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THE RUIN PROBLEM WITH A FINITE TIME HORIZON*

By Marc-Henri Amsler

University of Lausanne, Switzerland

Abstract

The paper presents an extension of the classical Cramér–Lundberg ruin theory: the famous upper bound for the ruin probability with an infinite time horizon can be extended in a certain sense to the short and middle term case. Furthermore, a relation between the average values of lifetime and ruin amount is given.

Keywords

Ruin theory, middle term horizon, lifetime, ruin amount.

1. Introduction

In order to assess the financial stability of an insurance portfolio, one usually utilizes the notion of "mathematical ruin". Ruin being the phenomenon by which a portfolio passes from the state "to be" to the state "to be no longer", actuaries have naturally sought to measure the danger of such a passage by its "probability". Numerous studies have unfortunately shown that the notion of "ruin probability" is not easy to handle, in theory as well as in practice. This difficulty, and it seems to be a major one, requires the search for another quantifier of the notion of ruin than that of probability.

The present article recalls firstly the notion of the ruin "counter-utility", proposed elsewhere, and which represents a more elaborate measure of danger than that of "probability". The ruin counter-utility takes into account three characteristics of ruin, that is:

the probability of its occurrence
the size of the ruin amount
the time of its occurrence.

The counter-utility is the greater, the larger the ruin amount, and is the smaller the more distant the event. The notion of counter-utility depends very closely on that of utility; in a certain way it reverses its properties.

Secondly, the article shows that the celebrated upper bound of the ruin probability, indicated by Lundberg, valid in an infinite time horizon, can be generalized to the case of a finite time horizon. For this purpose the future should not be separated in two distinct periods, the considered period, and the one left out, but should be considered in its totality with a progressive attenuation

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of what occurs, by a phenomenon similar to fog, limiting vision to a certain horizon (time-stumping phenomenon).

By means of this new approach to the ruin problem it is easier to acquire, without mathematical complications, some knowledge of the seriousness of ruin in a limited time horizon.

The considered portfolios will be characterised by the following symbols:

X	aggregate claim amount
$f(x)$	density function of X
$M(a)$	moment generating function of X
P	total risk premium per annum of the portfolio
P_E	Esscher premium of the portfolio
R_t	risk reserve, at time t
T	time elapsed until the first ruin
q_t	probability of the first ruin at time t
ψ	ruin probability in the future
Z	amount of the first ruin
Z_t	amount of the first ruin, at time t
$g_t(z)$	density function of Z_t
$u(x)$	utility function
$\bar{u}(x)$	counter-utility function
$\bar{U}(Z)$	ruin counter-utility
a	risk aversion coefficient
b	time stumping coefficient
θ	time horizon

The article considers, for means of simplification, portfolios that are stationary in time and create independent total claim amounts, and is based on exponential utility and counter-utility functions. Under these assumptions, the results are valid for an arbitrary process, not necessarily Poisson.

2. CLASSICAL RESULTS OF THE RUIN THEORY

The classical ruin theory is dominated by two notions: security margins and ruin probability. Here are some known properties:

Security Margins

The zero utility principle

$$U(R_{t+1}) = \int u(R_t + P - x) \cdot f(x) \cdot dx,$$

under the hypothesis of an exponential utility function $u(x)$, leads to the following formula for the premium P, margin included:

$$e^{aP} = M(a) \tag{1}$$

where

$$M(a)=\int e^{ax}\cdot f(x)\cdot dx$$

which is equivalent to

$$P=\frac{1}{a}\ln M(a). \tag{2}$$

Ruin Probability

The ruin probability (first ruin, with t an integer)

$$\psi=\sum_{t=1}^{\infty}\int_0^{\infty} g_t(z)\cdot dt \tag{3}$$

is limited by Lundberg's upper bound

$$\psi<e^{-a\ R_0}. \tag{4}$$

The coefficients a in (2) and (4) are identical.

3. THE NOTION OF COUNTER-UTILITY

The notion of utility is borrowed from economics: it allows the determination of preferences between many situations.

The notion of counter-utility is derived from that of utility; it adds, for insurance purposes, a possibility to measure singularity considered situations.

Let Y be a random variable. The expression

$$\bar{U}(Y)=\int \bar{u}(y)\cdot f(y)\cdot dy$$

in which the function $\bar{u}(y)$ satisfies

$$\bar{u}(y)>0; \qquad \bar{u}'(y)>0, \qquad \bar{u}''(y)\geqslant 0 \tag{5}$$

is called the counter-utility of Y. The function $\bar{u}(y)$ is the counter-utility function.

It is to be noted that the requirement $\bar{u}''\geqslant 0$ is the reverse of $u''\leqslant 0$, which the utility function is subjected to. $\bar{U}(Y)$ can be used to measure a risk: in $\bar{U}(Y)$, the big values of Y are weighted overproportionally.

The exponential function

$$\bar{u}(y)=e^{ay}$$

satisfies our exigences. The coefficient a is called the risk aversion coefficient.

The relation (1) expresses that, on the basis of an exponential counter-utility function, there is equivalence between the counter-utility of the premium P (left-hand term) and that of risk X (right-hand term):

$$e^{aP}=M(a).$$

This last relation formalises the "*counter-utility equivalence principle between premium and risk*". Through this interpretation of (1), the notion of counter-utility replaces that of utility and the counter-utility equivalence principle that of the zero utility principle.

The Counter-Utility of Risk

In order to estimate the risk situation of a portfolio, we consider the risk reserve, more exactly the value

$$\bar{R}_t = -R_t$$

which is representative of the danger (a positive danger if the risk reserve is negative and inversely), the counter-utility of this $\bar{R}_t$ is, at time t and for an exponential counter-utility function:

$$\bar{U}(\bar{R}_t) = \int e^{a\bar{r}} \cdot f_t(\bar{r}) \cdot d\bar{r}.$$

For $t = 0$, the risk reserve has a known value; therefore

$$\bar{U}(\bar{R}_0) = e^{a\bar{R}_0} = e^{-a\ R_0}.$$

If the premiums are determined by the zero utility principle, or by the counter-utility equivalence principle, it can be easily shown that the counter-utility of the risk situation is constant in time:

$$\bar{U}(\bar{R}_t) = \bar{U}(\bar{R}_0) \qquad t = 1, 2, 3, \ldots$$

therefore

(6) $$\bar{U}(\bar{R}_t) = e^{-a\ R_0}.$$

The value of Lundberg's upper bound (4) of the ruin probability ψ is thus equal to the counter-utility of the risk situation of the portfolio at the beginning of time, and, because of the constancy of this counter-utility in time, equal to the counter-utility of the risk situation at time t (always under the hypothesis of a counter-utility equivalence between premium and risk).

The Counter-Utility of Ruin

If Z_t represents the ruin amount (first ruin), at time t, it can be shown without difficulty that the counter-utility of the ruin situation for all future years, generalizing (3):

(7) $$\bar{U}(Z) = \sum_{t=1}^{\infty} \int_0^{\infty} e^{ez} \cdot g_t(z) \cdot dz$$

is equal to the value of the counter-utility of the risk situation:

(8) $$\bar{U}(Z) = \bar{U}(\bar{R}_t) = \bar{U}(\bar{R}_0) = e^{-a\ R_0}.$$

Thus the counter-utility $\bar{U}(Z)$ of the ruin situation for the entire future, the counter-utility $\bar{U}(\bar{R}_t)$ of the risk situation at t, notably when $t=0$, and the upper bound of the ruin probability according to Lundberg are identical.

4. INCREASING COUNTER-UTILITY PRINCIPLE

The formulae and properties stated so far are known.

The greater part of the following is new. The model referred to above can be generalized (always under the hypothesis of a stationary process and of an exponential utility function) in view of studying the equilibrium and the ruin conditions in the short and medium term.

The Counter-Utility of Risk

In reality, for a given aversion coefficient, premium P and risk X are not entirely equivalent. The relation (1) opens up three cases

$$e^{aP} \gtreqless M(a)$$

corresponding successively to an over-taxed premium, a premium equivalent in counter-utility and an under-taxed premium. We transform this last relation into an equation by the introduction of a supplementary factor

$$e^{aP} = M(a) \cdot e^{-b}. \tag{9}$$

The coefficient b measures the level of under-taxation of risk X by premium P. The coefficient b is positive in the case of under-taxation, which we will deal with later. Under these conditions, it can be easily shown that the counter-utility of the risk situation is no longer constant in time, but evolves as follows:

$$\bar{U}(\bar{R}_{t+1}) = \bar{U}(\bar{R}_t) \cdot e^{b}. \tag{10}$$

Given the initial value of $\bar{U}(\bar{R}_0)$ according to (8), we have

$$\bar{U}(\bar{R}_t) = e^{-aR_0+bt} \tag{11}$$

which generalises (6).

An under-taxed premium ($b>0$) leads therefore to an increase of the risk counter-utility, an over-taxed premium ($b<0$) to a decrease.

The recurrent relation (10) defines the *increasing counter-utility principle* (or decreasing if $b<0$).

The evolution of a portfolio with a constant counter-utility, seen under point 3 by the application of the zero utility principle, corresponds to the limit case $b=0$ between the two cases $b>0$ and $b<0$.

Formula (10) has an undoubtedly intuitive meaning.

The Counter-Utility of Ruin

In the case of an under-taxed portfolio (related to the counter-utility equivalence principle) it can be shown that if the definition (7) of the ruin counter-utility is generalized by the introduction of the factor e^{-b}, such that

$$\bar{U}(Z) = \sum_{t=1}^{\infty} e^{-b\,t} \cdot \int_0^{\infty} e^{a\,z} \cdot g_t(z) \cdot dz, \tag{12}$$

then the ruin counter-utility keeps its standard value (8)

$$\bar{U}(Z) = e^{-a\,R_0}. \tag{13}$$

In expression (12), the coefficients a and b are bound by relation (9). The introduction of the factor $e^{-b\,t}$ in (12) has the following meaning. The factor e^{-bt} $(b>0)$ reduces the weight of future ruins in $\bar{U}(Z)$: the more distant the ruin the greater the reduction of $\bar{U}(Z)$. This corresponds to a "time stumping" phenomenon. The coefficient b is the time stumping coefficient and e^{-b} the stumping factor.

For an aversion coefficient leading to the equivalence in counter-utility between premium and risk, the stumping coefficient b vanishes and (12) is identical to (7).

The expression $e^{-a\,R_0}$ according to (13) is thus a practical measure of the risk situation of a portfolio: it takes into account by means of the risk coefficient a the size of the ruin amount, and by means of the stumping coefficient b, the imminence of the ruin. The notion of ruin counter-utility (12) can thus be used to measure the financial equilibrium of an insurance portfolio. This notion is more elaborate than that of ruin probability, which only considers the alternative "to be or to be no longer".

5. FINITE TIME HORIZON

A second interpretation of formula (12) leads to an estimation of the risk situation of a portfolio limited to a finite time horizon.

If, in expression (12), we replace the ruin counter-utility at t, that is

$$\int_0^{\infty} e^{az} \cdot g_t(z) \cdot dz$$

by the length of the period (1 year) during which the said ruin might occur, expression (12) becomes

$$\sum_{t=1}^{\infty} e^{-bt} \cdot 1 \qquad (b>0)$$

whose signification is that of the future (up to infinity) subjected to the stumping process mentioned above.

Let us designate by θ this value, which we will call the "time horizon". Because

$$\sum_{t=1}^{\infty} e^{-bt} = \frac{1}{e^{b}-1}$$

we find for the period θ:

$$\theta = \frac{1}{e^b - 1} \tag{14}$$

or inversely

$$e^b = \frac{\theta + 1}{\theta}.$$

The greater the stumping coefficient b the shorter the horizon; this is a natural property of a stumping phenomenon.

If one accepts the notion of "time horizon", then expression (12) measures the ruin counter-utility in a "finite time horizon θ". The interpretation of expression (12) by means of the time horizon allows us to formulate an extension of the Cramér–Lundberg's theory when considering the short and medium term. The formula considers the entire future until infinity, but reduces the "weight" of future events in function of their distance in time, just as the discount phenomenon with regards to payments in a distant future.

6. RUIN AMOUNT AND PORTFOLIO LIFETIME

The method used above to estimate the financial equilibrium of insurance portfolios allows developments in various directions. Here follows what can be deduced from e.g. relations (12) and (13) about the ruin amount and portfolio life-time if ruin occurs.

In expression (12) $g_t(z)$ is the density function of the first ruin amount Z_t at time t. The expression

$$g_t^*(z) = e^{a\,z - b\,t} \cdot g_t(z) \Big/ \sum_{t=1}^{\infty} \int_0^{\infty} e^{a\,z-b\,t} \cdot g_t(z) \cdot dz \tag{15}$$

becomes the conditional density of amount Z_t at t, (under the hypothesis that the ruin occurs) which takes into consideration the size of the ruin (by the factor e^{az}) and the distance in time of the occurrence of the ruin (by the factor $e^{-b\,t}$).

Let us define

$$E^*(Z|T<\infty) = \sum_{t=1}^{\infty} \int_0^{\infty} z \cdot g_t^*(z) \cdot dz \tag{16}$$

and

$$E^*(T|T<\infty) = \sum_{t=1}^{\infty} t \int_0^{\infty} g_t^*(z) \cdot dz \tag{17}$$

as the "mathematical expectations" of, respectively, the first ruin, amount Z and the portfolio life-time T, if ruin occurs, calculated with the modified densities $g_t^*(z)$.

These two mathematical expectations are related!

Indeed, expressions (12) and (13) lead to the equality (18)

$$e^{-a\,R_0} = \sum_{t=1}^{\infty} e^{-b\,t} \int_0^{\infty} e^{a\,z} g_t(z)\, dz. \tag{18}$$

By logarithmic derivation with respect to a of the last equation (we are reminded that b is related to a by (9)), we have, after some elementary algebraic simplifications:

$$-R_0 = -b'(a) \cdot E^*(T|T<\infty) + E^*(Z|T<\infty). \tag{19}$$

By also taking the logarithmic derivative of (9) with respect to a, we find that

$$P = [\ln M(a)]' - b'(a)$$

that is

$$b'(a) = [\ln M(a)]' - P.$$

The first term of the right-hand expression is in fact

$$[\ln M(a)]' = \frac{M'(a)}{M(a)} = \frac{\int x \cdot e^{ax} \cdot f(x) \cdot dx}{\int e^{ax} \cdot f(x) \cdot dx} = P_E \tag{20}$$

which is equal to the Esscher premium corresponding to the aggregate claim amount X. Thus

$$b' = P_E - P.$$

Relation (19) becomes therefore

$$-R_0 = -(P_E - P) \cdot E^*(T|T<\infty) + E^*(Z|T<\infty)$$

or

$$\boxed{R_0 + E^*(Z|T<\infty) = (P_E - P) \cdot E^*(T|T<\infty).} \tag{21}$$

This formula can be interpreted as follows: left-hand expression: $R_0 + E^*(Z|T<\infty)$ is, at the time of ruin, the average total loss of the company; right-hand expression: $(P_E - P)$. $E^*(T|T<\infty)$ is, at the time of ruin, the average deficit in premiums in respect to the level of the Esscher premium and accumulated during the portfolio's life-time. It is to be noted that these are not average values in the usual statistical sense, but averages in the sense of the counter-utility theory, by means of the modified densities $g_t^*(z)$ which take into account the phenomena of risk aversion and time-stumping. That a relation should exist between the company's total loss and the deficit in premium is not unnatural. It is perhaps surprising that this relation is that simple.

In practice it is clear that it is not at all easy to calculate the expectations $E^*(Z|T<\infty)$ and $E^*(T|T<\infty)$. Formula (21) allows at least an estimation of one if there is a hint of the value of the other. It seems that the estimation of $E^*(T|T<\infty)$ is less tricky than that of $E^*(Z|T<\infty)$ to which many authors have applied themselves.

7. A NUMERICAL EXAMPLE

Given a portfolio with the following characteristics:

Risk X (millions of francs)

Claim Amount 1 year x	Prob $(X=x)$	
80	0.1	
90	0.2	
100	0.4	$E(X)=100$
110	0.2	$\text{Var}(X)=120$
120	0.1	

$$M(a)=\tfrac{1}{10}(e^{80a}+2e^{90a}+4e^{100a}+2e^{110a}+e^{120a}).$$

Finance

Risk premium	$P=110$
Initial risk reserve	$R_0=25$

Ruin

In the present example the annual surplus can only take values which are multiples of 10, and the initial risk reserve is 25, so that an eventual ruin amount will always be: $Z=z_0=5$. In order to simplify, we will designate by q_t the probability of the first ruin at T:

$$\int_0^\infty g_t(z)\,dz=q_t. \tag{22}$$

Probability of the First Ruin

A direct calculation, by repeated convolutions, gives the following values for the probabilities of the first ruin for $t=1,2,3,\ldots$.

The long-term ruin probability ψ is

$$\psi=\sum_{t=1}^{\infty}q_t=0.002\,446. \tag{23}$$

TABLE 1

RUIN PROBABILITIES

	Ruin Probabilities		Conditional Ruin probabilities	
t	q_t	Accumulated	$\frac{q_t}{\sum q_t} = q_t^*$	Accumulated
(1)	(2)	(3)	(4)	(5)
1	0.000 000	0 000 000	0	0
2	0.000 000	0 000 000	0	0
3	0 001 000	0 001 000	0.4088	0.4088
4	0 000 600	0 001 600	0 2453	0 6541
5	0 000 360	0 001 960	0 1472	0 8013
6	0.000 206	0 002 166	0.0842	0.8855
7	0 000 118	0 002 284	0 0482	0 9337
8	0 000 068	0.002 352	0 0278	0 9615
9	0 000 039	0.002 391	0 0160	0.9775
10	0.000 023	0 002 414	0 0094	0 9869
11	0 000 013	0 002 427	0 0053	0.9922
12	0 000 008	0.002 435	0 0033	0 9955
13	0.000 005	0 002 440	0 0020	0 9975
14	0 000 003	0 002 443	0 0013	0 9988
15	0.000 002	0 002 445	0.0008	0 9996
16	0 000 001	0 002 446	0.0004	1 0000
17	0	0 002 446	0	1.0000

First Case: Classical Theory, Infinite Time Horizon

Premium $P = 110$ and risk X are equivalent in counter-utility, in the sense of relation (1), for $a = 0.200\,4494$. According to (8), ruin counter-utility $\bar{U}(Z)$, risk counter-utility $\bar{U}(\bar{R}_t)$ and Lundberg's upper bound of the ruin probability are identical

$$\bar{U}(Z) = \bar{U}(\bar{R}_t) = e^{-aR_0} = 0.006\,663. \tag{24}$$

As the ruin amount is constant by nature ($Z = z_0 = 5$), the integral in (7) can be written because of (22)

$$\int_0^\infty e^{az} \cdot g_t(z) \cdot dz = e^{a\,z_0} \int_0^\infty g_t(z) \cdot dz = e^{az_0} \cdot q_t.$$

Expression (7) therefore becomes

$$\bar{U}(Z) = e^{a\,z_0} \cdot \sum_{t=1}^{\infty} q_t$$

from which we can conclude that

$$\sum_{t=1}^{\infty} q_t = \frac{\bar{U}(Z)}{e^{az_0}} = \frac{e^{-a\,R_0}}{e^{a\,z_0}} = \frac{0.006\,663}{2.724\,397} = 0.002\,446.$$

We find the value obtained by direct calculation, according to (23).

Second Case: Extended Theory, Finite Time Horizon

If we fix the horizon θ (Table 2, first column), the columns (2), (3) and (4) give, respectively, the values of the aversion coefficient a, the stumping coefficient b and the stumping factor e^{-b}. We find in column (5) the value of the ruin counter-utility, according to the formulae (12) or (13):

TABLE 2
RUIN COUNTER-UTILITIES

θ (1)	a (2)	b (3)	e^{-b} (4)	$\bar{U}(Z)$ (5)
3	0 239 340	0.287 68	0.750 00	0.002 520
5	0.225 743	0.182 32	0.833 33	0.003 540
10	0 214 008	0 095 31	0.909 09	0.004 747
∞	0 200 449	0 000 00	1 000 00	0 006 663

The above table states that the improvement of the measure $\bar{U}(Z)$ chosen to estimate the financial security of a portfolio is not radical when we bring forward the infinite horizon to a 10-year horizon, for example; the reduction is more appreciable if we switch to a horizon of 5 or 3 years. This is conform to the known property which states that if ruin occurs, it usually occurs in the near future. A comparison between the ruin probabilities accumulated over a period of t years (table 1, column 3) and the ruin counter-utilities in a horizon of θ years (table 2, column 5) gives the following:

TABLE 3
COMPARISON BETWEEN RUIN PROBABILITIES AND RUIN COUNTER-UTILITIES

t Years	Accumulated Ruin Probabilities	θ Years	Ruin Counter-Utility with Horizon θ
3	0 001 000	3	0 002 520
5	0 001 960	5	0 003 540
10	0 002 414	10	0 004 747
∞	0.002 446	∞	0 006 663

It can be stated that, for a common period $t = \theta$, the ratios between the two measures of ruin (probability and counter-utility) are rather stable.

