HOMOGENEOUS PREMIUM CALCULATION PRINCIPLES

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Abstract

A premium calculation principle π is called positively homogeneous if $\pi(cX) = c\pi(X)$ for all $c > 0$ and all random variables X. For all known principles it is shown that this condition is fulfilled if it is satisfied for two specific values of c only, say $c = 2$ and $c = 3$, and for only all two point random variables X. In the case of the Esscher principle one value of c suffices. In short this means that local homogeneity implies global homogeneity. From this it follows that in the case of the zero utility principle or Swiss premium calculation principle, the underlying utility function is of a very specific type.

A very general theorem on premium calculation principles which satisfy a weak continuity condition, is added. Among others the proof uses Kroneckers Theorem on Diophantine Approximations.

1. Introduction

In actuarial practice one generally uses only three premium calculation principles, namely the expected value principle, the variance principle and the standard deviation principle. Apart from these there are many other principles for determining a premium for a risk: all these are examined in the new textbook by Goovaerts, De Vylder and Haezendonck (1984). A central theme is the analysis of the principles which fulfil some desirable properties such as translation invariance, (sub-)additivity, iterativity, homogeneity etc. For example the expected value principle is always additive and homogeneous, but it is iterative or translation invariant only in the case of a vanishing loading.

If a premium principle is defined by a utility function, then the above mentioned, (so-called) plausible properties are in general very restrictive: The Swiss premium calculation principle e.g. is translation invariant if and only if the corresponding utility function is exponential or linear, and it is positively homogeneous if the utility function $u(x)$ is—up to linear transformations—a power of x. Therefore, e.g., the Swiss premium calculation principle is both translation invariant and homogeneous only in the case of a linear utility function. Such an analysis has been performed already for all known principles and all properties mentioned above. If π denotes a premium calculation principle, which therefore to any real random variable X assigns a real number $\pi(X)$—the premium of X—then in

all cases it turns out that, e.g., in the case of translation invariance (that means

$$(1) \qquad\qquad \pi(X + c) = \pi(X) + c$$

for all risks X and all real numbers c), it is sufficient to know equation (1) for all $c \in \mathbb{R}$ and only all $X \in D_2$ (D_2 the set of all 2-atomic random variables): If (1) is valid for all $c \in \mathbb{R}$, $X \in D_2$, then automatically (1) is fulfilled also for those X not in D_2. This reduction to the essence of a property has been worked out for the property of translation invariance by REICH (1984) in a definitive sense: Any principle is already translation invariant (i.e., (1) holds for all $c \in \mathbb{R}$ and all risks X), if (1) is fulfilled for all $X \in D_2$ and two specific values of c only, say $c = 1$ and $c = \sqrt{2}$. In case of the Orlicz principle (HAEZENDONCK and GOOVAERTS (1982)) a single value of c, e.g., $c = 1$, suffices. A further reduction is impossible as one can see from the counterexamples in REICH (1984).

In the case of the property of homogeneity (sometimes also called proportionality)

$$(2) \qquad\qquad \pi(cX) = c\pi(X)$$

(more exactly we will examine positively homogeneous principles, i.e., $c \in \mathbb{R}^+$) we will now give a similar analysis of the analogous problems. Equation (2) means for $c = \frac{1}{2}$, say, that the premium of X should be homogeneously divided in two equal parts, if the risk X is split up into two parts in a homogeneous way. The aim of this paper therefore is to give an answer to the question· How little does one really need to know, to have already property (2) in full generality (i.e., for arbitrary risk X and arbitrary $c \in \mathbb{R}^+$)? Of course, this leads to other conditions than in the case of translation invariance and other principles are now of special interest. A mere corollary from the results (still to be formulated and proved) should be mentioned here: Take for example the Swiss premium calculation principle. If (2) holds for all $X \in D_2$ only and for all $c \in [\frac{1}{3}, \frac{1}{2}]$, then (2) holds automatically for all risks X and all $c \in \mathbb{R}^+$. There is therefore no difference in homogeneity as a local or global property. This fact is a trivial consequence of theorem 2.2, which is best possible in the precise sense specified there. Moreover for every known premium calculation principle the following is true ($X \in D_2$): If (2) holds in the two special cases $c = \frac{1}{2}$ and $c = \frac{1}{3}$ only, then again (2) is fulfilled for all $c > 0$.

