SOME REMARKS ON A RECENT PAPER BY BORCH *)

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In his recent paper, "An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance", presented to the XVIth International Congress of Actuaries, Dr. Karl Borch considers the problem of minimizing the variance of the total claims borne by the ceding insurer. Adopting this variance as a measure of risk, he considers as the most efficient reinsurance scheme that one which serves to minimize this variance. If x represents the amount of total claims with distribution function $F(x)$, he considers a reinsurance scheme as a transformation of $F(x)$. Attacking his problem from a different point of view, we restate and prove it for a set of transformations apparently wider than that which he allows.

The process of reinsurance substitutes for the amount of total claims x a transformed value Tx as the liability of the ceding insurer, and hence a reinsurance scheme may be described by the associated transformation T of the random variable x representing the amount of total claims, rather than by a transformation of its distribution as discussed by Borch. Let us define an *admissible* transformation as a Lebesgue-measurable transformation T such that

$$\text{(a)} \quad 0 \leq Tx \leq x, \text{ and}$$

$$\text{(b)} \quad c = \int_0^\infty (x - Tx) \, dF(x),$$

where c is a fixed number between 0 and $m = E(x)$. Condition (a) implies that the insurer will never bear an amount greater than the actual total claims. In condition (b), c represents the reinsurance premium, assumed fixed, and is equal to the expected value of the difference between the total amount of claims x and the total retained amount of claims Tx borne by the insurer. Let us define

$m' = m - c$, and let us note then that, since $m = \int_0^\infty x dF(x)$, condition (b) could be rephrased as

$$m' = E(Tx) = \int_0^\infty Tx dF(x).$$

For this class of admissible transformations, we shall show that the transformation associated with stop loss reinsurance serves to minimize the variance of the cedent's liability. We shall then consider a modified set of transformations for which this result is not necessarily true, the set of n-transformations. We define an *n-transformation* as a Lebesgue-measurable transformation T_n which, for fixed n, is such that

(a) $T_n x = x$, for $x < n$, and

(b) $c = \int_0^\infty (x - T_n x) \, dF(x) = \int_n^\infty (x - T_n x) \, dF(x)$,

where c is, as before, the fixed reinsurance premium. Admissible transformations form a subset of the totality of all n-transformations for all $n \geq 0$; trivially, they form a subset of all o-transformations. For a fixed n, we shall determine that n-transformations which minimizes the variance of the total retained claims. Finally, we shall mention a particular kind of admissible n-transformation related to coinsurance.

Our problem is to determine the admissible transformation which minimizes the variance of the insurer's liability on the retained portfolio subject only to a fixed reinsurance premium c. As a first step in the solution, we give the following lemma.

Lemma I.   There exists a unique n_0 such that $c = \int_{n_0}^\infty (x - n_0)$ $dF(x)$. If we let $H(n) = c - \int_n^\infty (x - n) \, dF(x)$, the proof of this lemma follows easily from noticing that $H(n)$ is continuous, that $H'(n)$ is nonnegative, that $H(0) = c - m < 0$ and that $H(\infty) = c > 0$.

Let us next define the stop loss transformation $T*$ as

$$T* x = x, \quad \text{for } x < n_0 \text{ and}$$
$$T* x = n_0, \quad \text{for } x \geq n_0,$$

where n_0 is as in the above lemma. T^* represents a reinsurance plan whereby the insurer pays the total claims if they are less than n_0 and pays only n_0 for total claims in excess of n_0 while the reinsurer pays the excess $x - n_0$.

Throughout this paper, we shall abbreviate the variance of a random variable by V. We then note the following lemma.

Lemma 2. T^* is an admissible transformation and if $V^* = V(T^*x)$, then $V^* = \int_0^{n_0} (x - n_0)^2 \, dF(x) - (m' - n_0)^2$. The proof of this lemma readily follows from the well-known result that, if M is any number and x is any random variable with mean m, $E(x - m)^2 = E(x - M)^2 - (m - M)^2$.

We now proceed to the principal theorem, that the transformation T^* serves to minimize the variance of the amount of total claims on the retained portfolio.