Relation (21) *between Average Ruin Amount and Average Portfolio Life-Time, if Ruin Occurs*

The portfolio under consideration generating constant ruin amounts ($Z = z_0 = 5$), the conditional probabilities $g_t^*(z)$ (in the sense of the counter-utility theory) are reduced, for $b=0$ (that is without stumping phenomenon) to the usual conditional probabilities (Z takes only the value $z_0=5$)

$$g_t^*(z)=q_t\Big/\sum_{t=1}^{\infty} q_t=q_t^*.$$

The calculation of $E(T)$ on the basis of the probabilities in Table 1, column 4, gives us

$$E(T|T<\infty)=4.407.$$

The direct calculation of the Esscher premium according to (20) gives us

$$P_E=116.803.$$

The relation (21)

$$R_0+E^*(Z|T<\infty)=(P_E-P)\cdot E^*(T|T<\infty)$$

is verified, because

$$25+5=(116.803-110)\cdot 4.407.$$

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THE GENERAL ECONOMIC PREMIUM PRINCIPLE*

BY HANS BUHLMANN

Zürich

ABSTRACT

We give an extension of the Economic Premium Principle treated in Astin Bulletin, Volume 11 where only exponential utility functions were admitted. The case of arbitrary risk averse utility functions leads to similar quantitative results. The role of risk aversion in the treatment is essential. It also permits an easy proof for the existence of equilibrium.

KEYWORDS

Mathematical economics, equilibrium theory, premium principles.

1. THE PROBLEM

In BUHLMANN (1980) it was argued that in many real situations premiums are not only depending on the risk to be covered but also on the surrounding market conditions. The standard actuarial techniques are not geared to produce such a dependency and one has to construct a model for the whole market, if one wants to study the interrelationships between market conditions and premiums.

Such models exist in mathematical economics. For the purpose of this paper we borrow the model of mathematical economics for a pure exchange economy and we use the usual Walrasian equilibrium concept.

The more practically oriented reader might consider the model as an idealization of e.g., a reinsurance market where premiums of the contracts are determined by the market. Of course, the Walrasian model is not the only way to describe a reinsurance market. In oligopolistic situations one would rather have to rely on the theoretical framework provided by game theory. On the other hand the model used in this paper extends far beyond reinsurance.

The more theoretically minded reader will note that the model of an exchange economy used in the following has infinitely many commodities. The classical result of existence of equilibrium [see e.g., DEBREU (1959, 1974)] therefore does not hold. The existence proof given here is the theoretically most important aspect of the present paper.

2. THE MODEL FOR THE MARKET

We have agents i, $i = 1, 2, \ldots, n$ (typically reinsurers, insurers, buyers of direct insurance etc.).

* presented at the Meeting on Risk Theory September 1982 in Oberwolfach

The commodities to be traded are quantities of money, conditional on the random outcome ω, where ω stands for an element of a probability space $(\Omega, \mathfrak{A}, \Pi)$.

Let $Y_i(\omega)$ stand for the function as *traded* by agent i assigning to each state ω the payment received by i from the participants in the market. In insurance terminology Y_i describes an insurance policy or a reinsurance contract (Think of the sum of all insurance policies and reinsurance contracts bought and sold by i as if it were exactly one contract).

On the other hand we have conditional payments caused to agent i *from outside the market*. These payments—conditional on ω—are described by $X_i(\omega)$. In insurance terms X_i represents the risk of the agent i before (re-)insurance.

Using the terminology of BUHLMANN (1980) we call X_i the original risk of agent i, Y_i the exchange function (or exchange variable) of agent i. In addition we characterize each agent by his utility function $u_i(x)$ [as usual $u_i'(x)>0$, $u_i''(x)\leq 0$] and his initial wealth W_i.

Whereas the original risk X_i belongs to agent i from the start we imagine that Y_i can be freely bought by him at a price which is given by

$$\text{(1)} \qquad \text{Price}\,[Y_i] = \int_\Omega Y_i(\omega)\phi(\omega)\,d\Pi(\omega).$$

The function $\phi:\Omega\to\mathbb{R}$ appearing in (1) is called the *price density* The random vector $(Y_1, Y_2, \ldots, Y_n)$ representing the exchange variables bought by all agents will be denoted by $\boldsymbol{Y}$ in the sequel.

3. EQUILIBRIUM

DEFINITION. $(\tilde{\phi}, \tilde{\boldsymbol{Y}})$ is called an equilibrium if

(a) for all i: $E[u_i(W_i - X_i + \tilde{Y}_i - \int \tilde{Y}_i(\omega')\tilde{\phi}(\omega')\,d\Pi(\omega'))] = \max$ for all possible choices of the exchange variable Y_i.

(b) $\sum_{i=1}^n \tilde{Y}_i(\omega) = 0$ for all $\omega \in \Omega$.

TERMINOLOGY. If conditions (a) and (b) are satisfied we call $\tilde{\phi}$ equilibrium price density, $\tilde{\boldsymbol{Y}}$ equilibrium risk exchange.

Hint. It might be worthwhile to look up in BUHLMANN (1980) the definition in the special case of a finite probability space. The special case coincides with the standard equilibrium definitions in mathematical economics.

In BUHLMANN (1980) it was shown that for exponential utility functions $u_i(x) = 1 - e^{-\alpha_i x}$ the equilibrium price density has the following form

$$\text{(2)} \qquad \tilde{\phi}(\omega) = \frac{e^{\alpha Z(\omega)}}{E[e^{\alpha Z}]} \quad \text{where } \frac{1}{\alpha} = \sum_{i=1}^n \frac{1}{\alpha_i}$$

where Z has the precise meaning

$$\text{(3)} \qquad Z(\omega) = \sum_{i=1}^n X_i(\omega).$$

In this paper we show that equation (3) defines the "market conditions" also in the case of arbitrary utility functions. We shall see that locally (but not globally) even (2) carries over to the case of arbitrary utility functions.

REMARK. In the case of an arbitrary probability space existence of an equilibrium as defined is usually not discussed in the economic literature. Exceptions are BEWLEY (1972) and TOUSSAINT (1981) who treat the problem of existence for economies with infinitely many commodities by imposing some topological structure on the space of random variables Y_i. In this paper we shall prove that equilibrium exists making only risk theoretical assumptions. This is, however, postponed to section 8. Up to this section we therefore *assume* existence of an equilibrium.

4. PRICE EQUILIBRIUM AND PARETO OPTIMUM

It is shown in BUHLMANN (1980) that condition (a) is equivalent to condition

$$\text{(c) for all } i\text{:}\quad u_i'[W_i - X_i(\omega) + \tilde{Y}_i(\omega) - \int \tilde{Y}_i(\omega')\tilde{\phi}(\omega')\, d\Pi(\omega')]$$

$$= \tilde{\phi}(\omega) \underbrace{\int u_i'[W_i - X_i(\omega) + \tilde{Y}_i(\omega) - \int \tilde{Y}_i(\omega')\tilde{\phi}(\omega')\, d\Pi(\omega')]\, d\Pi(\omega)}_{C_i}$$

for almost all ω.

COROLLARY. *From* (c) *we see that* $\int \tilde{\phi}(\omega)\, d\Pi(\omega) = 1$.

As $\tilde{Y}_i$ is only determined up to an additive constant there is no loss of generality in assuming

$$\text{(d)}\qquad \int \tilde{Y}_i(\omega')\tilde{\phi}(\omega')\, d\Pi(\omega') = 0 \quad \text{for all } i.$$

For convenience we write $X_i - Y_i = Z_i$ (and quite naturally $X_i - \tilde{Y}_i = \tilde{Z}_i$) and use either the Y-variables or the Z-variables to describe the exchange. In the Z-language conditions (c) and (d) yield

$$\text{(4)}\qquad \text{for all } i\text{:}\quad u_i'[W_i - \tilde{Z}_i(\omega)] = C_i\tilde{\phi}(\omega) \qquad (C_i > 0)$$

which—according to Borch's theorem [see BORCH (1960)]—shows that an equilibrium risk exchange (conditions (b), (c), (d)) is automatically a Pareto optimum (condition (b)) plus (4)).

Conversely if we start with a Pareto optimum (condition (b) plus (4) because of Borch's theorem) all we need to render $(\tilde{\phi}, \tilde{\mathbf{Y}})$ an equilibrium is a change of the initial wealth W_i by the "free amounts" $A_i = E[\tilde{\phi}\tilde{Y}_i]$ where $\tilde{Y}_i = X_i - \tilde{Z}_i$. (Observe that $\sum_{i=1}^{n} A_i = 0$).

Before we continue our analysis it is important to note that the random variables $\tilde{Z}_i$ $(i=1,2,\ldots,n)$ and $\tilde{\phi}$ can be and very often must be chosen to *depend on ω only through* $Z(\omega)=\sum_{i=1}^{n} X_i(\omega)$. This result by BORCH (1962) can also be obtained from the following argument: Assume a Pareto optimal risk exchange $\tilde{\boldsymbol{Z}}$ with

(I) $$E[u_i(W_i-\tilde{Z}_i)]$$

and

(II) $$\sum_{i=1}^{n} \tilde{Z}_i(\omega)=\sum_{i=1}^{n} X_i(\omega)=Z(\omega) \quad \text{for all } \omega.$$

Define $\tilde{\tilde{Z}}_i=E[\tilde{Z}_i|Z]$ for each i.

$\tilde{\tilde{\boldsymbol{Z}}}$ is again a balancing risk exchange (i.e., satisfies (II)). From Jensen's inequality for the conditional expectation given Z we conclude that $\tilde{\tilde{\boldsymbol{Z}}}$ is at least as good as $\tilde{\boldsymbol{Z}}$ for all i. Namely

$$E[u_i(W_i-\tilde{Z}_i)|Z]\leqslant E[u_i(W_i-\tilde{\tilde{Z}}_i)|Z] \quad \text{for all } i$$

and hence

(I) $$E[u_i(W_i-\tilde{Z}_i)]\leqslant E[u_i(W_i-\tilde{\tilde{Z}}_i)] \quad \text{for all } i.$$

The inequality is strict unless either $\tilde{Z}_i=\tilde{\tilde{Z}}_i$ and/or $u_i(x)$ is linear on the probabilistic support of $\tilde{Z}_i$. Excluding linearity of u_i for all but one agent, $\tilde{Z}_i$ *must* depend on ω through Z for all i. In the case of linearity of u_i for several agents there is indifference of splitting the risk among them. Also in this case we *may* therefore assume that $\tilde{Z}_i$ depends on ω through Z for all i.

Finally if $\tilde{Z}_i$ is a function of Z for all i so must be $\tilde{\phi}$ as seen from (4).

Because of this we use also the notation $\tilde{Z}_i(\zeta)$, $\tilde{\phi}(\zeta)$, where ζ is the generic element of the probability space obtained by the mapping $Z:\Omega\to\mathbb{R}$.

5. RISK AVERSION

We rewrite (4) as

(5) $$\text{for all } i: \quad u_i'(W_i-\tilde{Z}_i(\zeta))=C_i\tilde{\phi}(\zeta) \quad \text{with} \quad \sum_{i=1}^{n} Z_i(\zeta)=\zeta.$$

Taking the logarithmic derivative on both sides we obtain

$$-\frac{u_i''(W_i-\tilde{Z}_i(\zeta))}{u_i'(W_i-\tilde{Z}_i(\zeta))}\tilde{Z}_i'(\zeta)=\frac{\tilde{\phi}'(\zeta)}{\tilde{\phi}(\zeta)}.$$

We introduce the *individual risk aversion* $\rho_i(x)=u_i''(x)/u_i'(x)$ and obtain

(6) $$\rho_i(W_i-\tilde{Z}_i(\zeta))\tilde{Z}_i'(\zeta)=\frac{\tilde{\phi}'(\zeta)}{\tilde{\phi}(\zeta)}$$

and because $\sum_{i=1}^{n} \tilde{Z}_i'(\zeta) = 1$ also

$$1 = \frac{\tilde{\phi}'(\zeta)}{\tilde{\phi}(\zeta)} \sum_i \frac{1}{\rho_i(W_i - \tilde{Z}_i(\zeta))}. \tag{7}$$

The sum on the right-hand side adds up the *individual risk tolerance units* and hence can be understood as the *total risk tolerance unit.* We express this by the abbreviated notation

$$\sum_i \frac{1}{\rho_i(W_i - \tilde{Z}_i(\zeta))} = \frac{1}{\rho(\zeta)}. \tag{8}$$

This notation suggests to call $\rho(\zeta)$ the *total risk aversion*. Observe, however, that this concept does not only depend on ζ but also on the functions $\tilde{Z}_i(\zeta)$ representing a particular fixed Pareto optimal splitting of the total risk.

With this understanding we also obtain from (6) and (7)

$$\tilde{Z}_i'(\zeta) = \frac{\rho(\zeta)}{\rho_i(W_i - \tilde{Z}_i(\zeta))} = \frac{1/[\rho_i(W_i - \tilde{Z}_i(\zeta))]}{1/\rho(\zeta)} \quad \text{Quotient of risk tolerance units} \tag{9}$$

This formula—as far as the author believes—not appearing elsewhere in the literature, is quite remarkable in two respects.

(a) Borch's condition (our (5) above) characterizes the Pareto optima by a system of differential equations with $n-1$ free parameters. In (9) these parameters have disappeared and we have a *unique system* of differential equations.

This means that one can now characterize the set of all Pareto optimal exchanges by the initial values $\tilde{Z}_i(0)$.

(b) The notion of risk aversion has been derived for the study of one single agent and the relationship between his certainty-equivalent and the risk variance [Pratt (1964)]. The appearance in the characterization of Pareto optimal risk exchanges is a surprise and gives the risk aversion a new additional meaning.

6. A NEW INTERPRETATION OF PARETO OPTIMAL RISK EXCHANGES

As just indicated, formula (9) allows us to characterize the set of all Pareto optima from their initial values. This shall now be done explicitly. Before we start we might, however, ask how these initial values $\tilde{Z}_i(0)$ should be interpreted.

Using the definitions as introduced in section 4

$$\tilde{Z}_i(0) = (X_i - \tilde{Y}_i)(0)$$

we see that $\tilde{Z}_i(0)$ stands for the total balance of payments to be made by i in the case when the total claims to the market $Z = \sum_{i=1}^{n} X_i$ are zero. This justifies the following

TERMINOLOGY. $T_i = -\tilde{Z}_i(0)$ is called *initial receipt* by agent i (before any positive or negative claims come in).

Any Pareto optimal risk exchange $\tilde{\boldsymbol{Z}} = (\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_n)$ can then be described as follows:

(A) Define arbitrary initial receipts T_i ($\sum_{i=1}^n T_i = 0$). (This is equivalent to choosing the constants C_i in equation (4)).

(B) Solve the system of differential equations (9) ($i = 1, 2, \ldots, n$) with initial conditions $\tilde{Z}_i(0) = -T_i$ ($i = 1, 2, \ldots, n$).

This mathematical characterization allows the following interpretation: After having distributed the initial receipts, the increases (decreases) $d\zeta$ of total risk ζ are split in the proportion of the risk tolerance units

$$(10) \qquad d\tilde{Z}_i(\zeta) = \frac{1/[\rho_i(W_i - \tilde{Z}_i(\zeta))]}{1/\rho(\zeta)} d\zeta.$$

It is clear how (10) would immediately allow for a numerical integration of the system of differential equations (9). In order to avoid any technical difficulties with the system of differential equations we make the hypothesis (from here on)

(H) The risk aversions $\rho_i(x)$ are positive continuous functions on $\mathbb{R}$, satisfying a Lipschitz condition $|\rho_i(x) - \rho_i(x')| \leq K|x - x'|$

Under (H) we have existence and uniqueness of the solution to (9) for arbitrary initial receipts $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$.

REMARKS

(1) The new interpretation of Pareto Optimum can also be used in the case of risk exchanges $\boldsymbol{Y}$ restricted by some bounds. In this case, however, not all the agents would always participate in the splitting of all increases (decreases) $d\zeta$.

(2) From our interpretation (10) it is clear that hypothesis (H) could be weakened to allow at most one function $\rho_i(x)$ to be zero for any specific argument x. We renounce this refinement.

7. THE GENERAL ECONOMIC PREMIUM PRINCIPLE DEPENDING ON THE INITIAL TRANSFER PAYMENTS

We start with an equilibrium $(\tilde{\boldsymbol{Z}}, \tilde{\phi})$ (remember $\tilde{Z}_i = X_i - \tilde{Y}_i$). As $\tilde{\boldsymbol{Z}}$ is Pareto optimal it can be constructed according to the description in section 6. The choice of the initial receipts T_i must be left open at the moment.

However, the equilibrium price density $\tilde{\phi}$ like the "after exchange" functions $\tilde{Z}_i$ ($i = 1, 2, \ldots, n$) can be determined from the basic equations in section 5 for any particular choice of $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$.

Combining (6) and (9) we obtain

$$(11) \qquad \rho_T(\zeta) = \frac{\tilde{\phi}'_T(\zeta)}{\tilde{\phi}_T(\zeta)}$$

with

$$\frac{1}{\rho_T(\zeta)} = \sum_{i=1}^{n} \frac{1}{\rho_i(W_i - \tilde{Z}_i^T(\zeta))}. \tag{12}$$

Observe that from here on $\tilde{Z}_i^T(\zeta)$ (for all i) stand for those *unique* Pareto optimal risk exchange functions with $\tilde{Z}_i^T(0) = -T_i$.

From (11) and the norming condition $\int \tilde{\phi}(\omega)\, d\Pi(\omega) = 1$ we obtain

$$\tilde{\phi}_T(Z(\omega)) = \frac{\exp \int_0^{Z(\omega)} \rho_T(\zeta)\, d\zeta}{E[\exp \int_0^{Z(\omega)} \rho_T(\zeta)\, d\zeta]}. \tag{13}$$

We easily recognize (13) as the global generalization of (2). The local behaviour, described by (11), is even the same as for exponential utilities. The basic difference is, of course, that in general the total risk aversion is not constant but depends on the total risk ζ and the way this total risk is split up among the agents.

For the practically minded reader we might add that the price density $\tilde{\phi}_T$ can be understood as a distortion of the actuarially correct probabilities. Formula (13) explains how this distortion comes about.

8. EXISTENCE OF EQUILIBRIUM

We have now—in a very natural way—come back to the question of existence of equilibrium. With the tools at our disposal we can now pose it as follows:

Are there initial receipts $\tilde{T} = (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n)$ such that

$$E[\phi_{\tilde{T}} \tilde{y}_i^T] = E[\phi_{\tilde{T}}(X_i - \tilde{Z}_i^T)] = 0 \quad \text{for all } i = 1, 2, \ldots, n? \tag{14}$$

Observe that for arbitrary initial receipts the resulting $(\tilde{\phi}_T, Y^T)$ satisfies (4) and (b). In order to be an equilibrium it must also satisfy (d) (which is the same as (14)). We could also say, in the spirit of section 3, that in equilibrium no change of initial wealth distribution by free amounts is needed.

THEOREM. *Under* (H) *and for bounded* X_i, $i = 1, 2, \ldots, n$ $\tilde{T}$ *exists.*

PROOF.

(i) Consider the mapping $\mathbb{R}^n \to \mathbb{R}^n$ which sends $T = (T_1, T_2, \ldots, T_n)$ into $S = (S_1, S_2, \ldots, S_n)$ by the rule

$$E[\tilde{\phi}_T(\tilde{Z}_i^T + T_i - X_i)] = S_i.$$

(ii) Observe that $(\tilde{Z}_i^T + T_i)(0) = 0$ by definition. In view of (10) and hypothesis H we must have for all i

$$(Z_i^T + T_i) \leq \zeta \quad \text{for } \zeta \geq 0$$
$$(Z_i^T + T_i) \geq \zeta \quad \text{for } \zeta \leq 0$$

which can be written as

$$|Z_i^T + T_i| \leq \left| \sum_{i=1}^{n} X_i \right| \leq nM \qquad (|X_i| \leq M \text{ for all } i, \text{ by assumption})$$

hence

$$|S_i| = E[\tilde{\phi}_T \cdot (\tilde{Z}_i^T + T_i - X_i)] \leq (n+1)M \quad \text{for arbitrary } \boldsymbol{T}.$$

(iii) Consider now the compact rectangle $|T_i| \leq (n+1)M$ for all i. Call it R.
Consider the hyperplane $\sum_{i=1}^{n} T_i = 0$. Call it E.
The intersection $R \cap E$ is non empty, compact and convex.
(iv) The mapping $\boldsymbol{T} \mapsto \boldsymbol{S}$ defined in (i) maps $R \cap E$ into $R \cap E$.

Check

$$\sum_{i=1}^{n} S_i = E\left[\tilde{\phi}_T \underbrace{\left(\sum_{i=1}^{n} Z_i^T + \sum_{i=1}^{n} T_i - \sum_{i=1}^{n} X_i \right)}_{0}\right] = 0$$

From (H) and boundedness of all X_i it follows by a standard theorem on differential equations that the solutions $\tilde{Z}_i^T$ $(i = 1, 2, \ldots, n)$ depend continuously on the initial conditions $\boldsymbol{T}$. Therefore the mapping $\boldsymbol{T} \mapsto \boldsymbol{S}$ is also continuous.

Applying BROUWER's Fixed-Point Theorem we have existence of $\tilde{\boldsymbol{T}}$ with

$$E[\tilde{\phi}_T \cdot (\tilde{Z}_i^{\tilde{T}} + \tilde{T}_i - X_i)] = \tilde{T}_i \quad \text{for all } i$$

and consequently

$$E[\tilde{\phi}_T(\tilde{Z}_i^{\tilde{T}} - X_i)] = 0 \quad \text{for all } i \qquad \text{q.e.d.}$$

REMARK. Boundedness of X_i is a rather strong technical assumption which one might want to weaken. The general idea would be to approximate arbitrary random variables X_i by truncation and to perform a limit argument. For the correctness of this limit argument, however, one needs again some technical assumptions.