From this one can prove that even an extremely weaker assumption than the homogeneity is (with the Orlicz principle as the only exception) very restrictive for all utility principles.

2. RESULTS AND REMARKS

Among the known principles the following are in every case (i.e., independent of the choice of the corresponding parameters or utility functions) positively homogeneous: Expected value principle, maximal loss principle, percentile principle, standard deviation principle and Orlicz principle. The variance principle

on the contrary is certainly not positively homogeneous in the case of a non-vanishing loading (cf. GOOVAERTS, DE VYLDER and HAEZENDONCK (1984)).

For the remaining cases of the Swiss Premium calculation principle, the zero utility principle (which is indeed a special case of the Swiss premium calculation principle, but has for technical reasons to be treated separately) and the Esscher principle it will now be proved for example: If one has for all $X \in D_2$

$$\pi(\tfrac{1}{2}X) = \tfrac{1}{2}\pi(X) \quad \text{and} \quad \pi(\tfrac{1}{3}X) = \tfrac{1}{3}\pi(X),$$

then

$$\pi(cX) = c\pi(X)$$

holds for all X and all $c \in \mathbb{R}^+$, i.e., π is positively homogeneous. More generally and more exactly:

2A. $\pi =$ zero utility principle

This principle was introduced by BUHLMANN (1970). One starts with a utility function u with $u'(x) \geq 0$, $u''(x) \leq 0$. For a given risk X the premium $P = \pi(X)$ is determined by

(3) $$E[u(P - X)] = u(0).$$

We prove

THEOREM 2.1. *For fixed, positive c_1, $c_2 \neq 1$ let $\log c_1 / \log c_2$ not be rational If for every $X \in D_2$*

$$\pi(c_1 X) = c_1 \pi(X) \quad and \quad \pi(c_2 X) = c_2 \pi(X)$$

hold, then u is linear. Conversely, if u is linear, then for all X and all $c \in \mathbb{R}^+$

$$\pi(cX) = c\pi(X)$$

holds, i.e. π is positively homogeneous.

REMARK Theorem 2.1 is best possible in the following sense: For any pair c_1, $c_2 \in \mathbb{R}^+$ ($c_1 = c_2$ is admissible), which the condition of theorem 2.1 (i.e., $\log c_1 / \log c_2 \in \mathbb{Q}$) does not fulfil, there is a non-linear utility function u such that

$$\pi(c_i X) = c_i \pi(X), \qquad i = 1, 2, X \in D_2,$$

holds. In this case the zero utility principle certainly is not positively homogeneous.

2B. $\pi =$ Swiss premium calculation principle

This principle was introduced by BUHLMANN, GAGLIARDI, GERBER and STRAUB (1977) If $z \in [0, 1]$ and u is a strictly monotonic, continuous function on $\mathbb{R}$ the

premium $P = \pi(X)$ for a risk X is given by the equation

(4) $$E[u(X - zP)] = u((1 - z)P).$$

In the case $z = 1$ and by the substitution $u(x) \to -u(-x)$ one just gets the zero utility principle. By a different proof than in the case 2A one proves for $0 \le z < 1$

THEOREM 2.2. *For fixed, positive* c_1, $c_2 \neq 1$ *let* $\log c_1/\log c_2$ *not be rational. If for every* $X \in D_2$

$$\pi(c_1 X) = c_1 \pi(X) \quad and \quad \pi(c_2 X) = c_2 \pi(X)$$

hold, then for suitable α, β, γ, $r \in \mathbb{R}$ *(with* $\beta\gamma > 0$, $r > 0$*)*

(5) $$u(x) = \begin{cases} \alpha + \beta x^r, & x \ge 0 \\ \alpha - \gamma(-x)^r, & x < 0. \end{cases}$$

Conversely, if u has the form (5), *then for all X and all* $c \in \mathbb{R}^+$

$$\pi(cX) = c\pi(X)$$

holds, i.e., π *is positively homogeneous.*

REMARK. Theorem 2.2 is best possible in the following sense· For any pair c_1, $c_2 \in \mathbb{R}^+$, which does not fulfil the conditions of theorem 2.2, there is an admissible utility function u, not of the form (5), such that

$$\pi(c_i X) = c_i \pi(X), \qquad i = 1, 2, X \in D_2$$

holds. In this case the Swiss premium calculation principle certainly is not positively homogeneous.