Theorem 1. If T is any admissible transformation and $V^T = V(Tx)$, then $V^* \leq V^T$.

Proof: Since T is admissible, $Tx \leq x$. Therefore, if $x \leq n_0$, then $(Tx - n_0)^2 \geq (x - n_0)^2$. Also, since $m' = \int_0^\infty Tx \, dF(x)$, we have that

$$V^T = \int_0^\infty (Tx - n_0)^2 \, dF(x) - (m' - n_0)^2 \geq \int_0^{n_0} (Tx - n_0)^2 \, dF(x) -$$

$$(m' - n_0)^2 \geq \int_0^{n_0} (x - n_0)^2 \, dF(x) - (m' - n_0)^2 = V^*.$$

This proves Dr. Borch's result for the class of admissible transformations, presumably a wider class than the one he considered. It shows that stop loss reinsurance produces a smaller variance than any other type of reinsurance associated with an admissible transformation. The above proof was suggested by Professor James G. Wendel who phrased the problem in general measure-theoretic terms.

We propose now to discuss the set of n-transformations defined above. This is a wider set than the class of admissible transformations. We note that the stop loss transformation T^* is an n_0-transformation; we wish to examine how its variance relates to that of an n-transformation with n not equal to n_0. Such n-trans-

formations arise in considering counter-examples to Dr. Borch's assertion, and, in fact, the stop loss transformation is no longer of minimum variance if choice from the set of n-transformations, $n < n_0$, is permitted.

An n-transformation represents a reinsurance plan whereby the cedent pays the full amount of all claims less than n and pays $T_n x$ on claims greater than n; as before, c represents the reinsurance premium. Let us note that, for $x \geq n$, $T_n x$ need not be less than x, the essential condition for a transformation to be admissible. Not all n-transformations have practical significance, but we shall investigate this set first in general and then attempt to limit the set by excluding certain transformations as impractical.

In considering these n-transformations, we shall make use of the function $N(n)$ defined by $N(n) = E(T_n x \mid x > n)$. If we denote $1 - F(n)$ by a_n, we see from the following equation that $N(n)$ is independent of the particular choice of n-transformation T_n and hence, for fixed n, is the same for all n-transformations.
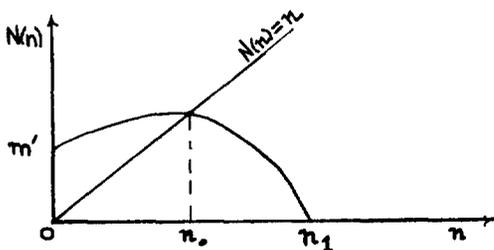
Figure 1

$N(n)$

$$N(n) = E(T_n x \mid x > n) = \frac{\int_n^\infty T_n x \, dF(x)}{a_n} = \frac{\int_n^\infty x \, dF(x) - c}{a_n}.$$

In the remaining discussion, we shall assume that $F(x)$ has a density function $f(x)$. Let us note that $N(0) = m - c = m'$, and that if n_1 is defined by $c = \int_{n_1}^\infty x \, dF(x)$ then $N(n_1) = 0$ since $F(n_1)$

is different from 1, for, if $F(n_1)$ were equal to 1, then $c = 0$, thereby contradicting our assumption that c is positive. Noting that

$$N'(n) = -\frac{f(n) H(n)}{a_n}$$ where $H(n)$ is defined in Lemma 1, we then

see easily that for $0 \leq n \leq n_1$, $N(n)$ has its maximum at n_0 and that $N(n_0) = n_0$. The graph of $N(n)$ is indicated in Figure 1.

If $n = n_1$, then for any n_1-transformation T_{n_1} we have that

$$c = \int_{n_1}^{\infty} (x - T_{n_1} x) \, dF(x) = c - \int_{n_1}^{\infty} T_{n_1} x \, dF(x) \text{ or that } \int_{n_1}^{\infty} T_{n_1} x \, dF(x)$$

$= 0$. For insurance with only positive risk sums as we consider here, n_1 is therefore a limiting case; if n were greater than n_1, we may

have $\int_{n}^{\infty} T_n x \, dF(x) < 0$, a most unrealistic situation.