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ON A CLASS OF SEMI-MARKOV RISK MODELS OBTAINED AS CLASSICAL RISK MODELS IN A MARKOVIAN ENVIRONMENT

JEAN-MARIE REINHARD
Groupe AG, Brussels, Belgium

ABSTRACT

We consider a risk model in which the claim inter-arrivals and amounts depend on a markovian environment process. Semi-Markov risk models are so introduced in a quite natural way. We derive some quantities of interest for the risk process and obtain a necessary and sufficient condition for the fairness of the risk (positive asymptotic non-ruin probabilities). These probabilities are explicitly calculated in a particular case (two possible states for the environment, exponential claim amounts distributions).

KEYWORDS
Semi-Markov processes, ruin theory.

1. INTRODUCTION

Several authors have used the semi-Markov processes in Queuing Theory and in Risk Theory [e.g., CINLAR (1967), NEUTS (1966), NEUTS and SHUN-ZER CHEN (1972), PURDUE (1974), JANSSEN (1980), REINHARD (1981)]. Besides, some duality results lead to nice connections between the two theories [FELLER (1971), JANSSEN and REINHARD (1982)].

Semi-Markov risk models may be defined as follows. Consider a risk model in continuous time; let B_n $(n \in N_0)$* and U_n $(n \in N_0)$ denote respectively the amount and the arrival time of the nth claim. Put $A_0 = B_0 = U_0 = 0$ and define $A_n = U_n - U_{n-1}$ $(n \in N_0)$. We suppose that the A_n and B_n are random variables defined on a complete probability space $(\Omega, \mathscr{A}, P)$; the variables A_n $(n \in N_0)$ are a.s. positive. Let now J_n $(n \in N)$ be random variables defined on $(\Omega, \mathscr{A}, P)$ and taking their values in $J = \{1, \ldots, m\}$ $(m \in N_0)$. Suppose finally that $\{(J_n, A_n, B_n); n \in N\}$ is a Markov chain with transition probabilities defined by a bivariate semi-Markov kernel:

$$P[J_{n+1} = j, A_{n+1} \leqslant t, B_{n+1} \leqslant x \mid J_k, A_k, B_k; k = 0, \ldots, n] = Q_{J_n j}(x, t) \quad \text{a.s.} \tag{1.1}$$
$$(j \in J, \quad t \geqslant 0, \quad x \in R, \quad n \in N)$$

where $Q_{ij}(x, \cdot)$ and $Q_{ij}(\cdot, t)$ are right continuous nondecreasing functions satisfying:

$$Q_{ij}(x, t) \geqslant 0, \quad Q_{ij}(\infty, 0) = 0 \qquad (i, j \in J; t \geqslant 0)$$
$$\sum_{j=1}^{m} Q_{ij}(\infty, \infty) = 1 \qquad (i \in J)$$
$$Q_{ij}(-\infty, \infty) = 0 \qquad (i, j \in J).$$

* $N_0 = \{1, 2, 3, \ldots\}$, $N = \{0, 1, 2, 3, \ldots\}$

Such processes, called (J–Y–X) processes, were studied by JANSSEN and REINHARD (1982) and REINHARD (1982). In the particular case where

$$Q_{ij}(x,t)=(1-e^{-\lambda t})Q_{ij}(x), \qquad \lambda>0, \tag{1.2}$$

the processes $\{A_n\}$ and $\{(J_n, B_n)\}$ being independent, JANSSEN (1980) interpreted the variables J_n as the types of the successive claims. The next section will show that another subclass of semi-Markov kernels appears if we assume that the risk depends on an environment process.

2. RISK PROCESSES IN A MARKOVIAN ENVIRONMENT

Suppose that the claim frequency and amounts depend on the external environment (economic situation . . .) and that the external environment may be characterized at any time by one of the m states $1,\ldots,m$ ($m\in N_0$). Let I_0 denote the state of the environment at time $t=0$ and let I_n, $n=1,\ldots$, be the state of the environment after its nth transition. Put $T_0=0$ and let T_n ($n\in N_0$) be the time at which occurs the nth transition of the environment process. We suppose that I_n and T_n ($n\in N$) are random variables defined on $(\Omega,\mathcal{A},P)$ and taking their values in J and R^+ respectively. Define now $Y_n=T_n-T_{n-1}$ ($n\in N_0$), $Y_0=0$ and assume that

$$P[I_{n+1}=j,\ Y_{n+1}\le t\,|\,(I_k,Y_k),k=0,\ldots,n,I_n=i]=h_{ij}(1-e^{-\lambda_i t}) \tag{2.1}$$

$$(i,j\in J;\quad t\ge 0;\quad n\in N)$$

where the λ_i are strictly positive real numbers and $H=(h_{ij})$ is a transition matrix:

$$h_{ij}\ge 0, \qquad \sum_{k=1}^{m} h_{ik}=1 \qquad (i,j\in J).$$

$\{I_n, n\in N\}$ is then a Markov chain with a matrix of transition probabilities $H=(h_{ij})$:

$$h_{ij}=P[I_{n+1}=j\,|\,I_n=i]. \tag{2.2}$$

Define $N_e(t)=\sup\{n: T_n\le t\}$ and $I(t)=I_{N_e(t)}$ ($t\ge 0$). The process $\{I(t), t\ge 0\}$ is a finite-state Markov process; it is known that the number of transitions of the environment process $\{I(t)\}$ in any finite interval $(s,t]$, i.e., $N_e(t)-N_e(s)$, is a.s. finite.

Denote now by J_n the state of the environment process at the arrival of the nth claim:

$$J_n=I(U_n) \qquad (n\in N). \tag{2.3}$$

We will suppose that the following assumptions are satisfied:

(H1) The sequences of random variables (A_n) and (B_n) are conditionally independent given the variables J_n.

(H2) The distribution of a claim depends uniquely on the state of the environment at the time of arrival of that claim. Let

$$F_i(x)=P[B_n\le x\,|\,J_n=i] \qquad (i\in J,\quad n\in N,\quad x\in R) \tag{2.4}$$

(H3) Let $N(t)$ be the number of claims occurring in $(0, t]$. If $I(u)=i$ for all u in some interval $(t, t+h]$, then the number of claims occurring in that interval, i.e., $N(t+h)-N(t)$, has a Poisson distribution with parameter α_i ($\alpha_i>0$); we assume further that given the process $\{I(t)\}$ the process $\{N(t)\}$ has independent increments. So

(2.5) $P[N(t+h)=n+1 \mid N(t)=n, I(u)=i \text{ for } t<u\leqslant t+h]=\alpha_i h+o(h).$

The process $\{N(t); t\geqslant 0\}$ appears thus as a Poisson process with parameter modified by the transitions of the environment process.

Under the above assumptions it may be shown that $\{(J_n, A_n, B_n), n\in N\}$ is a $(J\text{-}Y\text{-}X)$ process with semi-Markov kernel $\mathcal{Q}$ defined by (1.1). $\{(J_n, A_n), n\in N\}$ is a Markov renewal process [see PYKE (1961)]; we denote its kernel by $\mathcal{V}=(V_{ij}(\cdot))$:

$$V_{ij}(t)=P[J_{n+1}=j, A_n\leqslant t \mid (J_k, A_k), k=0,\ldots,n; J_n=i] \tag{2.6}$$

$$(i,j\in J,\quad n\in N,\quad t\geqslant 0).$$

Moreover it follows from the assumptions that

$$Q_{ij}(x,t)=V_{ij}(t)F_j(x) \qquad (i,j\in J,\quad t\geqslant 0,\quad x\in R). \tag{2.7}$$

$\{J_n, n\in N\}$ is a Markov chain with matrix P of transition probabilities defined by

$$P_{ij}=P[J_{n+1}=j \mid J_n=i]=Q_{ij}(\infty,\infty)=V_{ij}(\infty) \qquad (i,j\in J). \tag{2.8}$$

In the next section it will be shown how the semi-Markov kernel $\mathcal{Q}$ (or equivalently $\mathcal{V}$) can be deduced from the instantaneous rates α_i, the transition matrix H, the constants λ_i and the distributions $F_i(\cdot)$.

3. COMPUTATION OF THE KERNEL

Let us first introduce some notations: for any mass function (i.e., right continuous and non-decreasing) $G(t)$ defined on R^+ let

$$\tilde{G}(s)=\int_0^\infty e^{-st}G(t)\,dt, \qquad g(s)=\int_{0-}^\infty e^{-st}\,dG(t)$$

provided the above integrals converge.

The following system of integral equations may be easily deduced from the hypothesis

$$V_{ij}(t)=\delta_{ij}\frac{\alpha_i}{\alpha_i+\lambda_i}(1-e^{-(\alpha_i+\lambda_i)t})+\lambda_i\sum_{k=1}^m h_{ik}\int_0^t e^{-(\alpha_i+\lambda_i)u}V_{kj}(t-u)\,du \tag{3.1}$$

$$(i,j\in J,\quad t\geqslant 0).$$

The first term in the right side of (3.1) corresponds to the case where a claim occurs before the environment changes, the second term to the case where the environment changes before a claim occurs.

For $s\geqslant 0$, define now the following matrices:

$$L(s)=(h_{ij}\lambda_i/(\alpha_i+s+\lambda_i)), \qquad E(s)=(\delta_{ij}\alpha_i/(\alpha_i+s+\lambda_i)).$$

By taking the Laplace transforms of both sides in (3.1) we obtain

$$\tilde{V}_{ij}(s)=\delta_{ij}\frac{\alpha_i}{s(\alpha_i+\lambda_i+s)}+\frac{\lambda_i}{\alpha_i+\lambda_i+s}\sum_{k=1}^{m} h_{ik}\tilde{V}_{kj}(s) \tag{3.2}$$

$$(i,j\in J;\quad s>0),$$

or, in matrix notation,

$$[I-L(s)]\tilde{V}(s)=(1/s)E(s)\qquad (s>0) \tag{3.3}$$

(we will always use the same symbol for a matrix and its elements whenever this causes no ambiguity). As for any $s\geq 0$

$$L_i(s)=\sum_{j=1}^{m} L_{ij}(s)=\frac{\lambda_i}{\alpha_i+\lambda_i+s}<1,$$

$I-L(s)$ is regular for $s\geq 0$ and consequently (3.3) has as unique solution

$$\tilde{V}(s)=(1/s)[I-L(s)]^{-1}E(s)\qquad (s>0), \tag{3.4}$$

or equivalently

$$v(s)=[I-L(s)]^{-1}E(s)\qquad (s>0). \tag{3.5}$$

As $p_{ij}=V_{ij}(\infty)=\lim_{s\searrow 0} v_{ij}(s)$, the matrix P of the transition probabilities of the chain $\{J_n\}$ can be directly deduced from (3.5):

$$P=[I-L(0)]^{-1}E(0). \tag{3.6}$$

Notice that the semi-Markov kernel $\mathcal{V}$ is solution of a first order linear differential system: by deriving (3.1) with respect to t we obtain

$$V'_{ij}(t)=\alpha_i\delta_{ij}+\sum_{k=1}^{m}[\lambda_i h_{ik}-(\alpha_i+\lambda_i)\delta_{ik}]V_{kj}(t)\qquad (i,j\in J;\quad t\geq 0). \tag{3.7}$$

4. SOME RESULTS ABOUT QUANTITIES RELATED TO THE RISK PROCESS

In this section we derive some explicit expressions or equations related to the semi-Markov risk-process defined in the preceding sections.

4.1. *Stationary Probabilities of the Chain* $\{J_n\}$

From now on we suppose that the chain $\{J_n\}$ is irreducible. As m is finite there exists a unique probability distribution $\bar{\eta}=(\eta_1,\ldots,\eta_m)$ such that

$$\eta_i>0\qquad (i\in J), \tag{4.1}$$

$$\sum_{i=1}^{m}\eta_i h_{ij}=\eta_j\qquad (j\in J).$$

We have then:

THEOREM 1

The Markov chain $\{J_n; n \in N\}$ is irreducible and aperiodic (thus ergodic as $m < \infty$). Its stationary probabilities are given by

$$\pi_i = \frac{\alpha_i \eta_i}{\lambda_i} \left\{ \sum_{j=1}^{m} \frac{\alpha_j \eta_j}{\lambda_j} \right\}^{-1} \qquad (i \in J). \tag{4.2}$$

Proof

Let $i, j \in J$. As the chain $\{I_n\}$ is irreducible, there exists $n \in N$ such that $h_{ij}^{(n)} > 0$. It may be easily seen that this implies $(L^n(0))_{ij} > 0$. Now we obtain from (3.6):

$$p_{ij} = \sum_{n=0}^{\infty} (L^n(0))_{ij} \frac{\alpha_j}{\alpha_j + \lambda_j}. \tag{4.3}$$

The probabilities p_{ij} are thus strictly positive for all $i, j \in J$.

It remains to show that $\bar{\pi} P = \bar{\pi}$. Define the diagonal matrices

$$D = \left\langle \delta_{ij} \frac{\lambda_i}{\alpha_i + \lambda_i} \right\rangle, \qquad A = \left(\delta_{ij} \frac{\alpha_i}{\lambda_i} \right). \tag{4.4}$$

We have then $L(0) = DH$, $E(0) = I - D$, $\bar{\pi} = K\bar{\eta}A$ (where K is the norming factor in the right side of (4.2)), $AD = I - D$; (3.6) may be written as follows:

$$P = I - D + DHP. \tag{4.5}$$

Now

$$\bar{\pi}P = \bar{\pi} - \bar{\pi}D + \bar{\pi}DHP = \bar{\pi} - K[\bar{\eta}(I - D) - \bar{\eta}(I - D)HP].$$

As $\bar{\eta}H = \bar{\eta}$, we obtain

$$\bar{\pi}P = \bar{\pi} - K\bar{\eta}[(I - D) - (I - DH)P] = \bar{\pi}, \tag{4.6}$$

the last equality resulting from (4.5).

Note that (4.2) has an immediate intuitive interpretation: η_i is the asymptotic probability of finding the chain $\{I_n; n \in N\}$ in state i; $(\lambda_i)^{-1}$ is the mean time spent by the process $\{I(t); t \geq 0\}$ in state i before its next transition; α_i is the mean number of claims occurring per time unit when the process $\{I(t); t \geq 0\}$ sojourns in state i; π_i appears thus well as the asymptotic average number of claims occurring in environment i.

4.2. *Number of Claims Occurring in* $(0, t)$

The equations obtained here could be derived from the general theory of semi-Markov processes. It is, however, interesting to restate them directly as the semi-Markov kernel $\mathscr{V}$ is itself expressed as the solution of the differential system (3.7)

Define

$$N_j(t) = \begin{cases} \sum_{k=1}^{N(t)} 1_{[J_k = j]} & \text{if } N(t) > 0, \\ 0 & \text{if } N(t) = 0, \end{cases} \tag{4.7}$$

where as previously $N(t)$ is the number of claims occurring in $(0, t)$. $N_j(t)$ is clearly the number of claims occurring in environment j before t. Let

$$M_{ij}(t) = E[N_j(t) \mid J_0 = i]$$

and

$$M_i(t) = E[N(t) \mid J_0 = i] = \sum_{j=1}^{m} M_{ij}(t) \qquad (t \geq 0).$$

The following system of integral equations is easily obtained:

$$M_{ij}(t) = \delta_{ij} e^{-\lambda_i t} \alpha_i t + \int_0^t \lambda_i e^{-\lambda_i u} \Big[\delta_{ij} \alpha_i u + \sum_k h_{ik} M_{kj}(t-u) \Big] du$$

or

$$M_{ij}(t) = \delta_{ij} \alpha_i \frac{1 - e^{-\lambda_i t}}{\lambda_i} + \sum_{k=1}^{m} \lambda_i h_{ik} \int_0^t e^{-\lambda_i u} M_{kj}(t-u)\, du \qquad (t \geq 0). \tag{4.8}$$

Taking the derivatives of both sides with respect to t we obtain

$$M'_{ij}(t) = \alpha_i \delta_{ij} - \lambda_i M_{ij}(t) + \lambda_i \sum_{k=1}^{m} h_{ik} M_{kj}(t) \qquad (t \geq 0), \tag{4.9}$$

and after summation over j

$$M'_i(t) = \alpha_i - \lambda_i M_i(t) + \lambda_i \sum_{k=1}^{m} h_{ik} M_k(t) \qquad (t \geq 0). \tag{4.10}$$

(4.9) with the boundary condition $M_{ij}(0) = 0$ $(i, j \in J)$ has a unique solution.

4.3. *Further Properties of the Claim Arrival Process*

We extend first to the $(J\text{–}Y\text{–}X)$ processes a well known property of Markov chains and $(J\text{–}X)$ processes.

THEOREM 2

Let $\{(J_n, A_n, B_n);\ n \in N\}$ be a $(J\text{–}Y\text{–}X)$ process with state space $J \times R^+ \times R$ and kernel $\mathscr{Q}$ defined by (1.1). Suppose that the Markov chain $\{J_n\}$ is irreducible (and thus positive recurrent as m is finite). Let $Z_{ij}(x, t)$, $i, j \in J$, be real measurable functions defined on $R \times R^+$ such that the integrals

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} |Z_{ij}(x, t)| Q_{ij}(dx, dt) \qquad (i, j \in J)$$

are finite. Let

$$z_i = \sum_{j=1}^{m} \int_{-\infty}^{\infty}\int_{0}^{\infty} Z_{ij}(x, t) Q_{ij}(dx, dt) = E(Z_{J_{n-1}J_n}(B_n, A_n) \mid J_{n-1} = i).$$

Define then $n_{i,0} = 0$, $n_{i,k} = \inf\{n > n_{i,k-1}: J_n = i\}$ for $k \in N_0$ (recurrence indices of state i) and let

$$\zeta_{i,r} = E\left(\sum_{k=n_{i,r}+1}^{n_{i,r+1}} Z_{J_{k-1}J_k}(B_k, A_k)\right) \qquad (i \in J, \quad r \in N).$$

The random variables $\zeta_{i,r}$, $r = 1, 2, \ldots$, are i.i.d. and we have

$$E(\zeta_{i,r}) = \frac{1}{\pi_i} \sum_{j=1}^{m} \pi_j z_j \qquad (i \in J, \quad r \in N_0) \tag{4.11}$$

where the π_i are the stationary probabilities of the chain $\{J_n\}$.

Proof
Define

$${}_ip_{ij}^{(n)} = P[J_n = j, J_k \neq i \text{ for } k = 1, \ldots, n-1 \mid J_0 = i] \qquad (i, j \in J; \quad n \in N_0).$$

We have then

$$E(\zeta_{i,r}) = \sum_{k \neq i} \sum_{n=1}^{\infty} {}_ip_{ik}^{(n)} z_k + z_i \qquad (i \in J, \quad r \in N_0).$$

(4.11) follows since we know from Markov chain theory that $\sum_{n=1}^{\infty} {}_ip_{ik}^{(n)} = \pi_k/\pi_i$.