2C. $\pi = $ *Esscher principle*

This principle was introduced in BUHLMANN (1980) and so named in view of the formal similarity to the Esscher transform. Given $\alpha \ge 0$ the premium $P = \pi(X)$ is determined explicitly by the equation

$$\pi(X) = \frac{E[X \exp(\alpha X)]}{E[\exp(\alpha X)]}.$$

It is very easy to see (cf. GOOVAERTS, DE VYLDER and HAEZENDONCK (1984)), that the Esscher principle is positively homogeneous only in the case $\alpha = 0$, i.e., Esscher premium = net premium. A simple proof will give the following sharp result:

THEOREM 2.3. *If for a fixed* $c_0 \neq 1$, *a single (non-degenerated)* $X_0 \in D_2$ *the equation*

$$\pi(c_0 X_0) = c_0 \pi(X_0)$$

holds, then $\alpha = 0$.

2D. *General premium principles*

The result mentioned in the introduction, namely that the property of positive homogeneity is already fulfilled in the global sense if it is known only locally with respect to the variable c, now follows easily: If π denotes the zero utility principle, the Swiss premium calculation principle or the Esscher principle, then the theorems above yield at once

COROLLARY 2.4. *If for all $X \in D_2$ and all c in a given (arbitrarily small) bounded interval in $\mathbb{R}^+$ one has*

$$\pi(cX) = c\pi(X),$$

then $\pi(cX) = c\pi(X)$ holds for all X and all $c \in \mathbb{R}^+$.

Finally one should pay attention to a very general result, which on the one hand makes the results above more transparent, on the other hand is true for general, possibly still unknown principles: Denote by π any premium principle with the very weak and plausible continuity condition, that for every convergent sequence $(\gamma_k) \subset \mathbb{R}^+$ and every $X \in D_2$

$$\lim_{k \to \infty} \pi(\gamma_k X) = \pi(\lim_{k \to \infty} \gamma_k \cdot X).$$

For such principles one has throughout

THEOREM 2.5. *For fixed, positive c_1, $c_2 \neq 1$ let $\log c_1/\log c_2$ not be rational. If*

$$\pi(c_1 X) = c_1 \pi(X) \quad and \quad \pi(c_2 X) = c_2 \pi(X), \qquad X \in D_2,$$

then $\pi(cX) = c\pi(X)$ holds for all $X \in D_2$ and even all $c > 0$.

As a corollary (because in any interval I, however small it may be, there are of course always two numbers c_1, $c_2 \neq 1$ in I such that $\log c_1/\log c_2 \notin \mathbb{Q}$) we note: The (global) property of such premium calculation principles of being positively homogeneous is always a local property in the following sense:

If $\pi(cX) = c\pi(X)$ holds for only all $c \in I$, then automatically also for even all $c \in \mathbb{R}^+$.

REMARK. Simple and explicit examples for pairs of numbers c_1, $c_2 \neq 1$, which satisfy $\log c_1/\log c_2 \notin \mathbb{Q}$, are the following:

(i)	$c_1 = 2,$	$c_2 = 3,$
(ii)	$c_1 = \frac{1}{2},$	$c_2 = \frac{1}{3},$
(iii)	$c_1 = 1.1,$	$c_2 = 1.2,$
(iv)	$c_1 = 2,$	$c_2 = \pi,$
(v)	$c_1 = 2,$	$c_2 = e^{\pi},$
(vi)	$c_1 = e,$	$c_2 = e^{\pi},$
(vii)	$c_1 = 1.25,$	$c_2 = 1.1.$

As was pointed out by GERBER (1979), pp. 73–74, the global property of positive homogeneity is not reasonable for practical reasons. From example (vii) e.g., one can deduce by the corollary of theorem 2.5 more precisely that it is in the same way again unreasonable to accept the homogeneity property as a local property with respect to c only: If for all X

$$\pi(cX) = c\pi(X)$$

holds for, say all c between 1 and 1.25 (local property), then automatically by the results above also for all $c > 0$ (global property). In accordance with GERBER (1979) the quotient $\pi(cX)/\pi(X)$ should depend not only on c but also on X.