We wish now to consider the problem of finding, for a given n and fixed c, that particular n-transformation which minimizes the variance of the total claims borne by the cedent on the retained portfolio. Let us then define the transformation T_n^* by

$$T_n^* x = x, \qquad \text{for } x < n \text{ and}$$
$$T_n^* x = N(n), \text{ for } x \geq n.$$

Let us note that $N(n) > n$ for $n < n_0$ and $N(n) < n$ for $n > n_0$. We may then prove the following theorem.

Theorem 2.   If $V_n^* = V(T_n^* x)$ and $VT_n = V(T_n x)$, then, for a given $n \geq 0$, we have that $V_n^* \leq VT_n$.

Proof: $VT_n = \int_0^{\infty} (T_n x)^2 \, dF(x) - m'^2 = \int_0^{n} x^2 \, dF(x) +$

$$+ a_n \left\{ \int_{n}^{\infty} [T_n x - N(n)]^2 \frac{dF(x)}{a_n} + N^2 \right\} - m'^2 \geq \int_0^{n} x^2 \, dF(x) + a_n N^2(n) -$$

$- m'^2 = V_n^*.$

This theorem shows that, if we are to choose an n-transformation as the basis for a reinsurance scheme, n and c fixed, the transformation T_n^* serves to minimize the variance of the retained portfolio. We have therefore solved the problem of minimizing this variance in the set of n-transformations by choosing a reinsurance scheme which concentrates all claims above n at $N(n)$. We proceed to investigate V_n^* somewhat further.

Under the assumption that $F(x)$ and $N(n)$ are differentiable, we have that $V_0^* = 0$, $\dfrac{dV_n^*}{dn} = f(n)[N(n) - n]^2 \geq 0$ for all n, and hence V_n^* has an inflection point at $n = n_0$. Therefore V_n^* is a non-decreasing function of n; for $n > n_0$, i.e. $n > N(n)$, T_n^* is an admissible transformation and the minimum of V_n^* occurs at $n = n_0$. If $n < n_0$, i.e. $n < N(n)$, however, then $T_n^* x$ is greater than x for $n < x < N(n)$. In this case T_n^* fails to be admissible; the reinsurance scheme associated with it is somewhat impractical, for it requires the cedent to bear an amount $N(n)$ exceeding the total claims if the amount of total claims is between n and $N(n)$. The cedent would presumably pay the amount x to the insured and the amount $N(n) - x$ to the reinsurer and in effect would pay for stability by post-payment of reinsurance. In more detail, the insurer prepays c for reinsurance and pays the loss x if $x < n$ for a total payment of $c + x$, which varies with x. In addition, if $n < x < N(n)$, the insurerer pays $N(n) - x$ to the reinsurer, so that in this case a total of $c + x + (N - x) = c + N(n)$ is paid without variation, and this is true for $x > N(n)$; hence for all cases of total claims exceeding n, the insurer pays a total of $c + N(n)$. If $n = 0$, then $N(n) = m'$ and the insurer pays a total of $c + m' = m$, without variation. If $n = N(n)$, i.e. $n = n_0$, we have the stop loss reinsurance defined by T^*.

Let us consider a special set of admissible n-transformations which describe a type of reinsurance with a coinsurance feature, the transformations T_n^b, $0 < b \leq 1$, defined by

$$T_n^b x = x, \text{ for } x < n \text{ and}$$
$$T_n^b x = bn + (1 - b)x = n + (1 - b)(x - n), \text{ for } x \geq n,$$

where n is determined, for fixed reinsurance premium c, by $c = \int_0^\infty (x - T_n^b x)\, dF(x)$. In this scheme, the reinsurer bears b of the excess of claims above n and the cedent retains $(1 - b)$ of this excess over n, i.e. total claims in excess of n are shared in the proportion b by the reinsurer and $(1 - b)$ by the cedent.

The equation defining n may be reduced to $c = b \int_n^\infty (x - n)\, dF(x)$

which may be written as $\dfrac{c}{b} + a_n n = \int\limits_{n}^{\infty} x \, dF(x)$ or $N(n) - n = \dfrac{(1 - b) \, c}{a_n b}$.