Mean Recurrence Time of Claims Occurring in a Given Environment

We return now to the risk model. Define

$$G_{ij}(t) = P[N_j(t) > 0 \mid J_0 = i] \qquad (i, j \in J; \quad t \geq 0). \tag{4.12}$$

$G_{ij}(\cdot)$ is the distribution function of the first time at which a claim occurs in environment j given that the initial environment is i. Let

$$\gamma_{ij} = \int_{0-}^{\infty} t\, dG_{ij}(t) \qquad (i, j \in J). \tag{4.13}$$

We could obtain a system of integral equations for the distributions $G_{ij}(\cdot)$ and derive from it after passage to the Laplace–Stieltjes transforms a linear system for the γ_{ij}. We may, however, proceed more directly as follows:

$$(4.14)\qquad \gamma_{ij}=\sigma_{ij}\int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i t+\lambda_i\sum_{k=1}^m h_{ik}(t+\gamma_{kj})\right]dt$$

$$+(1-\delta_{ij})\int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i(t+\gamma_{ij})+\lambda_i\sum_{k=1}^m h_{ik}(t+\gamma_{kj})\right]dt;$$

we thus get a linear system:

$$(4.15)\qquad \frac{\lambda_i+\delta_{ij}\alpha_i}{\alpha_i+\lambda_i}\gamma_{ij}=\frac{1}{\alpha_i+\lambda_i}+\frac{\lambda_i}{\alpha_i+\lambda_i}\sum_{k=1}^m h_{ij}\gamma_{kj}\qquad (i,j\in J).$$

The diagonal elements γ_{ii} (mean recurrence time of claims occurring in state i) may be explicitly expressed by using Theorem 2. Define $Z_{ij}(x,t)=t$; then $z_i=E(A_1|J_0=i)$. We have

$$z_i=\int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i t+\lambda_i\sum_{j=1}^m h_{ij}(t+z_j)\right]dt\qquad (i\in J).$$

Hence

$$z_i=\frac{1}{\alpha_i+\lambda_i}+\frac{\lambda_i}{\alpha_i+\lambda_i}\sum_{j=1}^m h_{ij}z_j\qquad (i\in J),$$

or, if $\bar z=(z_1,\dots,z_m)^t$ and $\bar y=(\alpha_1^{-1},\dots,\alpha_m^{-1})^t$,

$$\bar z=(I-L(0))^{-1}E(0)\bar y=P\bar y;$$

we have thus

$$(4.16)\qquad z_i=E(A_1|J_0=i)=\sum_{j=1}^m p_{ij}\frac{1}{\alpha_j}\qquad (i\in J)$$

and consequently

$$(4.17)\qquad \sum_{i=1}^m \pi_i z_i=E_\pi(A_1)=\sum_{j=1}^m \pi_j\frac{1}{\alpha_j}.$$

Using finally theorem 2 we have:

THEOREM 3

For any $i\in J$:

$$(4.18)\qquad \gamma_{ii}=\frac{1}{\pi_i}\sum_{j=1}^m \pi_j\frac{1}{\alpha_j}.$$

Renewal Theorem—Stationary Probabilities

Given that $J_0=i$, the times at which claims occur in environment j form a pure renewal process if $i=j$ and a delayed renewal process if $i\neq j$. We have the classical renewal equations:

$$M_{ij}(t)=\int_0^t [1+M_{jj}(t-u)]\,d\dot{G}_{ij}(u) \qquad (i,j\in J;\quad t\geq 0). \tag{4.19}$$

As the distribution functions $G_{jj}(\cdot)$ are clearly not arithmetic, the expected number of claims occurring in environment j within $(t, t+h)$ tends to $h(\gamma_{jj})^{-1}$ when $t\to\infty$ whatever the initial environment i, i.e.,

$$\lim_{t\to\infty}[M_{ij}(t+h)-M_{ij}(t)]=\frac{h}{\gamma_{jj}} \qquad (i,j\in J;\quad h\geq 0). \tag{4.20}$$

[see FELLER (1971), Chapt. XI]. From (4.20) it follows that

$$\lim_{t\to\infty}\frac{M_{ij}(t)}{t}=\frac{1}{\gamma_{jj}} \qquad (i,j\in J). \tag{4.21}$$

Define now

$$F_{ij}(t)=(p_{ij})^{-1}V_{ij}(t) \tag{4.22}$$

$$R_{jk}^{(i)}(u,t)=P[J_{N(t)}=j, J_{N(t)+1}=k,\ U_{N(t)+1}\leq t+u\,|\,J_0=i];$$

the last quantity is thus the probability, given that $J_0=i$, that the last claim before t occurred in environment j and that the next claim will occur in environment k before time $t+u$. We deduce immediately from Theorem 7.1 of PYKE (1961b) that

$$\lim_{t\to\infty}R_{jk}^{(i)}(u,t)=p_{jk}\frac{1}{\gamma_{jj}}\int_0^u [1-F_{jk}(y)]\,dy, \tag{4.23}$$

which limit is independent of i; we denote it by $R_{jk}^0(u)$. Let now

$$V_{ij}^*(u)=\gamma_{ii}z_i^{-1}R_{ij}^0(u)$$

and define a chain $\{(\bar{J}_n, \bar{A}_n, \bar{B}_n);\ n\in N\}$ as follows:

$$\left\{\begin{array}{l}\bar{A}_0=\bar{B}_0=0 \quad \text{a.s.}\\ P[\bar{J}_1=j, \bar{A}_1\leq u, \bar{B}_1\leq x\,|\,\bar{A}_0, \bar{B}_0; \bar{J}_0=i]=V_{ij}^*(u)F_j(x)\\ P[\bar{J}_n=j, \bar{A}_n\leq u, \bar{B}_n\leq x\,|\,\bar{A}_k, \bar{B}_k, \bar{J}_k(k=0,\ldots,n-1); \bar{J}_{n-1}\\ \qquad =i]=V_{ij}(u)F_j(x)\end{array}\right. \tag{4.24}$$

$$(i,j\in J;\quad u\in R^+,\quad x\in R,\quad n\geq 2).$$

where z_i is defined by (4.16).

We define for that chain the same quantities and adopt the same notations as for the chain $\{(J_n, A_n, B_n);\ n\in N\}$. The risk processes associated with the two chains are identical except that for the second one the time of occurrence of the first claim is distributed according to the semi-Markov kernel $(V_{ij}^*(\cdot))$ instead of $(V_{ij}(\cdot))$. Suppose now that

$$a_i=P[\bar{J}_0=i]=\frac{z_i}{\gamma_{ii}} \qquad (i\in J). \tag{4.25}$$

Then [see PYKE (1961b)]:

(4.26) $$P[\bar{J}_{\bar{N}(t)}=j, \bar{J}_{\bar{N}(t)+1}=k, \bar{U}_{\bar{N}(t)+1}\leq t+u]=R^0_{jk}(u).$$

5. PREMIUM INCOME—RUIN PROBABILITIES

We assume that the company managing the risk receives premiums at a constant rate $c_i>0$ during any time interval the environment process remains in state i. The premium income process is thus characterized by a vector $(c_1,\ldots,c_m)$ with positive entries. Denote by $A^c(t)$ the aggregate premium received during $(0,t)$:

(5.1) $$A^c(t)=\sum_{k=1}^{N_e(t)} c_{I_{k-1}}(T_k-T_{k-1})+c_{I_{N_e(t)}}(t-T_{N_e(t)})$$

and by $B(t)$ the aggregate amount of the claims occurring in $(0,t)$:

(5.2) $$B(t)=\sum_{k=0}^{N(t)} B_k \qquad (t\geq 0).$$

Assume now that the initial amount of free assets of the company is $u\geq 0$. The amount of free assets at time t is then

(5.3) $$Z_u(t)=u+S(t)$$

where

(5.4) $$S(t)=A^c(t)-B(t).$$

Define then

(5.5) $$R_i(u,t)=P[Z_u(v)\geq 0 \text{ for } 0\leq v\leq t\,|\,J_0=i] \qquad (i\in J;\quad u,t\geq 0),$$

(5.6) $$R_i(u)=R_i(u,\infty)=P[Z_u(v)\geq 0 \text{ for all } v\geq 0\,|\,J_0=i] \qquad (i\in J,\quad u\geq 0).$$

We will refer to the probabilities (5.5) as to the finite time non-ruin probabilities and to the probabilities (5.6) as to the asymptotic non-ruin probabilities.

5.1. *Random Walk of the Free Assets*

Denote by A^c_n the premium received between the occurrences of the $(n-1)$th and nth claims $(n\geq 1)$. Define then

(5.7) $$X_k=A^c_k-B_k \qquad (k=1,2,\ldots);\quad X_0=0 \quad \text{a.s.},$$

(5.8) $$S_n=\sum_{k=0}^{n} X_k \qquad (n\in N).$$

Clearly the chain $\{(J_k,X_k); k\in N\}$ is a $(J\text{–}X)$ process, $\{S_n\}$ is a random walk defined on the finite Markov chain $\{J_n\}$ [see JANSSEN (1970); MILLER (1962); NEWBOULD (1973)]. The amount of free assets just after the occurrence of the

nth claim is given by

$$Z_u(A_0 + \cdots + A_n) = u + S_n$$

and clearly

(5.9) $$R_i(u) = P\left[\inf_k S_k \geq -u \mid J_0 = i\right].$$

From now on we assume that the d.f. $F_i(\cdot)$ has a finite expectation μ_i $(i \in J)$. We get then

(5.10) $$b_i = E[B_k \mid J_{k-1} = i] = \sum_{j=1}^{m} p_{ij}\mu_j$$

and

$$z_i^c = E[A_k^c \mid J_{k-1} = i] = \int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i c_i t + \lambda_i \sum_{j=1}^{m} h_{ij}(c_i t + z_j^c)\right] dt$$

so that, concluding as to obtain (4.16),

(5.11) $$z_i^c = \sum_{j=1}^{m} p_{ij}\frac{c_j}{\alpha_j} \qquad (i \in J).$$

If the premium rates are constant whatever the state of the environment, i.e., if $\bar{c} = (c, \ldots, c)$, we obtain naturally $z_i^c = cz_i$. We conclude from (5.10) and (5.11) that

(5.12) $$\zeta_i = E[X_k \mid J_{k-1} = i] = \sum_{j=1}^{m} p_{ij}\left(\frac{c_j}{\alpha_j} - \mu_j\right).$$

Notice that we would obtain the same result for a semi-Markov risk model with kernel $\mathcal{Q}^*$ defined by

(5.13) $$Q_{ij}^*(x, t) = p_{ij}(1 - e^{-\alpha_j t})F_j(x).$$

Define now

$$D_{i,r} = \sum_{k=n_{i,r}+1}^{n_{i,k+1}} X_k \qquad (i \in J, \quad r \in N_0)$$

where the $n_{i,r}$ are the recurrence indices of claims occurring in environment i as defined in section 4.3; for i fixed the variables $D_{i,r}$ $(r = 1, 2, \ldots)$ are i.i.d.; $D_{i,r}$ is clearly the variation of the free assets between the rth and $(r+1)$th claims occurring in environment i. We obtain from theorem 2

(5.14) $$E(D_{i,r}) = \frac{1}{\pi_i}\sum_{j=1}^{m} \pi_j\left(\frac{c_j}{\alpha_j} - \mu_j\right) \qquad (i \in J, \quad r \in N_0).$$

As the variables A_k^c are absolutely continuous and conditionally (given the J_k) independent of the variables B_k, the process $\{(J_n, S_n); n \in N\}$ is not degenerate [see NEWBOULD (1973)], i.e., there exist no constants $w_1, \ldots, w_m$ such that $P[X_n = w_j - w_i | J_{n-1} = i, J_n = j] = 1$, or equivalently there exists no i such that $D_{i,r} = 0$ a.s. (NEWBOULD (1973), lemma 2). Using Proposition 3A of JANSSEN (1970) we obtain then

THEOREM 4

Let

$$d = \sum_{j=1}^{m} \pi_j \left(\frac{c_j}{\alpha_j} - \mu_j \right). \tag{5.15}$$

Then (i) If $d > 0$, the random walk $\{S_n\}$ drifts to $+\infty$, i.e. $\lim_{n\to\infty} S_n = \infty$ a.s.; $R_i(u) > 0$, $\forall u \geqslant 0$, $i \in J$. (ii) If $d < 0$, the random walk $\{S_n\}$ drifts to $-\infty$, i.e. $\lim_{n\to\infty} S_n = -\infty$ a.s.: $R_i(u) = 0$, $\forall u \geqslant 0$, $i \in J$. (iii) If $d = 0$, the random walk $\{S_n\}$ is oscillating, i.e. $\limsup S_n = +\infty$ a.s. and $\liminf S_n = -\infty$ a.s.; $R_i(u) = 0$, $\forall u \geqslant 0$, $i \in J$.

Notice that when $m = 1$ theorem 4 reduces evidently to the classical result for the Poisson model.

5.2. *Distribution of the Aggregate Net Pay-out in* $(0, t)$

From now on we suppose that the claim amounts are a.s. positive:

$$F_i(0-) = 0, \qquad F_i(0) < 1 \qquad \forall i \in J. \tag{5.16}$$

Recall that $A^c(t)$ and $B(t)$ denote respectively the aggregate premium received and the aggregate amount of claims occurred during $(0, t)$. Then denote by $C(t)$ the net pay-out of the company in $(0, t)$:

$$C(t) = B(t) - A^c(t) = -S(t) \qquad (t \geqslant 0)$$

Let then

$$W_{ij}(x, t) = P[C(t) \leqslant x, I(t) = j | I(0) = i] \qquad (i, j \in J; \quad t \geqslant 0). \tag{5.17}$$

Define now

$$c_0 = \max\{c_i; i \in J\}, \qquad J_0 = \{i \in J : c_i = c_0\}.$$

It is easy to prove the following

LEMMA

(i) $W_{ij}(x, t) = 0$ for $i, j \in J$ and $x < -c_0 t$;
(ii) $W_{ij}(x, t) > 0$ for $i, j \in J$ and $x > -c_0 t$;
(iii) $W_{ij}(-c_0 t, t) > 0$ if $i, j \in J_0$ and either $i = j$ or there exist $r \in N_0$ and $i_1, \ldots, i_r \in J_0$ such that $h_{ii_1} h_{i_1 i_2} \ldots h_{i_r j} > 0$; $W_{ij}(-c_0 t, t) = 0$ otherwise.

Let now

$$\tilde{W}_{ij}(s,t)=\int_{-c_0t}^{\infty} e^{-sx}W_{ij}(x,t)\,dx;\qquad \tilde{W}(s,t)=(\tilde{W}_{ij}(s,t))\qquad (s>0),$$

$$w_{ij}(s,t)=\int_{-c_0t-}^{\infty} e^{-sx}\,d_xW_{ij}(x,t)=s\tilde{W}_{ij}(s,t);\qquad w(s,t)=(w_{ij}(s,t))\qquad (s>0),$$

$$\varphi_i(s)=\int_{0-}^{\infty} e^{-sx}\,dF_i(x)\qquad (s\geqslant 0).$$

The following theorem gives an explicit expression for the transform matrix $\tilde{W}(s,t)$.

THEOREM 5

For $s>0$ and $t\geqslant 0$,

$$\tilde{W}(s,t)=1/s\exp\{-T(s)t\}\tag{5.18}$$

where

$$T_{ij}(s)=\delta_{ij}(\alpha_i+\lambda_i-\alpha_i\varphi_i(s)-c_is)-\lambda_ih_{ij}.\tag{5.19}$$

Proof

For $x\geqslant -c_0t$, $t\geqslant 0$ and $h>0$ we obtain easily

$$\begin{aligned}W_{ij}(x,t+h)&=(1-(\alpha_i+\lambda_i)h)W_{ij}(x+c_ih,t)\\&\quad+\alpha_ih\int_{0-}^{x+c_ih+c_0t}W_{ij}(x+c_ih-y,t)\,dF_i(y)\\&\quad+\lambda_ih\sum_{k=1}^{m}h_{ik}W_{kj}(x+c_ih,t)+o(h).\end{aligned}\tag{5.20}$$

Dividing (5.20) by h and letting h tend to 0, we get

$$\begin{aligned}\frac{\partial}{\partial t}W_{ij}(x,t)-c_i\frac{\partial}{\partial x}W_{ij}(x,t)&=-(\alpha_i+\lambda_i)W_{ij}(x,t)\\&\quad+\alpha_i\int_{0-}^{x+c_0t}W_{ij}(x-y,t)\,dF_i(y)\\&\quad+\lambda_i\sum_{k=1}^{m}h_{ik}W_{kj}(x,t)\\&\qquad(x\geqslant -c_0t,\ t\geqslant 0).\end{aligned}\tag{5.21}$$

We multiply now each term in (5.21) by e^{-sx} and integrate from $-c_0t$ to ∞. We obtain so

$$\frac{\partial}{\partial t}\tilde{W}_{ij}(s,t)+\sum_{k=1}^{m}[\delta_{ik}(\alpha_i+\lambda_i-\alpha_i\varphi_i(s)-c_is)-\lambda_ih_{ik}]\tilde{W}_{kj}(s,t) = (c_0-c_i)\,e^{sc_0t}W_{ij}(-c_0t,t) \qquad (s>0,\,t\geqslant 0). \tag{5.22}$$

According to the above lemma the right side of (5.22) is always zero. In matrix notation, the solution of (5.22) is then easily seen to be

$$\tilde{W}(s,t)=\exp\{-T(s)t\}K \tag{5.23}$$

where

$$K=\tilde{W}(s,0)=(1/s)w(s,0)=(1/s)I \qquad (s>0).$$

The proof is complete.

Notice that when $m=1$ (5.18) reduces to the known result for the classical Poisson model.

5.3. *Seal's Integral Equation for the Finite Time non-ruin Probabilities*

We show in this subsection that the SEAL's integral equation (1974) may be extended to the here considered semi-Markov model. We still assume that the claim amounts are a.s. positive.

Define for $u,t\geqslant 0$ and $i,j\in J$

$$R_{ij}(u,t)=P[Z_u(v)\geqslant 0 \text{ for } 0\leqslant v\leqslant t,\, I(t)=j\,|\,I(0)=i]; \tag{5.24}$$

we have clearly

$$R_i(u,t)=\sum_{j=1}^{m}R_{ij}(u,t) \qquad (i\in J;\quad u,t\geqslant 0).$$

Define further for $s>0$ and $t\geqslant 0$

$$\tilde{R}_{ij}(s,t)=\int_0^\infty e^{-su}R_{ij}(u,t)\,du; \qquad \tilde{R}(s,t)=(\tilde{R}_{ij}(s,t)),$$

$$r_{ij}(s,u)=\int_{0-}^\infty e^{-su}\,d_uR_{ij}(u,t)=s\tilde{R}_{ij}(s,t); \qquad r(s,t)=(r_{ij}(s,t)).$$

We obtain easily for $u,t\geqslant 0$ and $h>0$

$$\begin{aligned}R_{ij}(u,t+h)=[1-(\alpha_i+\lambda_i)h]R_{ij}(u+c_ih,t)\\ +\alpha_ih\int_{0-}^{u+c_ih}R_{ij}(u+c_ih-y,t)\,dF_i(y)\\ +\lambda_ih\sum_{k=1}^{m}h_{ik}R_{kj}(u+c_ih,t)+o(h).\end{aligned} \tag{5.25}$$

Dividing (5.25) by h and letting h tend to 0, we find

$$(5.26)\qquad \frac{\partial}{\partial t}R_{ij}(u,t)-c_i\frac{\partial}{\partial u}R_{ij}(u,t)=-(\alpha_i+\lambda_i)R_{ij}(u,t)$$
$$+\alpha_i\int_0^u R_{ij}(u-y,t)\,dF_i(y)$$
$$+\lambda_i\sum_{k=1}^m h_{ik}R_{kj}(u,t)\qquad (u,t\geqslant 0).$$

Taking the Laplace transform of each term ir (5.26), we obtain

$$(5.27)\qquad \frac{\partial}{\partial t}\tilde{R}_{ij}(s,t)+\sum_{k=1}^m[\delta_{ik}(\alpha_i+\lambda_i-c_is-\alpha_i\varphi_i(s))-\lambda_ih_{ik}]\tilde{R}_{kj}(s,t)$$
$$+c_iR_{ij}(0,t)=0\qquad (s>0,\quad t\geqslant 0).$$

The solution of the differential system (5.27) is easily seen to be

$$(5.28)\qquad \tilde{R}(s,t)=\exp\{-T(s)t\}K-\int_0^t\exp\{-T(s)(t-u)\}CR(0,u)\,du$$
$$(s>0,\quad t\geqslant 0)$$

where $C=(\delta_{ij}c_i)$; the constant matrix K is determined by the boundary condition $r(s,0)=s\tilde{R}(s,0)=sI$. Thus $K=s^{-1}I$. Using finally (5.18), (5.28) may be written as follows

$$(5.29)\ \tilde{R}_{ij}(s,t)=\tilde{W}_{ij}(s,t)-s\sum_{k=1}^m\int_0^t\tilde{W}_{ik}(s,t-u)c_kR_{kj}(0,u)\,du\qquad (s>0,\quad t\geqslant 0).$$

Suppose now that the distributions $F_i(\cdot)$ are absolutely continuous and denote their densities by $f_i(\cdot)$. The mass functions $W_{ij}(\cdot,t)$ are then absolutely continuous too; we denote their densities by $W'_{ij}(\cdot,t)$ $(t\geqslant 0)$. Taking the inverse Laplace transforms in (5.29) we obtain then

$$(5.30)\quad R_{ij}(x,t)=W_{ij}(x,t)-\sum_{k=1}^m c_k\int_0^t W'_{ik}(x,u)R_{kj}(0,t-u)\,du\qquad (x,t\geqslant 0).$$

The unknown constants (with respect to x) $R_{kj}(0,u)$ are solutions of the Volterra type integral system obtained by putting $x=0$ in (5.30):

$$(5.31)\ R_{ij}(0,t)=W_{ij}(0,t)-\sum_{k=1}^m c_k\int_0^t W'_{ik}(0,u)R_{kj}(0,t-u)\,du\qquad (t\geqslant 0).$$

Define now

$$S_{ij}(x,t)=P[B(t)\leqslant x,I(t)=j\,|\,I(0)=i]\qquad (x,t\geqslant 0)$$

and denote the corresponding densities by $S'_{ij}(x,t)$. In the particular case where

$c_i = c$ ($i \in J$) we have clearly $W_{ij}(x, t) = S_{ij}(x + ct, t)$; (5.30) and (5.31) become then

$$(5.32)\quad R_{ij}(x, t) = S_{ij}(x + ct, t) - c \sum_{k=1}^{m} \int_0^t S'_{ik}(x + cu, u) R_{kj}(0, t-u)\, du \qquad (x, t \geq 0),$$

$$(5.33)\quad R_{ij}(0, t) = S_{ij}(ct, t) - c \sum_{k=1}^{m} \int_0^t S'_{ik}(cu, u) R_{kj}(0, t-u)\, du \qquad (t \geq 0).$$

When $m = 1$ (5.32) and (5.33) reduce exactly to Seal's system.