3. PROOFS

Ad 2A: First of all we consider the zero utility principle π with strictly monotonic utility function u, such that $u'(x) \geqslant 0$, $u''(x) \leqslant 0$. To prove theorem 2.1 we assume $u(0) = 0$ without loss of generality, because for a given risk X the premium $P = \pi(X)$ does not change if in (3) u is substituted by $u - u(0)$.

LEMMA 3.1. *If for a fixed $c_0 > 0$*

$$(6) \qquad \pi(c_0 X) = c_0 \pi(X), \qquad X \in D_2,$$

then there exists a $\beta_0 = \beta_0(c_0)$ such that

$$u(c_0 x) = \beta_0 u(x)$$

holds for all $x \in \mathbb{R}$.

PROOF. For the present let be $x > 0$, and for $a, b \in \mathbb{R}$, $q \in [0, 1]$ let $X = X_{a,b,q} \in D_2$ be defined by

$$\mathrm{pr}\,(X = a) = 1 - q, \qquad \mathrm{pr}\,(X = b) = q.$$

With the abbreviation $P = P(a, b, q) = \pi(X)$ one has by (3)

$$(7) \qquad (1 - q)u(P - a) + qu(P - b) = u(0),$$

and by (6)

$$(8) \qquad (1 - q)u(c_0(P - a)) + qu(c_0(P - b)) = u(0).$$

(7) and (8) yield

$$(9) \qquad \frac{1 - q}{q} = \frac{u(P - b) - u(0)}{u(0) - u(P - a)},$$

$$(10) \qquad \frac{1 - q}{q} = \frac{u(c_0(P - b)) - u(0)}{u(0) - u(c_0(P - a))}.$$

Putting $x' = -1$, $a = 2$, $\gamma = 1$, $b = 1 - x$ one gets

$$(11) \qquad a = \gamma - x', \qquad b = \gamma - x \quad \text{and} \quad b < \gamma < a.$$

If

(12)
$$q = \frac{u(0) - u(x')}{u(x) - u(x')},$$

then $0 < q < 1$ in view of the strict monotonicity of u. Because of $x = \gamma - b$, $x' = \gamma - a$ and by (12) one concludes

$$(1 - q)u(\gamma - a) + qu(\gamma - b) = u(0),$$

therefore $\gamma = P(a, b, q)$ according to (7). From this, in view of (10), (11) it follows

(13)
$$\frac{1 - q}{q} = \frac{u(c_0 x) - u(0)}{u(0) - u(c_0 x')}$$

for arbitrary $x > 0$. Together with (9) this leads to

$$\frac{u(c_0 x) - u(0)}{u(0) - u(c_0 x')} = \frac{u(x) - u(0)}{u(0) - u(x')}$$

Therefore

$$u(c_0 x) - u(0) = \frac{u(0) - u(c_0 x')}{u(0) - u(x')} [u(x) - u(0)],$$

$$u(c_0 x) = \frac{u(0) - u(c_0 x')}{u(0) - u(x')} u(x) + u(0) \left[1 - \frac{u(0) - u(c_0 x')}{u(0) - u(x')} \right]$$

$$= \frac{u(c_0 x')}{u(x')} u(x)$$

for all $x > 0$, respecting the normalization $u(0) = 0$. With $\beta_0 = u(c_0 x')/u(x')$ this is the assertion for $x > 0$.

In the case $x \leq 0$ one proves in an analogous way the existence of a real number γ_0 such that

(14)
$$u(c_0 x) = \gamma_0 u(x)$$

holds for all $x \leq 0$. Now certainly $\beta_0 = \gamma_0$ (this is exactly the statement of lemma 3.1), because with regard to (9) and $u(0) = 0$ one has

(15)
$$\frac{1 - q}{q} = -\frac{u(P - b)}{u(P - a)}$$

Correspondingly by (10)

(16)
$$\frac{1 - q}{q} = -\frac{u(c_0(P - b))}{u(c_0(P - a))} = -\frac{\beta_0}{\gamma_0} \cdot \frac{u(P - b)}{u(P - a)}$$

if $b < P < a$, which is true in view of (11) and $P = \gamma$. Comparing (15) and (16) one has $\beta_0 = \gamma_0$.