These equations define b as a function $b(n)$ of n. Since T_n^b is an admissible transformation, we may apply Theorem I to find that, if $V_n^b = V(T_n^b x)$, then V_n^b is minimized at $n = n_0$, $b(n_0) = 1$, for these values of n and b clearly give the stop loss transformation $T*$.

Although the variance is minimized at $b = 1$, an insurer or re-insurer may still prefer a scheme where $b < 1$, for underwriting reasons, for example. Let us note that $b(0) = \dfrac{c}{m}$ and $V_n^{b(0)} = (1 - \dfrac{c}{m})^2 V(x)$. It is also easily seen that $\dfrac{db(n)}{dn} = \dfrac{a_n b^2}{c}$, and as n in-creases from 0 to n_0, $b(n)$ increases from $\dfrac{c}{m}$ to 1. From this it may be

noted that values of n larger than n_0 are impractical, for they would imply values of b greater than 1, and hence, for sufficiently large claims, $T_n^b x$ would be negative.

Let us define the function y_x of the random variable x by

$$y_x = x, \text{ for } x \geq n$$

and let us note that it has distribution function $[F_{(y)} - F_{(n)}]/a_n$. Assuming that $V_n^{b(n)}$ is a differentiable function of n, we find that

$$\frac{dV_n^b}{dn} = -\frac{2 (a_n b)^2 (1 - b)}{c} V(y_x).$$

Since $b(n) > 1$ for $n > n_0$ and $b(n) < 1$ for $n < n_0$, we have $\dfrac{dV_n^b}{dn} \leq 0$ for $n < n_0$ and $\dfrac{dV_n^b}{dn} \geq 0$ for $n > n_0$. If $n = n_0$, then $b(n_0) = 1$ and $\dfrac{dV_n^b}{dn} = 0$; this confirms our finding that V_n^b is mini-mized at $n = n_0$. Hence as n increases from 0 to n_0, V_n^b decreases from $(1 - \dfrac{c}{m})^2 V(x)$ to $V*$.

Coinsurance is an important element of reinsurance, and for this reason we have chosen to investigate it in detail. For coinsurance

with $b < 1$, the insurer's variance is somewhat greater than it is under stop loss reinsurance, and the relations obtained here provide some indication of the increase in the insurer's variance for various degrees of coinsurance.

We briefly discuss an alternate criterion to that involving the variance. Let c again be a fixed positive reinsurance premium and let T be a continuous, nondecreasing, admissible transformation.

Let $M = \inf \{ x \mid T_x = m' \}$, where $m' = \int_0^\infty Tx dF(x)$. Define the gain $G(x)$ and the loss $L(x)$ by

$$G(x) = m' - Tx, L(x) = 0, \text{ for } x < M, \text{ i.e. } Tx < m'$$
$$G(x) = 0, \qquad L(x) = Tx - m', \text{ for } x \geq M.$$

Let $R = R(T) = E(G^2) - E(L^2)$, let us choose R as the basis for our new criterion, the larger R, the better the reinsurance scheme T; R gives some measure of how favorable the scheme is to the insurer. Preliminary study indicates that there may not be any simple solution to finding a transformation T maximizing R. In certain special cases of practical interest, we can, however, find a solution. We shall assume that $m' \leq n$, a not very restrictive condition, since one is usually interested in reinsuring only claims above a certain limit, usually beyond the mean m, and hence beyond m'. This condition implies that $M = m'$, if we consider the coinsurance transformation T_n^b.

We find that $R = 2 \int_0^M (Tx - m')^2 dF(x) - V^T$, in general. For $T = T_n^b$, this reduces to $R = 2 \int_0^{m'} (x - m')^2 dF(x) - V_n^b$, the first term of which is independent of T. Therefore, to maximize R, we must minimize V_n^b; but this problem was solved above by taking $n = n_0$ and $b(n_0) = 1$, i.e. for T_n^b the stop loss transformation T^*. This argument has been stated in terms of the coinsurance transformation because it is a practical illustration of a continuous, nondecreasing, admissible n-transformation. One may assert, however, that with choice permitted from the set of all such transformations with $m' \leq n$, the stop loss transformation gives maximal R.