5.4. *Asymptotic Non-ruin Probabilities*

We suppose here that the number d defined by (5.15) is strictly positive; then for all $i \in J$ and $u \geq 0$, $R_i(u) > 0$ and $R_i(\cdot)$ is a probability distribution. After summation over j (5.26) gives for $t = \infty$:

$$(5.34)\quad c_i R'_i(u) = (\alpha_i + \lambda_i) R_i(u) - \alpha_i \int_{0-}^{u} R_i(u-y)\, dF_i(y) - \lambda_i \sum_{k=1}^{m} h_{ik} R_k(u)$$
$$(i \in J; \quad u \geq 0).$$

It can be shown that (5.34) has a unique solution such that $R_i(\infty) = 1$, $\forall i \in J$. Integrating (5.34) from 0 to t we get

$$(5.35)\qquad c_i R_i(t) = c_i R_i(0) + \alpha_i \int_0^t R_i(t-y)[1 - F_i(y)]\, dy$$
$$+ \lambda_i \int_0^t \left[R_i(u) - \sum_{k=1}^{m} h_{ik} R_k(u) \right] du \qquad (i \in J, \quad t \geq 0).$$

For $m = 1$ (5.35) is the well known defective renewal equation from which the famous Cramer estimate may be derived (see FELLER, Chapter XI). For $m > 1$, (5.35) is unfortunately not more a renewal type equation. Letting t tend to ∞ in (5.35) does not give an explicit value for the probabilities $R_i(0)$ as is the case when $m = 1$:

$$(5.36)\qquad R_i(0) = 1 - \frac{\alpha_i \mu_i}{c_i} - \frac{\lambda_i}{c_i} \int_0^\infty \left[R_i(u) - \sum_{k=1}^{m} h_{ik} R_k(u) \right] du.$$

However, when the claim amounts distributions are exponential,

$$F_i(x) = 1 - e^{-x/\mu_i} \qquad (x \geq 0),$$

a further differentiation of both sides of (5.34) shows that the asymptotic non-ruin probabilities are solution of the differential system

$$(5.37)\qquad R''_i(u) = \left(\frac{\alpha_i + \lambda_i}{c_i} - \frac{1}{\mu_i} \right) R'_i(u) - \frac{\lambda_i}{c_i} \sum_{j=1}^{m} h_{ij} R'_j(u) + \frac{\lambda_i}{c_i \mu_i} R_i(u)$$
$$- \frac{\lambda_i}{c_i \mu_i} \sum_{j=1}^{m} h_{ij} R_j(u) \qquad (i \in J, \quad u \geq 0)$$

with the boundary conditions

$$R_i(\infty)=1; \qquad R_i'(0)=\frac{\alpha_i+\lambda_i}{c_i}R_i(0)-\frac{\lambda_i}{c_i}\sum_{j=1}^{m} h_{ij}R_j(0) \qquad (i\in J). \tag{5.38}$$

6. EXAMPLE

Assume that

$$m=2, \qquad h_{12}=h_{21}=1, \qquad h_{11}=h_{22}=0; \tag{6.1}$$

there are thus two possible states for the environment, the sojourn times in each state being exponentially distributed.

The solution of system (3.7) is then

$$\begin{cases} V_{11}(t)=-\dfrac{\alpha_1(\alpha_1+\lambda_2+r_1)}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\alpha_1(\alpha_2+\lambda_2+r_2)}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{12}(t)=-\dfrac{\lambda_1\alpha_2}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\lambda_1\alpha_2}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{22}(t)=-\dfrac{\alpha_2(\alpha_1+\lambda_1+r_1)}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\alpha_2(\alpha_1+\lambda_1+r_2)}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{21}(t)=-\dfrac{\lambda_2\alpha_1}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\lambda_2\alpha_1}{r_2(r_1-r_2)}(1-e^{r_2t}) \qquad (t\geqslant 0), \end{cases} \tag{6.2}$$

where r_1 and r_2 are the solutions (always distinct and negative as α_i, $\lambda_i>0$) of

$$(\alpha_1+\lambda_1+r)(\alpha_2+\lambda_2+r)=\lambda_1\lambda_2. \tag{6.3}$$

The stationary probabilities for the chain $\{J_n\}$ are given by (4.2) which becomes here

$$\pi_1=\frac{\alpha_1\lambda_2}{\alpha_1\lambda_2+\alpha_2\lambda_1}, \qquad \pi_2=\frac{\alpha_2\lambda_1}{\alpha_1\lambda_2+\alpha_2\lambda_1}. \tag{6.4}$$

Expectations of the number of claims occurring in environment i $(i=1,2)$ before t are obtained by solving system (4.9) with the boundary conditions $M_{ij}(0)=0$:

$$\begin{aligned} M_{11}(t)&=\frac{\alpha_1\lambda_2}{\lambda_1+\lambda_2}t+\frac{\alpha_1\lambda_1}{(\lambda_1+\lambda_2)^2}(1-e^{-(\lambda_1+\lambda_2)t}), \\ M_{12}(t)&=\frac{\alpha_2\lambda_1}{\lambda_1+\lambda_2}t-\frac{\alpha_2\lambda_1}{(\lambda_1+\lambda_2)^2}(1-e^{-(\lambda_1+\lambda_2)t}). \end{aligned} \tag{6.5}$$

$M_{22}(t)$ and $M_{21}(t)$ are obtained by replacing in the expressions of $M_{11}(t)$ and $M_{12}(t)$ respectively $\alpha_{1(2)}$ by $\alpha_{2(1)}$ and $\lambda_{1(2)}$ by $\lambda_{2(1)}$.

The mean recurrence time of claims occurring in environment i ($i=1,2$) is given by (4.18).

$$\gamma_{11}=\frac{\lambda_1+\lambda_2}{\alpha_1\lambda_2}, \qquad \gamma_{22}=\frac{\lambda_1+\lambda_2}{\alpha_2\lambda_1}; \tag{6.6}$$

We obtain then from (4.15)

$$\gamma_{12}=\frac{\alpha_2+\lambda_1+\lambda_2}{\alpha_2\lambda_1}, \qquad \gamma_{21}=\frac{\alpha_1+\lambda_1+\lambda_2}{\alpha_1\lambda_2}. \tag{6.7}$$

The characteristic number d defined by (5.15) takes the following form:

$$d=\frac{\lambda_2(c_1-\alpha_1\mu_1)+\lambda_1(c_2-\alpha_2\mu_2)}{\alpha_1\lambda_2+\alpha_2\lambda_1}. \tag{6.8}$$

From now on we assume that $d>0$ and that the claim amount distributions $F_i(\cdot)$ are exponential, i.e.,

$$F_i(x)=1-e^{-x/\mu_i} \qquad (x\geqslant 0;\quad i=1,2). \tag{6.9}$$

From (5.37) and (5.38) we obtain that the asymptotic non-ruin probabilities are solution of the following differential system

$$\begin{cases} c_1R_1''(u)=\left(\alpha_1+\lambda_1-\dfrac{c_1}{\mu_1}\right)R_1'(u)+\dfrac{\lambda_1}{\mu_1}R_1(u)-\dfrac{\lambda_1}{\mu_1}R_2(u)-\lambda_1R_2'(u) \\ c_2R_2''(u)=\left(\alpha_2+\lambda_2-\dfrac{c_2}{\mu_2}\right)R_2'(u)+\dfrac{\lambda_2}{\mu_2}R_2(u)-\dfrac{\lambda_2}{\mu_2}R_1(u)-\lambda_2R_1'(u) \end{cases} \tag{6.10}$$

$$(u\geqslant 0)$$

with the boundary conditions

$$\begin{cases} R_1(\infty)=R_2(\infty)=1 \\ c_1R_1'(0)-(\alpha_1+\lambda_1)R_1(0)+\lambda_1R_2(0)=c_2R_2'(0) \\ \qquad -(\alpha_2+\lambda_2)R_2(0)+\lambda_2R_1(0)=0. \end{cases} \tag{6.11}$$

Define

$$\rho_i=\frac{1}{\mu_i}-\frac{\alpha_i}{c_i} \qquad (i=1,2) \tag{6.12}$$

and assume without restriction that $\rho_1\geqslant\rho_2$.
The condition $d>0$ is then equivalent to the following

$$\frac{\lambda_2}{c_2\mu_2}\rho_1+\frac{\lambda_1}{c_1\mu_1}\rho_2>0. \tag{6.13}$$

As $\rho_1 \geqslant \rho_2$, then ρ_1 is clearly strictly positive. We obtain then that the general solution of (6.10) takes the form

$$\text{(6.14)} \qquad \begin{cases} R_1(u) = A_0 + A_1 e^{k_1 u} + A_2 e^{k_2 u} + A_3 e^{k_3 u}, \\ R_2(u) = A_0 - D(k_1) A_1 e^{k_1 u} - D(k_2) A_2 e^{k_2 u} \\ \qquad - D(k_3) A_3 e^{k_3 u}, \end{cases}$$

where

$$\text{(6.15)} \qquad D(k_i) = \frac{c_1\mu_1 k_i^2 + (c_1 - \alpha_1\mu_1 - \lambda_1\mu_1)k_i - \lambda_1}{\lambda_1\mu_1 k_i + \lambda_1}$$
$$= \frac{\lambda_2\mu_2 k_i + \lambda_2}{c_2\mu_2 k_i^2 + (c_2 - \alpha_2\mu_2 - \lambda_2\mu_2)k_i - \lambda_2},$$

and where k_1, k_2, k_3 are the roots of the characteristic equation

$$\text{(6.16)} \qquad P(k) = k^3 + \left(\rho_1 + \rho_2 - \frac{\lambda_1}{c_1} - \frac{\lambda_2}{c_2}\right)k^2$$
$$+ \left[\left(\rho_1 - \frac{\lambda_1}{c_1}\right)\left(\rho_2 - \frac{\lambda_2}{c_2}\right) - \frac{\lambda_2}{c_2\mu_2} - \frac{\lambda_1}{c_1\mu_1} - \frac{\lambda_1\lambda_2}{c_1c_2}\right]k$$
$$- \left(\frac{\lambda_2}{c_2\mu_2}\rho_1 + \frac{\lambda_1}{c_1\mu_1}\rho_2\right) = 0.$$

From (6.13) we see that $k_1k_2k_3 > 0$. It is easily verified that

$$P(-\rho_1) = \frac{\alpha_1\lambda_1}{c_1^2}(\rho_1 - \rho_2) \geqslant 0; \qquad P(-\rho_2) = \frac{\alpha_2\lambda_2}{c_2^2}(\rho_2 - \rho_1) \leqslant 0;$$
$$P(0) < 0.$$

From this we may deduce that $P(k)$ has a negative root, say k_2, between $-\rho_1$ and $-\rho_2$. As the product of the three roots is positive we deduce further that the two other roots, k_1 and k_3, are real (if k_1 and k_3 were complex conjugate roots, their product would be positive; we would then have $k_1k_2k_3 < 0$). As $P(+\infty) = +\infty$ and $P(-\infty) = -\infty$, we conclude finally that when $\rho_1 > \rho_2$ one of the roots, say k_1, is strictly less than $-\rho_1$ and that the other, k_3, is positive. When $\rho_1 = \rho_2 = \rho$ (we have then $k_2 = -\rho$), we obtain the same conclusions by verifying that $P'(-\rho) < 0$. We summarize this as follows:

$$\text{(6.17)} \qquad \begin{array}{lll} k_1 < -\rho_1 < k_2 < \min\{0, -\rho_2\}, & k_3 > 0 & \text{if } \rho_1 > \rho_2, \\ k_1 < k_2 = -\rho < 0 < k_3 & & \text{if } \rho_1 = \rho_2 = \rho. \end{array}$$

From the boundary conditions (6.11) we obtain that

$$\text{(6.18)} \qquad A_0 = 1, \qquad A_3 = 0$$

and that A_1 and A_2 are the solutions of

$$[c_1k_1-\alpha_1-\lambda_1-\lambda_1D(k_1)]A_1+[c_1k_2-\alpha_1-\lambda_1-\lambda_1D(k_2)]A_2=\alpha_1$$

$$[(-c_2k_1+\alpha_2+\lambda_2)D(k_1)+\lambda_2]A_1+[(-c_2k_2+\alpha_2+\lambda_2)D(k_2)+\lambda_2]A_2=\alpha_2$$

or, which is equivalent in view of (6.15),

$$\left\{\begin{aligned}&\frac{A_1}{\mu_1k_1+1}+\frac{A_2}{\mu_1k_2+1}=-1\\&\frac{D(k_1)}{\mu_2k_1+1}A_1+\frac{D(k_2)}{\mu_2k_2+1}A_2=1.\end{aligned}\right.\tag{6.19}$$

We can obtain a lower bound for k_1. Verify first that $P(\mu_1^{-1})<0$ if $\mu_1\leq\mu_2$ and that $P(\mu_2^{-1})<0$ if $\mu_2\leq\mu_1$. We can then easily conclude that

$$-\min\{\mu_1,\mu_2\}^{-1}<k_1.\tag{6.20}$$

We summarize the above results in

THEOREM 6

If $m=2$, $h_{12}=h_{21}=1$, $d>0$ and if the claim amount distributions are exponential, the asymptotic non-ruin probabilities are given by

$$R_1(u)=1+A_1e^{k_1u}+A_2e^{k_2u},$$

$$R_2(u)=1-D(k_1)A_1e^{k_1u}-D(k_2)A_2e^{k_2u}\qquad(u\geq 0),$$

where k_1 and k_2 are the two negative roots of (6.16), where the constants $D(k_i)$ are given by (6.15) and where A_1 and A_2 are solutions of (6.19).

When $\alpha_1=\alpha_2=\alpha$, $\mu_1=\mu_2=\mu$, $c_1=c_2=c$ and if λ_1 and λ_2 are arbitrary positive numbers, then $k_2=-\rho$ and k_1 is the negative root of

$$k^2+\left(\rho-\frac{\lambda_1+\lambda_2}{c}\right)k-\frac{\lambda_1+\lambda_2}{c\mu}=0.\tag{6.21}$$

When obtain then $D(k_2)=-1$, $D(k_1)=\lambda_2/\lambda_1$ and the solution of (6.19) is $A_1=0$, $A_2=-\alpha\mu/c$. As expected the ruin probabilities $R_1(u)$ and $R_2(u)$ are in this case identical and equal to the ruin probabilities obtained for the classical Poisson model with exponentially distributed claim amounts:

$$R_1(u)=R_2(u)=1-\frac{\alpha\mu}{c}e^{-\rho u}.\tag{6.22}$$

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A MULTIVARIATE MODEL OF THE TOTAL CLAIMS PROCESS

BY J. DAVID CUMMINS AND LAUREL J. WILTBANK

University of Pennsylvania, Philadelphia

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Much of the risk theory literature deals with the total claims distribution $F(x) = \sum_{k=0}^{\infty} p_k S^{k^*}(x)$, where p_k = the probability of k claims and $S(x)$ is the distribution function of severity. Both p_k and $S(x)$ are univariate probability distributions. Thus, $F(x)$ can be interpreted as a model of claims from one class of policies or as an aggregate model where p_k and $S(x)$ represent mixed probability distributions from a heterogeneous portfolio of policies. An alternative approach to modelling total claims in the latter case would be to recognize explicitly that total claims are the result of the interaction of multivariate processes. In the most general case, total claims arise from a multivariate accident process where each accident triggers multivariate claims frequency and severity processes.

The purpose of this article is to present a multivariate model of total claims and to develop the cumulant generating function of this distribution. Such a model is superior to the traditional model in two respects: (1) It permits explicit recognition of shifts in the overall portfolio composition. Applications of the traditional model, in contrast, rely on the assumption that the portfolio composition is relatively constant over time. (2) It facilitates the evaluation of the effects of reinsurance on the total claims distribution when the reinsurance arrangements are not the same in different segments of the portfolio.

THE TOTAL CLAIMS DISTRIBUTION

As indicated, the total claims distribution involves three multivariate processes: the accident process, claims frequency processes, and claims severity processes. Each type of accident can be assigned unique multivariate claims frequency and severity processes. For example, automobile accidents can give rise to bodily injury liability, property damage liability, and physical damage claims; workers' compensation accidents can give rise to wage loss and medical claims.

Dependencies can arise at various stages of the process. For example, bodily injury and property damage liability claims severity from a given accident may be dependent. The model presented in this article recognizes dependencies of three types: dependencies among different types of accident frequencies, among different types of claims frequencies for a given accident of a particular type, and among claims severities for a given accident of a particular type. The authors believe that these are the types of dependencies most likely to arise in practice. Dependencies may exist among different types of accidents due to weather conditions, business cycles, or other factors. Further, accidents with more (or more severe) claims of one type also may be likely to have more (or more severe) claims of other types and vice versa. Assuming independence among the accident frequency, claims frequency, and claims severity distributions is analogous to the usual assumption of independence between frequency and severity. Dependencies may exist among severities from different types of accidents but this is likely to be attributable to a common inflationary effect, which can be handled more satisfactorily through the use of forecasting models.

The Accident Process

Let $i = 1, 2, \ldots, A$ index the *types of accidents*, and let N_i be the random total number of *accidents of type i* in a given period. The N_i may be statistically dependent, with joint density:

$$(1) \quad \Pr\{N_1 = n_1; N_2 = n_2; \ldots, N_A = n_A\} = q(n_1, n_2, \ldots, n_A), \qquad n_i = 0, 1, 2, \ldots.$$

The Claim Frequency Process

For a single accident of type i, claims of B_i different *claim types* can arise. Let K_{ij} be the random variable, number of *claims* of claim-type j from a single accident of type i. The K_{ij} may be dependent for a given accident of type i, with joint density:

$$(2) \qquad \Pr\{K_{i1} = k_1; K_{i2} = k_2; \ldots, K_{iB_i} = k_{B_i}\} = p_i(k_1, k_2, \ldots, k_{B_i}),$$
$$(k_j = 0, 1, 2, \ldots; i = 1, 2, \ldots, A).$$

The total *number of claims* of all types from a *single* accident of type i is $K_i = K_{i1} + K_{i2} + \cdots + K_{iB_i}$. The numbers of claims from *successive* accidents of the same type or from accidents of different types are assumed to be independent.

The Claim Severity Process

Each claim of claim-type j in an accident of accident-type i is assumed to have a random severity X_{ijl}, where l indexes the individual claim, $l = 1, 2, \ldots, K_{ij}$, in a different accident and claim type. Thus, the total severity in claim category j in a single accident of type i is the random sum:

$$(3) \qquad X_{ij} = \begin{cases} 0, & K_{ij} = 0 \\ X_{ij1} + X_{ij2} + \cdots + X_{ijK_{ij}}, & K_{ij} \neq 0 \end{cases}$$

where $X_{ijl}, X_{ij} \geq 0$; $i = 1, 2, \ldots, A$; and $j = 1, 2, \ldots, B_i$. Then, the *total severity* of a *single accident* of type i over all claim categories is:

$$(4) \qquad X_i = X_{i1} + X_{i2} + \cdots + X_{iB_i}.$$

The random variable X_i is clearly conditional on the outcome of the random vector of claim frequencies, $\mathbf{K}_i = (K_{i1}, K_{i2}, \ldots, K_{iB_i})$, associated with a single accident of type i. The individual claim severities from a given accident of type

i, $X_{ijl}(j=1, 2, \ldots, B_i;\ l=1, 2, \ldots, K_{ij})$, can be treated as mutually statistically dependent. The joint severity distribution function can be written as:

$$S_i(x_{i11}, \ldots, x_{i1K_{i1}}; x_{i21}, \ldots, x_{i2K_{i2}}; \ldots; x_{iB_i1}, \ldots, x_{iB_iK_{iB_i}}|\boldsymbol{k}_i). \tag{5}$$

The marginal distributions of (5) can be written as:

$$S_{ijl}(x_{ijl}|\boldsymbol{k}_i) = S_{ij\cdot}(x_{ijl}|\boldsymbol{k}_i), \quad \text{all } l. \tag{6}$$

This permits the distributions to vary depending upon the claims frequency vector $\boldsymbol{k}_i$, e.g., more dangerous distributions may characterize accidents with larger numbers of claims. The notation $S_{ij\cdot}(x_{ijl}|\boldsymbol{k}_i)$ reflects the assumption that the marginals are identical (but not necessarily independent) for different claims of the same type arising out of an accident with a given claim vector $\boldsymbol{k}_i$.

One can also define the conditional distribution of the sum of claims from an accident of type i (equation (4)):

$$S_i(x_i|\boldsymbol{k}_i) = \Pr\{X_i \leq x_i|\boldsymbol{K}_i = \boldsymbol{k}_i\}. \tag{7}$$

This distribution is a convolution of simpler distributions *only* in the special case where the X_{ijl} are statistically mutually independent for all (j, l) with i fixed. It is assumed that the X_i are independent between different accident types. Independence is also assumed among claim severities for different accidents of the same type.

Distribution of Accidents Among Claim Categories

Given the foregoing, the next step is to obtain the distribution of the total severity of all accidents of accident-type i. First, note that the vector of outcomes of $\boldsymbol{K}_i$ can be thought of as a selection of one of a countable number of *patterns*: $\boldsymbol{k}_i(0) = (0, 0, \ldots, 0)$; $\boldsymbol{k}_i(1) = (1, 0, \ldots, 0)$; $\boldsymbol{k}_i(2) = (0, 1, \ldots, 0); \ldots;$ $\boldsymbol{k}_i(B_i) = (0, 0, \ldots, 1)$; $\boldsymbol{k}_i(B_i+1) = (1, 1, \ldots, 0); \ldots$, etc. Indexing this set by $\pi = 0, 1, 2, \ldots, \Pi$, where Π may be infinite, we observe that (2) provides the probability distribution of these patterns:

$$p_i(\boldsymbol{k}_i(\pi)) = \Pr\{\boldsymbol{K}_i = \boldsymbol{k}_i(\pi)\} = p_i(\pi). \tag{8}$$

These are the probabilities of patterns of claim numbers for a *single* accident. If the patterns generated by each of the n_i accidents of this type are mutually independent and independent of all patterns of other accident types, the distribution of patterns over all accidents of the ith type follows a multinomial law.

Let $\boldsymbol{N}_i = (N_i(\pi); \pi = 0, 1, \ldots, \Pi)$ be the random vector describing the distribution of the N_i accidents of type i over the various claim category patterns where $N_i(\pi)$ is the number of accidents with claim pattern $\boldsymbol{k}_i(\pi)$. Then

$$\begin{aligned}\Pr\{\boldsymbol{N}_i = \boldsymbol{n}_i|N_i = n_i\} &= p_i(n_i(0), n_i(1), \ldots, n_i(\Pi)|n_i) \\ &= \binom{n_i}{n_i(0)n_i(1)\cdots n_i(\Pi)}[p_i(0)]^{n_i(0)} \\ &\quad \times [p_i(1)]^{n_i(1)} \cdots [p_i(\Pi)]^{n_i(\Pi)}\end{aligned} \tag{9}$$

where

$$p_i(\pi) \geq 0; \qquad \sum_{\pi=0}^{\Pi} p_i(\pi) = 1 \quad \text{and} \quad \sum_{\pi=0}^{\Pi} n_i(\pi) = n_i.$$

Note that the $p_i(\pi)$ may come from any appropriate probability distribution. It is the *allocation* of accidents among claim category patterns and not the probabilities of patterns which is multinomial.