PROOF OF THEOREM 2.1. Under the assumptions of theorem 2.1 there are by lemma 3.1 real numbers β_1, β_2 such that

(17) $u(c_1 x) = \beta_1 u(x)$ and $u(c_2 x) = \beta_2 u(x)$, $x \in \mathbb{R}$.

Successive application of these relations gives for $x = 1$

$$u(c_1^n c_2^m) = \beta_1^n \beta_2^m u(1), \qquad n, m \in \mathbb{Z}.$$

Together with (17) it follows for $x \in \mathbb{R}$, n, $m \in \mathbb{Z}$

(18) $u(c_1^n c_2^m x) = u(c_1^n c_2^m) u(x)/u(1)$.

By assumption one has $\log c_1 / \log c_2 \notin \mathbb{Q}$, therefore according to Kronecker's approximation theorem (cf. REICH (1984), Appendix) the set

$$\{k \log c_1 + l \log c_2 \ \ k, l \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. From this it follows at once that for every given number $y > 0$ there are two sequences $k(n)$, $l(n) \in \mathbb{Z}$ such that

(19) $y = \lim_{n \to \infty} c_1^{k(n)} c_2^{l(n)}$.

By (18) and the continuity of u one concludes

(20) $u(yx) = u(y)u(x)/u(1)$.

for arbitrary $y > 0$, $x \in \mathbb{R}$. The only continuous solution of this functional equation are

(21) $u(x) = u(1)x^r$, $x > 0$,

with some $r \in \mathbb{R}$, as is well known. Because u is strictly increasing, (21) holds for all $x > 0$ with suitable $r > 0$. Moreover, if $x < 0$ then it follows from (20)

$$u(x) = u(-1)(-x)^r,$$

so indeed there are numbers $\beta > 0$, $\gamma < 0$ such that

$$u(x) = \begin{cases} \beta x^r, & x \geq 0 \\ \gamma(-x)^r, & x < 0 \end{cases}.$$

In the case $r = 1$ one has certainly $\beta = -\gamma$ for continuous u', therefore u is linear. The case $r \geq 2$ is impossible in view of $u''(x) \leq 0$, the case $0 < r < 2$, $r \neq 1$, is impossible according to the existence of $u''(0)$. Because of the assumed normalization of u theorem 2.1 is proved.

The remark after theorem 2.1 can be easily proved.

Ad 2B:

PROOF OF THEOREM 2.2 (Swiss premium calculation principle). Let be $z < 1$ and without any restriction of generality let u be strictly increasing. Assume as

in theorem 2.2 that two numbers c_1, $c_2 \neq 1$ are given and that $\pi(c_i X) = c_i \pi(X)$ holds for $i = 1, 2$. According to (4) one has more precisely

$$(22) \qquad E[u(c_i X - zc_i P)] = u((1 - z)c_i P), \qquad X \in D_2, \qquad i = 1, 2.$$

Defining $g_i(x) = u(c_i x)$ equation (22) gives

$$E[g_i(X - zP)] = g_i((1 - z)P).$$

By GOOVAERTS, DE VYLDER and HAEZENDONCK (1984), theorem 2, p. 72, there exist real numbers α_i, β_i, $i = 1, 2$, such that

$$(23) \qquad u(c_i x) = \alpha_i + \beta_i u(x), \qquad x \in \mathbb{R}.$$

Without any restriction one can assume u to be normalized, especially $u(0) = 0$. Then, of course, $\alpha_i = 0$ and

$$(24) \qquad u(c_i x) = \beta_i u(x), \qquad x \in \mathbb{R}.$$

From this it follows immediately for arbitrary n, $m \in \mathbb{Z}$, $x \in \mathbb{R}$

$$(25) \qquad u(c_1^n c_2^m x) = \beta_1^n \beta_2^m u(x) = u(c_1^n c_2^m) u(x) / u(1).$$

The condition $\log c_1 / \log c_2 \notin \mathbb{Q}$ leads via Kronecker's approximation theorem and the continuity of u to

$$(26) \qquad u(yx) = u(y)u(x) / u(1)$$

for all $y > 0$, $x \in \mathbb{R}$.

