why insurance companies would want to charge anyone a lower premium... weren't they interested in earning as much money as possible?

The discussion of credibility estimation for classes with small exposure is very general, emphasising the need to find a reliable and appropriate "credibility complement" (i.e., the term following the $(1-z)$). No model is given to help the actuary in finding a credibility complement.

Reinsurance, by Gary S. Patrik

In his introduction, the author explains the nature of reinsurance, its objectives, different reinsurance forms, cost considerations to the cedant. A thorough treatment of reinsurance pricing is given. Pricing formulae are derived under the assumption of pareto or lognormal claim size distributions (or their censored counterparts, for loss degrees). The peculiarities of all the common forms of reinsurance are discussed in detail.

Gary Patrik has also included a section on reinsurance loss reserving. The problems encountered in estimating the outstanding losses of a reinsurer, are recounted and explained. A general procedure of attacking the estimation problem is sketched, which begins with partitioning the data into meaningful blocks of reasonably homogeneous contracts. The chain ladder and Bornhuetter-Ferguson methods are given as possible estimation tools.

This chapter is the first one in this book which formally takes stochastic variation into account. The properties of the pareto and lognormal distributions, and the aggregate loss model, are given in an appendix.

In my opinion, Gary Patrik has written an excellent treatise on reinsurance, an area which is notoriously difficult to describe comprehensively, and comprehendibly.

Credibility, by Gary G. Venter

Charles C. Hewitt has written the prologue and postlogue for Gary Venter's chapter on Credibility. Both are a defense of Bayesian estimation and its linear counterpart, credibility.

Gary Venter, in his introduction, discusses alternative ways of viewing the prior distribution in credibility theory, mentioning both the frequentist and the formal view. He gives a short outline of the history of credibility theory. A review of the necessary probability theory is given, including: several lucid examples of the use of Bayesian inference outside insurance, a discussion of diffuse priors, the NP approximation to the aggregate claims distribution. The limited fluctuation approach to credibility is briefly outlined.

The least squares approach to credibility is then introduced, using the Bühlmann model. Estimation of the structural parameters is discussed within that framework, including the correction needed to make the estimated credibility factor unbiased, and Bayesian estimation of the credibility factor.

The next section is on incorporating risk size, giving the Bühlmann-Straub model. Empirical and Bayesian estimation of the credibility factor is discussed.

The last section is on assessing the linearisation error incurred when the unrestricted Bayes estimator is replaced by a credibility estimator. The author uses the example of a lognormal distribution with a lognormal prior.

A survey of further topics is given. In an appendix, the properties of a great number of distributions are tabulated, including an overview of less well-known pairs of conditional distribution/conjugate prior.

Though this chapter does not pursue credibility theory to its utmost generality, it mentions a number of interesting aspects (e.g., linearisation error), which other textbooks do not address explicitly.

Investment Issues in Property-Liability Insurance, by Stephen P. D'Arcy

The author discusses the role of investment income. He begins with an overview of the common assets (bonds, equities, real estate, others) and their peculiarities. Investment and tax strategies are then discussed. The former are of general interest, while the latter will be of most interest to actuaries practicing in the U.S.A. Different measures of the rate of return of an insurance business are presented, including combined ratio, underwriting profit margin, operating ratio, return on equity, and the effect of discounting losses. The impact (practical and statutory) of investment income on pricing is discussed, including use of the CAPM.

Special Issues, by Stephen P. D'Arcy

The following topics are briefly discussed in this chapter: measurement, allocation and uses of surplus; insurer solvency issues, including NAIC Early Warning tests and other rating systems, and guarantee funds; the risk theory approach to insurer solvency; planning and forecasting; sources of industry data and forecasts.

While not giving any detailed guidance for work in any of these areas, the chapter outlines the considerations which will have to be made.

General Review

For a long time, stochastic modelling was most prominent within casualty insurance. Only recently has the stochastic approach to life insurance been "officially" sanctioned by the publication of the Society of Actuaries' textbook *Actuarial Mathematics*. Even while it was being taught in a deterministic framework, the theory of life insurance offered techniques and equations of wide applicability, explanatory value and considerable elegance.

Casualty actuarial "science", as described in *Foundations of Casualty Actuarial Science*, lags several evolutionary steps behind life actuarial science, as expounded in *Actuarial Mathematics*. Not only is a deterministic view taken throughout most of the book (all but two chapters); it also lacks the unifying theory, model framework and other paradigms, which are the hallmarks of a true science.

On the other hand, the book gives a comprehensive overview of the tasks which may be asked of a practicing casualty actuary, and how one may attack them. It has a great deal of useful advice for the novice. In particular, it explains most of the concepts and the terminology of casualty insurance, and discusses their application. As a source of practical inspiration, the book can be recommended.

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