We can now find the distribution of Y_i, the total value of claims in accident class i, conditional on the realized number of accidents, $N_i = n_i$. Note that the single-accident conditional severity distribution function, (7), can be rewritten as $S_i(x_i|\pi)$, $\pi = 0, 1, \ldots \Pi$. It follows that:

$$(10) \qquad \Pr\{Y_i \leq y_i|n_i\} = H_i(y_i|n_i) = \sum_{\mathbf{n}_i} p_i(n_i(0), n_i(1), \ldots, n_i(\Pi)|n_i) \times [S_i(y_i|0)]^{n_i(0)*} * [S_i(y_i|1)]^{n_i(1)*} * \cdots * [S_i(y_i|\Pi)]^{n_i(\Pi)*}$$

where $\sum_{\mathbf{n}_i}$ indicates summation over all possible realizations of $\mathbf{N}_i$ such that $\sum_{\pi=0}^{\Pi} n_i(\pi) = n_i$.

The Total Claims Distribution

The unconditional grand total value of claims over all accident classes can be written as:

$$(11) \qquad Y = Y_1 + Y_2 + \cdots + Y_A.$$

The distribution function of Y is easy to specify because the severities of different accident classes are assumed to be independent. From (1) and (10),

$$(12) \qquad \Pr[Y \leq y] = F(y) = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_A} q(n_1, n_2, \ldots, n_A) \times [H_1(y|n_1)] * [H_2(y|n_2)] * \cdots * [H_A(y|n_A)].$$

The Cumulant Generating Function

The formula for $F(y)$ is mathematically intractable for most probability distributions encountered in practice. However, the cumulant generating function of $F(y)$ can be written quite compactly, facilitating the derivation of cumulants for use in the Normal-Power or Gamma approximations. The cumulant generating function is preferable to the moment generating function since moments and cumulants can be obtained much more simply using the former function. The function is analogous to that developed by Brown (1977) for the univariate accident frequency-claims frequency-claims severity case.

To obtain the cumulant generating function, we first derive the moment generating function. Let

$$(13) \qquad M_Y(t) = \int e^{ty}\, dF(y) \quad \text{and} \quad \Upsilon_i(t|n_i) = \int e^{ty_i}\, dH_i(y_i|n_i).$$

From (12), one obtains:

$$M_Y(t)=\sum_{n_1}\sum_{n_2}\cdots\sum_{n_A} q(n_1,n_2,\ldots,n_A)\prod_{i=1}^{A} \Upsilon_i(t|n_i). \tag{14}$$

But if

$$\Psi_i(t|\pi)=\int e^{tx_i}\,dS_i(x_i|\pi) \tag{15}$$

we find from (9), (10), and the expansion of a multinomial:

$$\begin{aligned}\Upsilon_i(t|n_i)&=\sum_{n_i} p_i(n_i(0),\ldots,n_i(\pi)|n_i)[\Psi_i(t|0)]^{n_i(0)}\\ &\quad\times[\Psi_i(t|1)]^{n_i(1)}\cdots[\Psi_i(t|\Pi)]^{n_i(\Pi)}\\ &=[p_i(0)\Psi_i(t|0)+p_i(1)\Psi_i(t|1)+\cdots+p_i(\Pi)\Psi_i(t|\Pi)]^{n_i}.\end{aligned} \tag{16}$$

Using (16), the moment generating function (14) can be written:

$$M_Y(t)=\sum_{n_1}\sum_{n_2}\cdots\sum_{n_A} q(n_1,n_2,\ldots,n_A)\prod_{i=1}^{A}\left[\sum_{\pi=0}^{\Pi} p_i(\pi)\Psi_i(t|\pi)\right]^{n_i}. \tag{17}$$

This is the moment generating function of the multivariate accident frequency distribution $q(n_1,n_2,\ldots,n_A)$ with auxiliary parameters $\log[\sum_{\pi=0}^{\Pi} p_i(\pi)\Psi_i(t|\pi)]$, $i=1,\ldots,A$. Thus, (17) can be written as:

$$M_Y(t)=M_{N_1,N_2,\ldots,N_A}\left\{\log\left[\sum_{\pi=0}^{\Pi} p_1(\pi)\Psi_1(t|\pi)\right],\ldots,\log\left[\sum_{\pi=0}^{\Pi} p_A(\pi)\Psi_A(t|\pi)\right]\right\}. \tag{18}$$

The definition of the cumulant generating function is $C_Y(t)=\log M_Y(t)$. Using this definition and (18), one can write

$$C_Y(t)=C_{N_1,N_2,\ldots,N_A}\left\{\log\left[\sum_{\pi=0}^{\Pi} p_1(\pi)\Psi_1(t|\pi)\right],\ldots,\log\left[\sum_{\pi=0}^{\Pi} p_A(\pi)\Psi_A(t|\pi)\right]\right\}. \tag{19}$$

Next, notice that $\Psi_i(t)=\sum_{\pi=0}^{\Pi} p_i(\pi)\Psi_i(t|\pi)$, $i=1,2,\ldots,A$, is the moment generating function of the mixed severity distribution:

$$S_i(x_i)=\sum_{\pi=0}^{\Pi} p_i(\pi)S_i(x_i|\pi). \tag{20}$$

Hence, (19) can be rewritten as:

$$C_Y(t)=C_{N_1,N_2,\ldots,N_A}\{C_{X_1}(t),\ldots,C_{X_A}(t)\} \tag{21}$$

where $C_{X_i}(t)=$ the cumulant generating function of $S_i(x_i)$.

An interesting special case occurs when the claim severities within each accident type are mutually independent. Recalling (3) and (6), one obtains:

$$\Psi_i(t)=\sum_{\pi=0}^{\Pi} p_i(\pi)[\Psi_{i1\cdot}(t|\pi)]^{k_1}\cdots[\Psi_{iB_i}(t|\pi)]^{k_{Bi}} \tag{22}$$

where $\Psi_{ij\cdot}(t)=$ the moment generating function of $S_{ij\cdot}(x_{ijl})$. Equation (22) is recognizable as the moment generating function of a multivariate claims frequency process with auxiliary parameters $\log \Psi_{ij\cdot}(t)$, $j = 1, 2, \ldots, B_i$. Using (22), equation (21) becomes:

$$(23) \qquad C_Y(t) = C_{N_1, \ldots, N_A}\{C_{K_{11}, \ldots, K_{1B_1}}[C_{X_{11}}(t), \ldots, C_{X_{1B_1}}(t)], \ldots, C_{K_{A1}, \ldots, K_{AB_A}}[C_{X_{A1}}(t), \ldots, C_{X_{AB_A}}(t)]\}$$

where $C_{X_{ij}}(t)$ = the cumulant generating function of $S_{ij\cdot}(x_{ijl})$, and $C_{K_{i1}, \ldots, K_{iB_i}}[\cdot]$ = the cumulant generating function of the multivariate claims frequency distribution applicable to accident type i.

Examples of Cumulants

The first and second cumulants of $F(y)$ are straightforward generalizations of the usual formulas for the first two cumulants of sums of random variables. The third cumulant, while also a generalization, is more interesting and is shown below:

$$(24) \quad \kappa_{3Y} = \sum_{i=1}^{A} [\kappa_{3N_i}\kappa_{1X_i}^3 + \kappa_{1N_i}\kappa_{3X_i} + 3\kappa_{2N_i}\kappa_{1X_i}\kappa_{2X_i}] + 3\sum\sum_{i \neq g} [\kappa_{1N_i 1N_g}\kappa_{2X_i}\kappa_{1X_g}]$$
$$+ 3\sum\sum_{i \neq g} [\kappa_{2N_i 1N_g}\kappa_{1X_i}^2\kappa_{1X_g}] + 6\varepsilon \sum_{i \neq g \neq h} \kappa_{1N_i 1N_g 1N_h}\kappa_{1X_i}\kappa_{1X_g}\kappa_{1X_h}$$

where $i, g = 1, \ldots, A$; and $\varepsilon = 1$ for $A \geq 3$, 0 otherwise. The double summation $\sum\sum_{i \neq g}$ means the summation over both subscripts, omitting terms where the subscripts are equal. The summation $\sum_{i \neq g \neq h}$ means the summation over all combinations $i \neq g \neq h$, where $i, g, h = 1, 2, \ldots, A$.

The κ's are cumulants. Numerical subscripts refer to the cumulant number, while letter subscripts refer to random variables. Symbols with more than one of each type of subscript are cross-cumulants. For example, $\kappa_{1N_1 1N_2}$ is the first cross-cumulant (covariance) of the accident frequency random variables N_1 and N_2.

Cumulants of the mixed severity distributions $S_i(x_i)$, $i = 1, 2, \ldots, A$, can be obtained directly using (2), (3), (4), (7), and (20). The formulas for the first three cumulants are as follows:

$$(25) \quad \kappa_{1X_i} = \sum_{\pi} p_i(\pi)\mu_i(X_i|\pi)$$

$$(26) \quad \kappa_{2X_i} = \sum_{\pi} p_i(\pi)\alpha_{2i}(X_i|\pi) - \left[\sum_{\pi} p_i(\pi)\mu_i(X_i|\pi)\right]^2$$

$$(27) \quad \kappa_{3X_i} = \sum_{\pi} p_i(\pi)\alpha_{3i}(X_i|\pi) - 3\left[\sum_{\pi} p_i(\pi)\mu_i(X_i|\pi)\right] \cdot \left[\sum_{\pi} p_i(\pi)\alpha_{2i}(X_i|\pi)\right]$$
$$+ 2\left[\sum_{\pi} p_i(\pi)\mu_i(X_i|\pi)\right]^3$$

where

$$\mu_i(X_i|\pi) = \int x_i \, dS_i(x_i|\pi) \quad \text{and} \quad \alpha_{mi}(x_i|\pi) = \int x_i^m \, dS_i(x_i|\pi).$$

The formulas for the moments of $S_i(x_i|\pi)$ are straightforward but cumbersome. The moments of individual claim severities are permitted to vary according to the claim pattern, π, allowing for the possibility of larger claim severity means, variances, etc. for accidents involving large numbers of claims.

If the individual claim severity moments are assumed to be equal within a claim type (regardless of π), the cumulant formulas are simplified. The second cumulant, for example, is:

$$\text{(28)} \qquad \kappa_{2X_i} = \sum_j \left[\kappa_{1K_{ij}}\kappa_{2X_{ij}} + \kappa_{2K_{ij}}\kappa^2_{1X_{ij}} + (\kappa_{2K_{ij}} + \kappa^2_{1K_{ij}} - \kappa_{1K_{ij}})\kappa_{1X_{ijl}1X_{ijg}}\right]$$
$$+ \sum_{j \neq h}\sum \left[\kappa_{1K_{ij}1K_{ih}}\kappa_{1X_{ij}}\kappa_{1X_{ih}} + (\kappa_{1K_{ij}1K_{ih}} + \kappa_{1K_{ij}}\kappa_{1K_{ih}})\kappa_{1X_{ij}1X_{ih}}\right]$$

where j, $h = 1, 2, \ldots, B_i$; $\kappa_{mK_{ij}}$ = the cumulants of the marginals of (2), here $m = 1, 2$; $\kappa_{1K_{ij}1K_{ih}}$ = first cross-cumulant of (2), $j \neq h$; $\kappa_{mX_{ij}}$ = cumulants of $S_{ij}(x_{ijl})$, assumed identical for all l within claim type j, $m = 1, 2$, where $S_{ij}(x_{ijl})$ is (6) without the condition; $\kappa_{1X_{ijl}1X_{ijg}}$ = first cross-cumulant of claim severities of the same type, $l \neq g$, assumed identical for all l, g; and $\kappa_{1X_{ij}1X_{ih}}$ = first cross-cumulant of claim severities of different types, $j \neq h$.

In practical applications, it generally will be necessary to combine the higher order claim patterns to permit the estimation of severity distributions. For example, with two claim types, analysis might be confined to the following patterns: $\boldsymbol{k}_i'(0) = (0, 0)$; $\boldsymbol{k}_i'(1) = \{(K_{i1}, 0); K_{i1} = 1, 2, \ldots\}$; $\boldsymbol{k}_i'(2) = \{(0, K_{i2}), K_{i2} = 1, 2, \ldots\}$; and $\boldsymbol{k}_i'(3) = \{(K_{i1}, K_{i2}); K_{i1}, K_{i2} \geq 1\}$. The probabilities of each revised pattern, $\boldsymbol{k}_i'(\pi)$, $\pi = 1, 2, 3$, can be obtained by summing the appropriate probabilities from (8). The severity distributions $S_i(x_i|1)$ and $S_i(x_i|2)$ are univariate distributions estimated on a set of observations on X_{i1} and X_{i2}, respectively, from accidents with the designated claim patterns $\pi = 1$ and $\pi = 2$, respectively. (Recall (3).) Thus, in estimating the severity distributions, no distinction is made among accidents which have different numbers of claims, K_{ij}. Rather, the sum of claim severities of a particular type from a given accident is considered a single observation from the appropriate severity distribution. Cumulants of $S_i(x_i|3)$ are obtained from a bivariate severity distribution estimated on a set of observations on (X_{i1}, X_{i2}), where $K_{i1}, K_{i2} \geq 1$. $S_i(x_i|0)$ is a degenerate distribution.

Conclusion

This article has presented a multivariate model of the total claims distribution. The model could be used in conjunction with the Normal-Power or Gamma approximations to model the total claims of an insurance company by estimating cumulants for each segment of the portfolio and combining the cumulants according to the appropriate formulas. This approach should be superior to the traditional $F(x)$ model for some applications because it focuses directly on individual segments of the portfolio and clarifies the interactions among the segments. Empirical research is needed on the types of distributions that are appropriate for modeling the claims process in the multivariate context and about the nature and magnitude of the dependencies among the variables comprising the process.

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A STABLE RECURSIVE ALGORITHM FOR EVALUATION OF ULTIMATE RUIN PROBABILITIES

BY MARC GOOVAERTS AND FLORIAN DE VYLDER*

K.U. Leuven and U.C. Louvain, Belgium

ABSTRACT

Probabilities of ruin are solutions of differential or integrodifferential equations. Solving such equations numerically can be performed by means of approximate quadrature formulae for the convolution part of the equation. In this contribution it is shown how applicable recursion formulae, giving results within a prescribed tolerance level, can be obtained. Some numerical results are displayed.

KEYWORDS

Ruin probability, numerical analysis, ordering of risks.

1. INTRODUCTION

Gerber (1982) introduced a method for approximating the distribution of aggregate claims and their corresponding stop-loss premium by means of a discrete compound Poisson distribution and its corresponding stop-loss premium. This discretization is an important step in the numerical evaluation of the distribution of aggregate claims, because recent results on recurrence relations for probabilities by PANJER (1981) and SUNDT and JEWELL (1981) only apply to discrete distributions. The discretization technique is efficient in a certain sense, because a properly chosen discretization gives raise to numerical upper and lower bounds on the stop-loss premium, giving the possibility of calculating the numerically estimates for the error on the final numerical results. For calculating the infinite time ruin probability numerically one has to solve the following integral equation, according to GERBER (1979):

$$\psi(x)=\frac{\lambda}{c}\int_0^x \psi(x-y)(1-F(y))\,dy+\frac{\lambda}{c}\int_x^\infty (1-F(y))\,dy \tag{1}$$

with

$$\psi(0)=\frac{\lambda p}{c}, \qquad c=\lambda p(1+\eta) \tag{2}$$

and where

$$p=\int_0^\infty (1-F(y))\,dy. \tag{3}$$

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In the sequel we write $h(x)$ for

$$h(x)=\int_x^\infty (1-F(y))\,dy \tag{4}$$

such that $h(0)=p$.

In fact, for solving equation (1), use could be made of a discretization technique for approximating the integrals in the equation in order to get a system of linear equations in the unknown probabilities $\psi(x_1)$, $\psi(x_2)$, ..., $\psi(x_n)$. However, not every discretization is effective to get numerically stable results, on the contrary the propagation of the error induced by the discretization technique provides us, in general, with unstable numerical results. In this contribution we will derive an efficient discretization technique which allows us to calculate $\psi(x)$ numerically within a given tolerance by means of a stable recusive algorithm. As by-product of the method an estimate for the error in the numerical result is obtained.

2. AN EFFICIENT DISCRETIZATION FOR THE CONVOLUTION INTEGRAL

In the sequel we define the convolution product by

$$\varphi * H(x)=\int_0^x \varphi(x-y)\,dH(y)$$

where the integral is taken over the closed interval. The iterative procedure for calculating numerically the infinite time ruin probabilities is based on the following result:

THEOREM. *The function ψ (denoting the infinite time ruin probability) is a solution of the equation*

$$\psi*(p-h)=p(1+\eta)\psi-h. \tag{5}$$

In that equation or in any equivalent equation replace (h,p) by (h_0,p_0) and let ψ_0 be the corresponding solution. Then $\psi\le\psi_0(\psi\ge\psi_0)$ if h_0 is decreasing $h_0\ge 0$, and $h/p\le h_0/p_0(h/p\ge h_0/p_0)$.

PROOF. See appendix.

COROLLARY. *In case $p=p_0$ the inequality condition of the theorem reduces to $h\le h_0(h\ge h_0)$.*

REMARKS. For the proof of the theorem to hold it is not necessary that h_0 can be written as $h_0(x)=\int_x^\infty (1-F_0(x))\,dx$ where F_0 is a distribution function.

In this extended version of the theorem, we do not have the additional assumption $h_0(0)=p$. Indeed, $h(0)=p$, so in order to get an upper bound we have to suppose $p\le h_0(0)$ and in order to get a lower bound we have to suppose $p\ge h_0(0)$.

The theorem, or another version of it, has given us a possibility to obtain analytical upper and lower bounds on infinite time ruin probabilities in case of constraints on claim size distribution, as explained in GOOVAERTS and DE VYLDER (1983). Now it will enable us to deduce numerical bounds on infinite time ruin probabilities because an application of the theorem will provide us with a stable recursive algorithm. Our aim consists in calculating $\psi(x)$. In order to obtain numerical upper and lower bounds (to obtain an error estimate) the following procedure is applied. The underlying motivation of it follows directly from an inspection of figure 1 and an application of the theorem.

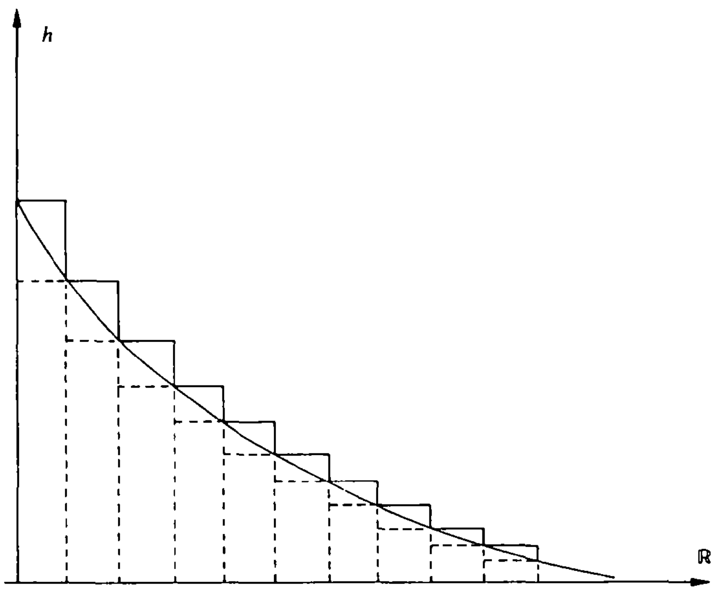

FIGURE 1

Practical Procedure

(i) Consider the following subdivision of the interval $[0, x]$

$$\left[0, \frac{x}{n}\right], \left[\frac{x}{n}, 2\frac{x}{n}\right], \ldots, \left[\frac{n-1}{n}x, x\right].$$

(ii) As indicated in fig. 1 we consider

$$\forall y \in \left[\frac{i}{n}x, \frac{i+1}{n}x\right] \qquad h_u(y) = h\left(i\frac{x}{n}\right)$$

$$h_l(y) = h\left((i+1)\frac{x}{n}\right).$$

Consequently

$$h(y) \leq h_u(y) \qquad \forall y \geq 0$$

$$h(y) \geq h_l(y) \qquad \forall y \geq 0.$$

(iii) Let $\psi_u(x)$ be the solution of

$$\psi_u(k)\left(1-\frac{\lambda^p 1}{c}\right)=\frac{\lambda}{c}\left(1-\frac{\lambda^p 1}{c}\right)h_u(k)-\frac{\lambda}{c}\int_0^k \psi_u'(k-y)h_u(y)\,dy.$$

Because $h_u(y)$ is a piecewise constant function, the integral appearing in the r.h.s. can be worked out as follows: let $k = j(x/n)$, then:

$$\psi_u\left(j\frac{x}{n}\right)\left(1-\frac{\lambda^p 1}{c}\right)=\frac{\lambda}{c}\left(1-\frac{\lambda^p 1}{c}\right)h\left(j\frac{x}{n}\right)$$
$$-\frac{\lambda}{c}\sum_{i=0}^{j-1} h\left(i\frac{x}{n}\right)\int_{i(x/n)}^{(i+1)} \psi_u'\left(j\frac{x}{n}-y\right)dy.$$

This equation can be cast into the form:

$$\psi_u\left(j\frac{x}{n}\right)=\frac{\lambda}{c}h\left(j\frac{x}{n}\right)+\frac{\lambda}{c}\sum_{i=1}^{j}\left[h\left((i-1)\frac{x}{n}\right)-h\left(i\frac{x}{n}\right)\right]\psi_u\left((j-i)\frac{x}{n}\right). \tag{6}$$

We also get, proceeding along the same lines,

$$\begin{aligned}\psi_l\left(j\frac{x}{n}\right)=\frac{\lambda}{c}\left[1-\frac{\lambda}{c}\left(p_1-h\left(\frac{x}{n}\right)\right)\right]^{-1}&\left[h\left(j\frac{x}{n}\right)\right.\\ &\left.+\sum_{i=1}^{j-1}\left[h\left(i\frac{x}{n}\right)-h\left((i+1)\frac{x}{n}\right)\right]\psi_l\left((j-i)\frac{x}{n}\right)\right]\end{aligned} \tag{7}$$

with of course

$$\psi_u(0)=\psi_l(0)=\frac{\lambda p}{c}. \tag{8}$$

Starting from (8) we calculate recursively by means of (6) and (7) the couple $(\psi_u(j(x/n)), \psi_l(j(x/n))$ for $j = 1, 2, \ldots, n$ to obtain two approximations to $\psi(x)$, namely $\psi_u^{(n)}(x)$, $\psi_l^{(n)}(x)$ where we added explicitly the index n to denote the dependence on n. The following inequalities are obtained

$$\psi_l^{(n)}(x) \leq \psi(x) \leq \psi_u^{(n)}(x). \tag{9}$$

Also from the result of the above quoted theorem we conclude that

$\psi_l^{(2^n)}(x)$ is not decreasing in n

$\psi_u^{(2^n)}(x)$ is not increasing in n

Hence $\psi(x)$ can be approximated by

$$\psi(x) \approx \tfrac{1}{2}\psi_l^{(n)}(x) + \tfrac{1}{2}\psi_u^{(n)}(x)$$

with an upper bound for the error given by

$$\psi_u^{(n)}(x)-\psi_l^{(n)}(x).$$

In order to obtain a result within a prescribed tolerance level ε, n is chosen large enough such that

$$\psi_u^{(n)}(x)-\psi_l^{(n)}(n)<\varepsilon$$

and $n=2^k$ (k integer).