In the case $x \geq 0$ one introduces $u_1(x) = u(x) / u(1)$ and gets by (26)

$$u_1(yx) = u_1(y)u_1(x), \qquad x, y > 0.$$

As is well known for continuous u_1 it follows that u_1 is monomial, therefore u too.

$$u(x) = u(1)x^r, \qquad x \geq 0,$$

with suitable $r > 0$.

In the case $x < 0$ one defines $z = -x$ and $u_2(z) = -u(z)$. By (26)

$$u_2(yz) = u_2(y)u_2(z) / u_2(1),$$

therefore it follows in a similar way that u_2 is a monomial. This means

$$u(x) = -u(-1)(-x)^s, \qquad x < 0,$$

for suitable $s > 0$. In view of (26) $r = s$ holds, so the first part of theorem 2.2 is proved. The second part is trivial.

The remark after theorem 2.2 is easily proved and the proof is omitted.

Ad 2C:

PROOF OF THEOREM 2.3 (Esscher principle). Let $X_0 \in D_2$ not be degenerated, say

$$\text{pr}(X_0 = a) = 1 - q, \qquad \text{pr}(X_0 = b) = q$$

for some $a, b \in \mathbb{R}$, $a \neq b$, $q \in (0, 1)$. If for a fixed $c_0 \neq 1$

$$\pi(c_0 X_0) = c_0 \pi(X_0)$$

is true, then

$$\frac{c_0 a(1-q) \exp(\alpha c_0 a) + c_0 bq \exp(\alpha c_0 b)}{(1-q) \exp(\alpha c_0 a) + q \exp(\alpha c_0 b)} = c_0 \frac{a(1-q) \exp(\alpha a) + bq \exp(\alpha b)}{(1-q) \exp(\alpha a) + q \exp(\alpha b)}.$$

Multiplication yields

$$(a-b) \exp[\alpha(c_0 a + b)] = (a-b) \exp[\alpha(c_0 b + a)],$$

therefore in view of $a \neq b$

$$\exp[\alpha(c_0 a + b)] = \exp[\alpha(c_0 b + a)].$$

Assume $\alpha > 0$, then $c_0 a + b = c_0 b + a$, therefore $a = b$ according to $c_0 \neq 1$. This is a contradiction, so necessarily $\alpha = 0$.

Ad 2D:

If $I = (a, b)$, $0 \leqslant a < b \leqslant \infty$, is any interval in $\mathbb{R}^+$, then by trivial arguments there are two numbers $c_1, c_2 \in I$ such that $\log c_1 / \log c_2 \notin \mathbb{Q}$. From this it is clear that Corollary 2.4 follows from the preceding Theorems.

PROOF OF THEOREM 2.5. By induction one immediately proves for $n, m \in \mathbb{N}$

(27) $$\pi(c_1^n X) = c_1^n \pi(X), \qquad \pi(c_2^m X) = c_2^m \pi(X).$$

Because of

$$\pi(X) = \pi\left(c_i \cdot \frac{1}{c_i} X\right) = c_i \pi\left(\frac{1}{c_i} X\right), \qquad i = 1, 2,$$

equation (27) even holds for $n, m \in \mathbb{Z}$. Then one has

(28) $$\pi(c_1^n c_2^m X) = c_1^n \pi(c_2^m X) = c_1^n c_2^m \pi(X).$$

Given an arbitrary $c > 0$ there are according to Kronecker's approximation theorem (cf. REICH (1984), Appendix) two sequences $n(k), m(k) \in \mathbb{Z}$ such that

$$\gamma_k = c_1^{n(k)} c_2^{m(k)} \to c.$$

The continuity condition for π and (28) yields

$$\pi(cX) = \pi(\lim_{k \to \infty} \gamma_k \cdot X) = \lim_{k \to \infty} \pi(\gamma_k X) = \lim_{k \to \infty} \gamma_k \cdot \pi(X) = c\pi(X),$$

therefore the assertion.

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