In order to examine the stability of the numerical procedure, in fact, in order to examine the propagation of errors induced by this recursive algorithm, we suppose that $\psi(x/n),\ldots,\psi((j-1)x/n)$ are calculated with an error $\varepsilon_1,\ldots,\varepsilon_{j-1}$. Let $\varepsilon=\max\{\varepsilon_1,\ldots,\varepsilon_{j-1}\}$ then the error ε_j on $\psi(j(x/n))$ is bounded by:

$$\varepsilon_j<\frac{\lambda}{c}\sum_{i=1}^{j}\left(h\left((i-1)\frac{x}{n}\right)-h\left(i\frac{x}{n}\right)\right) \qquad \varepsilon_j<\varepsilon.$$

Consequently there is no cumulative effect of propagation of errors. Let us compare the kind of recursion relation with the recursive algorithm of PANJER (1981) and SUNDT and JEWELL (1981) for the calculation of the distribution function of a compound Poisson variable, where

$$g_j=\frac{\lambda}{j}\sum_{i=1}^{j} if_i g_{j-i}.$$

In case λ is relatively small no problem arises as far as the propagation of errors is concerned.

In case λ is relatively large however (and this is exactly the case where it is interesting to apply such kind of a scheme) the recursive algorithm is unfortunately not very stable as far as the propagation of errors is concerned. Indeed let $\varepsilon_1,\ldots,\varepsilon_{j-1},\varepsilon_j$, denote the errors on $g_1,\ldots,g_j$ respectively. In case $\varepsilon=\max(\varepsilon_1\ldots,\varepsilon_{j-1})$ then

$$\varepsilon_j\leqslant\frac{\lambda}{j}\sum_{i=1}^{j} if_i \qquad \varepsilon\approx\mu\frac{\lambda}{j}\cdot\varepsilon.$$

Consequently as long as $j<\lambda$ the upper bound of the error behaves like

$$\varepsilon_j\simeq\lambda^j\cdot c$$

which of course can cause a lot of unexpected difficulties in actual application of the recursion algorithm.

3. ILLUSTRATION OF THE METHOD

In VAN WOUWE, DE VYLDER and GOOVAERTS (1982) the present results are successfully applied to the numerical calculation of bounds on infinite time ruin probabilities in case of constraints on claim size distributions. Use has been made of some of the analytical upper and lower bounds on stop-loss premiums.

Let us still remark that the rate of convergence of the recursive algorithm is determined by the rate of convergence to zero of the ruin probability when $x \to \infty$. Therefore we have selected an application which from the numerical point of view has a relatively low speed of convergence.

We consider the case of Pareto claims, $F_X(x) = 1 - 1/(1+x)^2$, $p = 1$, $\eta = 0.2$ and of course $h(x) = 1/(1+x)$. The following results are obtained.

Upper Bound

n \ x	10	50	100
20	0.455529	0.193577	0.119406
40	0.449979	0.164704	0.087263
80	0.439944	0.153144	0.076432
160	0.439944	0.148211	0.072358

Lower Bound

n \ x	10	50	100
20	0.411083	0.121643	0.058221
40	0.422112	0.129821	0.061631
80	0.428309	0.135709	0.064429
160	0.431619	0.139413	0.066421

APPENDIX PROOF OF THE THEOREM

Use will be made of the following result well known from renewal theory:

LEMMA. *If H is a strictly defective distribution function and if f is bounded, then the renewal equation*

(A1) $$\xi = f + \xi \times H$$

has a unique bounded solution ξ. If $f \geq 0$, then $\xi \geq 0$. If $f \leq 0$, then $\xi \leq 0$.

By means of one partial integration the equation (1) can be cast into the form

(A2) $$\psi * (p-h) = p(1+\eta)\psi - h.$$

This relation can still be displayed as:

(A3) $$p - p(1+\eta)\psi = (1-\psi) * (p-h).$$

By the definition of ψ_0, we still have

(A4) $$p_0 - p_0(1+\eta)\psi_0 = (1-\psi_0) * (p_0 - h_0).$$

From (A3) and (A4) we deduce

(A5) $(1+\eta)(\psi-\psi_0)=h/p-h_0/p_0+(\psi-\psi_0)*(1-h/p)+\psi_0*(h_0/p_0-h/p)$.

Then the above mentioned lemma can be applied with

$$\xi=\psi-\psi_0$$

$$H=(1-h/p)/(1+\eta)$$

(A6) $$f=(1-\psi_0)*(h/p-h_0/p)/(1+\eta).$$

Then of course f is bounded. Moreover because $h/p\geqslant h_0/p_0$ we have $f\geqslant 0$. The function H is a distribution function (it is increasing and $H\geqslant 0$), in fact it is a strictly defective distribution function because $H(\infty)=1/(1+\eta)$. Hence the result of the theorem follows.

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AN APPLICATION OF GAME THEORY: COST ALLOCATION

By Jean Lemaire

Université Libre de Bruxelles

Summary

The allocation of operating costs among the lines of an insurance company is one of the toughest problems of accounting; it is first shown that most of the methods used by the accountants fail to satisfy some natural requirements. Next it is proved that a cost allocation problem is identical to the determination of the value of a cooperative game with transferable utilities, and 4 new accounting methods that originate from game theory are proposed. One of those methods, the proportional nucleus, is recommended, due to its properties. Several practical examples are discussed throughout the paper.

Keywords

Game theory, cost allocation.

1. Cost allocation in practice

Cost allocation is one of the toughest problems of accounting. It occurs whenever cooperation between several departments of a company produces economies of scale: the benefits of cooperation have to be allocated to the participating departments. In insurance, such problems are numerous, especially in countries where companies are allowed to operate on a multi-class basis; the accountants of the company are then compelled to divide the operating costs between the different classes. The amount of time spent and the complexity of the methods used in cost allocation are absolutely startling: for instance a large Belgian company that operates in three classes (life, fire and accident) uses no less than 11 different criteria or "keys".

Key No. 1: Direct Imputation

Some operating costs can be directly assigned to a class: the salary of the employees that work exclusively in that class, the brokers' commissions, the surveyors' fees, ... Note that only 57% of the operating costs of the company can be allocated directly.

Key No. 2: In Proportion to Key No. 1

The salaries of the employees who do not work exclusively for one class, the premiums of their insurance policies, the employer's contribution to the Social Security system, ... are allocated in proportion of the total observed under key No 1.

Key No. 3: *In Proportion to the Number of Files*

The salaries, the telephone bills, the travel expenses of the administrative inspectors of the company are allotted according to the number of files they have to consider monthly.

Key No. 4: *In Proportion to the Number of Policies and Endorsements*

Costs allocated according to this key include the salaries of the producing inspectors, of the premium collectors, the agents' solidarity fund in case of illness, etc.....

Key No. 5: *One Third to Each Class*

The company operates a training center, where its agents now and then come for a full week of lessons All costs relating to this activity (instructors' salaries, food and beverages, caretaker's wage, heating of the center,...) are simply distributed evenly among the classes!

Key No. 6: *Average of Keys Nos.* 3, 4 *and* 5

The premiums of the insurance policies of the inspectors are the only costs allocated by this key.

Key No. 7: *In Proportion to the Surface Occupied*

Heating costs, water, electricity, telephone bills, cleaners' salaries, lift maintenance,... are apportioned according to the surface occupied by the three classes in the building.

Key No. 8: *In Proportion to Premium Income*

The list of costs divided according to this key is nearly endless and very diversified: subsidies to various organizations, subscriptions to papers and magazines, gifts for the employees' children at Christmas, prizes for competitions between the agents, advertising, travel costs of the directors, maintenance of the company cars, reception costs of the foreign visitors, printing of the company's newsletter,....

Key No. 9: *In Proportion to the Average Number of Employees of each Class*

In this section we have the maintenance costs of the printing department, the operating costs of the restaurant, the stationery supplies,....

Key No. 10: *In Proportion to the Number of Computing Hours + the Average Number of Disks and Tapes*

This key was selected to subdivide the computer costs.

Key No. 11: *In Proportion to the Total of Keys No.* 1 *to* 10

This last key includes the postage costs, the operating costs of the company's local offices, the insurance policies of the company cars, the medical aid for the employees, ...

In addition to this complexity, quite large amounts (millions of Belgian francs!) are arbitrarily transferred from one class to another whenever it is felt that one of the keys acts unfairly.

The accountants unanimously acknowledge that their methods are extremely complex and in some ways completely arbitrary. They admit that the grand total for one class may be wrong by quite a few percent, but pretend that this is not too important: since the total profit of the company is the sum of its three components, they claim that an allocation error simply increases the profit of one class at the expense of another, and does not influence the total result. This is not correct: unfair allocations may lead to actions that decrease the total profit of the company, as shown by the following examples.

EXAMPLE 1. In the case where service department costs are allocated to producing divisions, the overcharged division has an incentive to independently contract out such services, and avoid the use of the service department. While the division reports a cost savings from such a move, overall corporate profits may suffer. For instance, in one company, some of the policies of one class are printed outside the printing department: the manager of this class has noticed that, due to the selected allocation key, it is cheaper to have its policies printed outside than at the company's printing department. This is a nice example of an individually optimal decision that turns out to be a collective error: the class manager has increased his profit, but the company profit has decreased, since the printing department's salaries and maintenance have to be paid anyway.

EXAMPLE 2. Key No. 10 penalizes the computer programs that use a lot of disks and tapes. So there is an incentive for class managers to have those programs run outside the company: this reduces the operating expenses of the class, but increases the company's expenses.

EXAMPLE 3. In many countries the technical results of a class influence the commissions paid to the agents and/or the bonus paid at the end of the year to the employees. Also the profit of a class is one of the criteria for the evaluation of the performance of its manager. All those persons would certainly not be very happy if they were to learn that their class is subsidizing another one, by way of some unfair cost allocation procedures that have distorted the relative profitabilities of their products.

EXAMPLE 4: The worst error that could be induced by an incorrect allocation procedure is under-pricing, selling a type of policy below the "break-even" price, without being aware of it. Typically this may happen if the selected key fails to identify the high operating costs of a line, like travel assistance or familial responsibility, that produces numerous small claims. For example, if an allocation key is: "For all policies of the accident class, the operating costs equal 20% of the commercial premium", that amounts to have the travel assistance line subsidized by motorcar third-party liability.

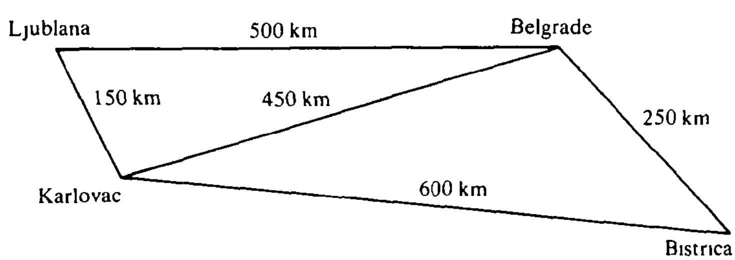
Those examples show that it is of uttermost importance to develop "fair" cost allocation techniques. We shall attempt to show that game theory may be "the" solution to this problem. First (Section 2) an introductory example shows why the classical cost allocation methods, failing to satisfy some important properties, have to be rejected. Then, we show (Section 3) that the cost allocation problem is identical to the problem of computing the value of a n-person cooperative game with transferable utilities. We propose (Section 4) four new methods, adapted from game theory, and compare them (Section 5) by means of three important properties. In Section 6 the case of games without core is briefly considered. In Section 7 we present an extensive list of other applications of game theory that show that cost allocation is an area where game theoretic ideas are effectively implemented. Finally, in Section 8, we completely solve a practical example.

2. AN INTRODUCTORY EXAMPLE

EXAMPLE 1. During the first week of April 1983, three Belgian drivers were involved in an accident in Yugoslavia.

Policy-holder	Company	Place of accident	Amount of the claim (× 1 000 dinars)
J_1	1	Ljublana	$s_1 = 300$
J_2	2	Karlovac	$s_2 = 1000$
J_3	3	Bistrica	$s_3 = 200$

The three concerned companies need a damage survey of the cars. They happen to have the same local correpondent in Belgrade, the appraisal bureau Y. Observing the location of the three claims on the map, Y notices that it is much cheaper



(in total mileage) to sens an expert for a round trip, than to come back to Belgrade after each evaluation.

Let S be any subset of $N=\{1,2,3\}$. Denote by $c(S)$ the total mileage driven in order to inspect the vehicle(s) of S.

$$c(1)=1000$$
$$c(2)=900$$
$$c(3)=500$$
$$c(12)=1100$$
$$c(13)=1500$$
$$c(23)=1300$$
$$c(N)=c(123)=1500$$

(for simplicity we denote $c(12)$ for $c(\{1,2\})$, *etc*).

So a round trip produces a total gain of $1000+900+500=900$ km. This however creates a problem to Y: what amount x_i should be charged to each company? Clearly the fixed costs (hotel nights in each city, adjuster's fee for each vehicle, ..) can be assigned directly to the corresponding claim, so we only need to consider the repartition of the variable costs, the travel expenses. We suppose that the expert's reimbursement indemnity is proportional to the mileage driven. The classical cost allocation methods used in accounting are the following.

Method 1: *Equal Repartition of the Total Gain*

$$x_i=c(i)-\frac{1}{3}\left[\sum_j c(j)-c(N)\right].$$

This leads to the allocation vector $\bar{x}=(x_1, x_2, x_3)$:

$$\bar{x}=(700,600,200).$$

Method 2: *Proportional Repartition of the Total Gain* (*or Moriarity's Method*)

$$x_i=c(i)-\frac{c(i)}{\sum_j c(j)}\left[\sum_k c(k)-c(N)\right]=\frac{c(i)}{\sum_j c(j)}c(N).$$

In our example $\bar{x}=(625,562.5,312.5)$.

Method 3: *Equal Repartition of the Non-Marginal Costs*

Define the marginal cost for i:

$$CM(i)=c(N)-c(N\backslash\{i\}).$$

$CM(i)$ (sometimes called the separable cost) is the additional mileage to be driven if $\{i\}$ is considered to be the last claim, if it is added to the group $N\backslash\{i\}$, already formed. The method advocates

$$x_i = CM(i) + \frac{1}{3}\left[c(N) - \sum_k CM(k)\right]$$

i.e., an allocation $\bar{x} = (500, 300, 700)$ (the marginal costs are $(200, 0, 400)$).

Method 4: *Proportional Repartition of the Non-Marginal Costs*

$$x_i = CM(i) + \frac{CM(i)}{\sum_j CM(j)}\left[c(N) - \sum_k CM(k)\right] = \frac{CM(i)}{\sum_j CM(j)} c(N)$$

We obtain $\bar{x} = (500, 0, 1{,}000)$.

Method 5: *Repartition Proportional to the Claim Amounts*

$$x_i = \frac{s_i}{\sum_j s_j} c(N)$$

i.e., $\bar{x} = (300, 1{,}000, 200)$.

The five methods recommend wildly different allocations. They can be compared by their properties. In order for a method to be "fair", it certainly has to satisfy the two following natural properties.

Property 1: *Individual Rationality*

$$x_i \leqslant c(i).$$

A company cannot be charged more than if its policy-holder had been alone to cause an accident. It is inconceivable that a company should suffer from a global saving.

Property 2: *Collective Rationality* (*or Marginality Principle*)

$$x_i \geqslant c(N) - c(N\backslash\{i\}) = CM(i).$$

No company should be charged less than its marginal cost; if the property is not satisfied for a company, it is effectively subsidized by the other two, who have interest to secede.

The two properties limit the range of the acceptable values for x_i:

$$200 \leqslant x_1 \leqslant 1000$$

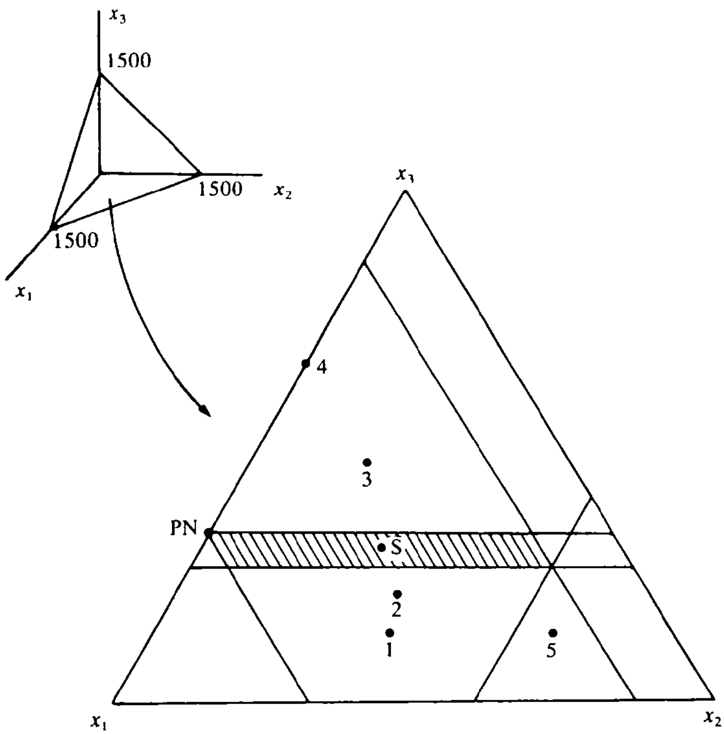
$$0 \leqslant x_2 \leqslant 900$$

$$400 \leqslant x_3 \leqslant 500.$$

Consequently *all* of the above methods have to be rejected.

The different allocations can be represented in the so-called "fundamental triangle of costs".

$$\left\{(x_1, x_2, x_3) \middle| x_i \geqslant 0, \sum_i x_i = 1500\right\}.$$



The hatched surface is the set of the acceptable allocations, delimited by the two properties. The repartitions are indicated by the number of the method.

3. LINK WITH COOPERATIVE GAME THEORY

We shall show in this section that the cost allocation problem is identical to the determination of the value of a game with transferable utilities.

Cost Allocation

Let N be a set of n departments $\{1, 2, \ldots, n\}$ involved in a given job or project. A cost $c(S)$ is attached to each subset or coalition S of departments. A consequence of scale economies is that the set function $c(S)$ has to be sub-additive

$$c(S)+c(T)\geq c(S\cup T) \quad \forall S, T \ni S\cap T=\varnothing:$$

it is cheaper for two departments to collaborate on a job than to act independently.

A cost allocation is a vector $\bar{x}=(x_1,\ldots,x_n)$, such that $x_i\geq 0$, $\forall i$ and $\sum_{i=1}^n x_i = c(N)$.

$\bar{x}$ is said to be individually rational if $x_i \leq c(i) \quad \forall i$.

$\bar{x}$ is said to be collectively rational if, $\forall S$, $\sum_{i\in S} x_i \leq c(S)$.

Imputation of a Game

A n-person cooperative game with transferable utilities is a pair $[N, v(S)]$, where $N=\{1,2,\ldots,n\}$ is the set of the players, and $v(S)$, the characteristic function of the game, is a super-additive set function that associates a real number $v(S)$ to each coalition S of players.

$$v(S)+v(T)\leq v(S\cup T) \quad \forall S, T \ni S\cap T=\varnothing$$

(it is not limitative to assume that $v(i)=0\ \forall i$).

An imputation is a vector $\bar{y}=(y_1,\ldots,y_n)$ such that $y_i\geq v(i)$, $\forall i$ and $\sum_{i=1}^n y_i = v(N)$.

The core is the set of imputations such that $\sum_{i\in S} y_i \geq v(S)$, $\forall S$.

Clearly the two problems are identical if we define

$$v(S)=\sum_{i\in S} c(i)-c(S):$$

the characteristic function associated to each coalition is the saving it can achieve. An imputation of this game defines a cost allocation by

$$x_i = c(i)-y_i.$$

So it is equivalent to define a cost allocation game by $[N, v(S)]$ or $[N, c(S)]$ In the sequel, all formulas will be expressed in terms of $c(S)$.

Note that properties 1 and 2 define the core of the game (in the 3-player case). Obviously none of the preceding five methods will provide a point that always belongs to the core, since none explicitly considers all the $c(S)$.

NOTE. The core of a game may be void (a necessary and sufficient condition for a non void core in a 3-person game is $c(12)+c(13)+c(23)\geq 2c(123)$). In that case, there exists no acceptable cost allocation: there is always at least a set of players who have right to complain and who have interest to separate from the rest of the group. Fortunately, in most of the applications, economies of scale are so large that the game is convex.

DEFINITION. A game is convex if, $\forall S$, T (not necessarily disjoint)

$$c(S)+c(T)\geq c(S\cup T)+c(S\cap T).$$

In the three-player case, convexity reduces to 3 conditions

$$c(12)+c(13) \geqslant c(123)+c(1)$$

$$c(12)+c(23) \geqslant c(123)+c(2)$$

$$c(13)+c(23) \geqslant c(123)+c(3).$$

In the four-player case, there are already 30 conditions!

An equivalent definition of convexity is

DEFINITION. A game is convex if, $\forall i$, $\forall S \subseteq T \subseteq N$

$$c(T \cup \{i\})-c(T) \leqslant c(S \cup \{i\})-c(S).$$

So in a convex game there is a "snow-balling" effect: it becomes more and more interesting to enter a coalition as its number of members increases, since the "admission cost" $c(S \cup \{i\})-c(S)$ decreases. Particularly, it is always preferable to be the last to enter the grand coalition N (this justifies our definition of the marginal cost in Section 2: it is only in the case of a convex game that one can assert that the sum of the marginal costs is less than or equal to the total cost $c(N)$).

In a convex game, the study of the different value concepts is considerably easier, since one can show that the core of such a game is always non void and that it satisfies interesting regularity properties: it is a compact convex polyhedron, of dimension at most $n-1$ (Shapley, 1971). Moreover, it coincides with the bargaining set and the Von Neumann and Morgenstern solution (Maschler, Peleg and Shapley, 1972).

4. FOUR NEW COST ALLOCATION METHODS

4.1. *The Shapley Value*

Shapley (1953) has proved that there exists one and only one allocation $\bar{x}$ that satisfies the following 3 axioms.

AXIOM 1. *Symmetry. For all permutations* Π *of players such that* $c[\Pi(S)]=c(S)$, $\forall S$, $x_{\Pi(i)}=x_i$.

A symmetric problem has a symmetric solution. If there are two players that cannot be distinguished by the cost function, if their contribution to each coalition is the same, it is normal to award them the same amount (this axiom is sometimes called "anonymity").

AXIOM 2. *Inessential players. If, for a player* i, $c(S)=c(S \backslash \{i\})+c(i)$ *for each coalition* S *to which he can belong, then* $x_i=c(i)$.

Such a player does not contribute any scale economy to any coalition; he is called an inessential player, and cannot claim to receive a share of the total gain.

AXIOM 3. *Additivity. Let* $[N, c(S)]$ *and* $[N, c'(S)]$ *be two games, and* $x_i(c)$ *and* $x_i(c')$ *the associated allocations. Then*

$$x_i(c+c') = x_i(c) + x_i(c') \quad \forall i.$$

This axiom has been subject to a lot of criticisms, since it excludes the interactions between both games. In the present case, however, those critiques do not appear to have much ground; it is indeed quite natural, in accounting, to add profits that originate from different sources.

Denote by s the number of members of a coalition S. The only imputation that satisfies the axioms is

$$x_i = \frac{1}{n!}\sum_S (s-1)!(n-s)![c(S) - c(S\backslash\{i\})].$$

INTERPRETATION. The Shapley value is the mathematical expectation of the admission cost when all orders of formation of the grand coalition are equiprobable. Everything happens as if the players enter the coalition one by one, each of them receiving the entire saving he offers to the coalition formed just before him. All orders of formation of N are considered and intervene with the same weight $1/n!$ in the computation. The Shapley value can also be written

$$x_i = c(i) - \frac{1}{n!}\sum_S (s-1)!(n-s)![c(S\backslash\{i\}) + c(i) - c(S)].$$

The term between square brackets is the saving achieved by incorporating i to coalition S. The cost charged to i is consequently his individual cost less a weighted sum of savings.

The allocation, proposed by Shapley, for example 1, is

$$\bar{x} = (600, 450, 450).$$

It is represented by an S in the fundamental triangle of costs.

4.2. *The Nucleolus* (*Schmeidler*, 1969)

The nucleolus measures the attitude of a coalition towards a proposed allocation by the difference between the cost it can secure and the proposed cost Define the excess

$$e(\bar{x}, S) = c(S) - \sum_{i \in S} x_i$$

that measures the "happiness degree" of each coalition S. If the excess is negative, the proposed allocation is outside the core; if it is positive, the allocation is acceptable, but the coalition nevertheless has an interest in obtaining the highest possible $e(\bar{x}, S)$. The nucleolus is the imputation that maximizes (lexicographically) the minimal excess.

Let $z(\bar{x})$ be the vector (with 2^{n-1} components) of the excesses of all coalitions $S \subset N$ $(S \neq \varnothing, S \neq N)$, ordered by increasing magnitude. A lexicographic ordering of the vectors $z(\bar{x})$ [i.e., $z(\bar{x}) \geqslant_L z(\bar{x}')$ if $\bar{x}=\bar{x}'$ or if $z_k(\bar{x}) > z_k(\bar{x}')$ for the first component k for which $\bar{x}$ differs from $\bar{x}'$] defines a semi order L. The nucleolus is the first element (=the maximal element) of this semi-order: $z(\bar{x}) \geqslant_L z(\bar{x}')\ \forall \bar{x}'$. To compute the nucleolus amounts to award a subsidy δ, as large as possible, to each proper sub-coalition of N. So one has to solve the linear program

$$\max \delta$$

$$\sum_{i\in S} x_i + \delta \leqslant c(S) \quad \forall S \subset N, S \neq \varnothing, S \neq N,$$

$$\sum_{i=1}^{n} x_i = c(N) \qquad x_i \geqslant 0 \quad \forall i.$$

In the case of example 1, the maximal value of δ is 50; this leads to the same allocation

$$\bar{x} = (600, 450, 450)$$

as the one proposed by Shapley.

4.3. *The Proportional Nucleolus* (*Young et al.*, 1980).

The proportional nucleolus is obtained when the excess is defined by the formula

$$e(\bar{x}, S) = \frac{c(S) - \sum_{i\in S} x_i}{c(S)};$$

instead of granting the same amount to each proper coalition of N, a subsidy proportional to $c(S)$ is awarded. One has to solve the linear program

$$\max s$$

$$\sum_{i\in S} x_i \leqslant c(S)(1-s), \quad \forall S \subset N, S \neq \varnothing, S \neq N,$$

$$\sum_{i\in N} x_i = c(N) \qquad x_i \geqslant 0 \quad \forall i.$$

In the case of example 1, we obtain the allocation (denoted PN on the fundamental triangle)

$$\bar{x} = (1000, 0, 500):$$

all the profit of cooperation goes to the second player, who makes the most out of his veto right; without him, indeed, players 1 and 3 cannot achieve any saving.

4.4. *The Disruptive Nucleolus* (*Littlechild and Vaidya*, 1976) (*Michener, Yuen and Sakurai*, 1981)

For each allocation $\bar{x}$ define the propensity to disrupt for coalition S as the ratio between what $N\backslash S$ and S would lose if $\bar{x}$ were to be abandoned.

$$d(\bar{x}, S) = \frac{c(N \backslash S) - \sum_{i \in N \backslash S} x_i}{c(S) - \sum_{i \in S} x_i}.$$

The disruptive nucleolus is computed like the nucleolus, replacing $e(\bar{x}, S)$ by $d(\bar{x}, S)$: let $z(\bar{x})$ be the vector whose components are the $d(\bar{x}, S)$, $\forall S \neq \emptyset$, N, ranged in increasing order. By lexicographically ordering the $z(\bar{x})$, we obtain a semi-order; its first element is the disruptive nucleolus.

In the case of a 3-person game, we obtain the allocation

$$x_i = CM(i) + \frac{c(i) - CM(i)}{\sum_{j=1}^{3} [c(j) - CM(j)]} \cdot \left[c(N) - \sum_{k=1}^{3} CM(k) \right].$$

This leads, for example 1, to

$$\bar{x} = (600, 450, 450),$$

the same allocation as the Shapley value.

5. PROPERTIES

In Section 4, we have proposed 4 new cost allocation methods, that originate from game theory. Which of them should be selected? The study of the following theoretical properties will help us in this choice.

PROPERTY 1. *Collective rationality. The method should provide an imputation within the core (when it is non void).*

Examples 1 and 2 of Section 1 show that this is a very desirable property. An allocation outside the core effectively means that some departments are unwillingly subsidizing some others; therefore the department managers are enticed to quit the grand coalition and to have the work done outside the company. Allocations within the core are necessary to remove the incentive for sub-coalitions to act independently of the grand coalition.

By construction, the three lexicographic concepts always belong to the core. On the other hand, the Shapley value may fall outside. For instance, in the 3-person game defined by $c(1) = c(2) = c(3) = c(12) = 12$, $c(13) = c(23) = 20$, $c(123) = 23$, the Shapley allocation is $\bar{x} = (6\frac{1}{3}, 6\frac{1}{3}, 10\frac{1}{3})$, while the core is defined by the inequalities $3 \leqslant x_1$, $x_2 \leqslant 12$, $11 \leqslant x_3 \leqslant 12$. In the case of a convex game, however, the Shapley value always belongs to the core (Shapley, 1971); it even lies in its center, since it is the center of gravity of the core's extremal points.

PROPERTY 2. *Monotonicity in costs. All the players contribute to an increase in the project's global cost $c(N)$.*

More often than not, negotiations related to the allocation of the cost of a project take place before it is even started: an electric power company will accept to contribute to the cost of erecting a dam only if it knows in advance how much it will cost (or at least if a good estimation of the total cost is known). But it is rather infrequent that the final cost of a project is known as early as the first discussions: the general rule is rather that it exceeds the forecasts. The monotonicity property demands that each player participates to a rise in the total cost: it would be unfair to have a player benefit from an increase of $c(N)$ (it is assumed that $c(S)$, $\forall S \subset N$ is not modified).

The Shapley value is monotonic. Suppose $c(N)$ increases by a. In the expression

$$x_i = \frac{1}{n!} \sum_S (s-1)!(n-s)![c(S) - c(S \backslash \{i\})],$$

$c(N)$ appears only once, when i enters coalition $N \backslash \{i\}$ to form N. This term (and thus x_i)

$$\frac{1}{n!}(n-1)!1![c(N) - c(N \backslash \{i\})]$$

increases by $[(n-1)!/n!]a = a/n$. Consequently, any budget overstepping is spread evenly among the participants. This is open to criticism: it does not seem fair that all players must contribute equally to unforeseen costs, while their shares in the project may be very different; a "small" department, that only has to pay a small share of the initial allocation, gets the same increase as a "large" participant.

The proportional nucleolus is also monotonic: each increase of the global cost is shared among the players in proportion of their profit $c(i) - x_i$: this is intuitively far more satisfying (see Young, Okada and Hashimoto (1980) for the proof).

On the other hand the nucleolus and the disruptive nucleolus are not monotonic. In the case of the nucleolus, a counter-example was presented by Megiddo (1974). As for the disruptive nucleolus, consider the following example

$$c(1) = 4, \quad c(2) = c(3) = 6, \quad c(12) = c(13) = 7.5, \quad c(23) = 12, \quad c(123) = 13.$$

One verifies that the disruptive nucleolus proposes the allocation

$$\bar{x} = (1.75, 5.625, 5.625).$$

If we now put $c(123) = 13.1$, we obtain

$$\bar{x} = (1.727, 5.6865, 5.6865):$$

while the total cost of the project has increased, the contribution of player 1 has decreased.

PROPERTY 3: *Additivity. A subdivision of a player into two should not affect the allocation.*

Let $[N, c(S)]$ be an allocation game and $\bar{x} = (x_1, \ldots, x_n)$ the proposed allocation. Let $[N^*, c^*(S)]$ be the game that results from the splitting of the cost center j into two centers j_1 and j_2. The cost functions has to be such that, for all $S \subset N\backslash\{j\}$,

$$c^*(S) = c(S) \quad \text{and} \quad c^*(S \cup \{j_1\}) = c^*(S \cup \{j_2\}) = c^*(S \cup \{j_1, j_2\}) = c(S \cup \{j\})$$

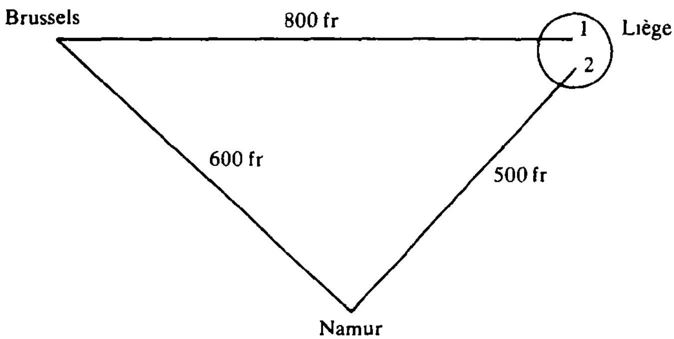
(in words: either fragment, in the absence of the other, incurs the same costs that the two together would incur). Then additivity demands that the allocation $\bar{x}^* = (x_1^*, \ldots, x_{j_1}^*, x_{j_2}^*, \ldots, x_n^*)$ satisfies

$$x_{j_1}^* + x_{j_2}^* = x_j,$$

while for the remaining players i,

$$x_i^* = x_i.$$

Example 2. An insurance company whose head office lies in Brussels wants to install two computer terminals in its local office in Liège, and one in Namur. The renting costs of the telephone lines are indicated in the following figure.



What amount should be charged to each local office? If we reason in terms of terminals, we face a 3-person game, with the cost function

$$c^*(1) = c^*(2) = c^*(12) = 800$$
$$c^*(3) = 600$$
$$c^*(13) = c^*(23) = c^*(123) = 1100.$$

If we think in terms of offices, we have a 2-person (L and N) game

$$c(L) = 800$$
$$c(N) = 600$$
$$c(LN) = 1100.$$

A solution concept is additive iff it amounts to the same to reason in terms of terminals or of offices. The values of the four concepts proposed in Section 4 are

	3-person game			2-person game	
	1	2	3	Liège	Namur
Shapley	350	350	400	650	450
Nucleolus	325	325	450	650	450
Prop Nucl	$314\frac{2}{7}$	$314\frac{2}{7}$	$471\frac{3}{7}$	$628\frac{4}{7}$	$471\frac{3}{7}$
Disr Nucl	$336\frac{16}{19}$	$336\frac{16}{19}$	$426\frac{6}{19}$	650	450

So the Shapley value and the disruptive nucleolus do not satisfy the property. The nucleolus and the proportional nucleolus are additive. Let us check this for the nucleolus in the case of a 3-person game (the proof is similar for the proportional nucleolus and can be easily generalized to any number of players).

Consider the 3-person game $[\{1,2,3\}, c(S)]$ and assume player 3 is split into 3_1 and 3_2 to form the 4-person game $[\{1,2,3_1,3_2\}, c^*(S)]$, where

$$c^*(1) = c(1)$$
$$c^*(2) = c(2)$$
$$c^*(3_1, 3_2) = c^*(3_1) = c^*(3_2) = c(3)$$
$$c^*(1,2) = c(1,2)$$
$$c^*(1,3_1,3_2) = c^*(1,3_1) = c^*(1,3_2) = c(1,3)$$
$$c^*(2,3_1,3_2) = c^*(2,3_1) = c^*(2,3_2) = c(2,3)$$
$$c^*(1,2,3_1) = c^*(1,2,3_2) = c^*(1,2,3_1,3_2) = c(123).$$

The linear program to compute the nucleolus of the 4-person game is
max δ

(1) $x_1^* + \delta \leq c^*(1)$

(2) $x_2^* + \delta \leq c^*(2)$

(3) $x_{3_1}^* + \delta \leq c^*(3_1)$

(4) $x_{3_2}^* + \delta \leq c^*(3_2)$

(5) $x_1^* + x_2^* + \delta \leq c^*(1,2)$

(6) $x_1^* + x_{3_1}^* + \delta \leq c^*(1,3_1)$

(7) $x_1^* + x_{3_2}^* + \delta \leq c^*(1,3_2)$

(8) $x_2^* + x_{3_1}^* + \delta \leq c^*(2,3_1)$

(9) $x_2^* + x_{3_2}^* + \delta \leq c^*(2,3_2)$

(10) $x_{3_1}^* + x_{3_2}^* + \delta \leq c^*(3_1,3_2)$

(11) $x_1^* + x_2^* + x_{3_1}^* + \delta \leq c^*(1,2,3_1)$

(12) $x_1^* + x_2^* + x_{3_2}^* + \delta \leq c^*(1,2,3_2)$

(13) $x_1^* + x_{3_1}^* + x_{3_2}^* + \delta \leqslant c^*(1, 3_1, 3_2)$

(14) $x_2^* + x_{3_1}^* + x_{3_2}^* + \delta \leqslant c^*(2, 3_1, 3_2)$

(15) $x_1^* + x_2^* + x_{3_1}^* + x_{3_2}^* = c^*(1, 2, 3_1, 3_2)$.

Given the symmetry of $c^*(S)$, $x_{3_1}^*$ will be equal to $x_{3_2}^*$ So conditions (4), (7), (9) and (12) are unnecessary. (10), that can be written $2x_{3_1}^* + \delta \leqslant c^*(3_1)$, is stronger than (3), so the latter can be deleted. Also (6) and (8) are superfluous, due to (13) and (14). Finally (11) is automatically satisfied, due to (15). Consequently only 7 constraints remain, namely

$$x_1^* + \delta \leqslant c^*(1) = c(1)$$

$$x_2^* + \delta \leqslant c^*(2) = c(2)$$

$$x_1^* + x_2^* + \delta \leqslant c^*(1, 2) = c(1, 2)$$

$$x_{3_1}^* + x_{3_2}^* + \delta \leqslant c^*(3_1, 3_2) = c(3)$$

$$x_1^* + x_{3_1}^* + x_{3_2}^* + \delta \leqslant c^*(1, 3_1, 3_2) = c(1, 3)$$

$$x_2^* + x_{3_1}^* + x_{3_2}^* + \delta \leqslant c^*(2, 3_1, 3_2) = c(2, 3)$$

$$x_1^* + x_2^* + x_{3_1}^* + x_{3_2}^* = c^*(1, 2, 3_1, 3_2) = c(1, 2, 3).$$

Setting $x_{3_1}^* + x_{3_2}^* = x_3$, these are the constraints of the linear program that computes the nucleolus of the 3-person game $[\{1, 2, 3\}, c(S)]$.

In summary, the only method that satisfies the three properties is the proportional nucleolus; we propose it as the best cost allocation method.

6. GAMES WITH EMPTY CORE

If the core of the game is empty, any cost allocation proposal is unstable, since at least one coalition has an incentive to back out of the group. Cooperation between the players is not spontaneous any more, it has to be enforced by an external authority. If one wishes to single out one point, it is necessary to relax some of the collective rationality conditions until a core appears. One can for instance impose a uniform tax ε to each proper subcoalition of N. The least core is obtained by computing the smallest acceptable tax by means of the linear program

$$\min \varepsilon$$

$$\sum_{i \in S} x_i \leqslant c(S) + \varepsilon \quad \forall S \subset N$$

$$\sum_{i=1}^{n} x_i = c(N).$$

If one feels that the tax has to be proportional to $c(S)$, one obtains the proportional least core by introducing a tax rate t and solving the program

$$\min t$$

$$\sum_{i \in S} x_i \leqslant c(S)(1 + t) \quad \forall S \subset N$$

$$\sum_{i=1}^{n} x_i = c(N).$$

Notice the similarity with the nucleolus and the proportional nucleolus: in one case coalitions are taxed in order to make the core exist, in the other case coalitions are subsidized in order to reduce the core to a single imputation.

Contrary to the nucleolus and the proportional nucleolus, the Shapley value and the disruptive nucleolus always exist, whether the core is empty or not.

7. COMMENTS

Cooperative game theory presently faces an interesting turning-point of its history. It was born out of practical problems of considerable importance; for instance engineers of the Tennessee Valley Authority (Ramsmeier, 1943), as early as 1930, have considered several cost allocation methods to share among the beneficiaries of the project the costs of improving the existing water communications and constructing dams. The concepts of core, nucleolus and disruptive nucleolus were formulated in an embryonic form, a quarter of a century before those notions were presented in game theory, several years before the publication of the celebrated book of Von Neumann and Morgenstern (1944).

As the problem of the repartition of scale economies occurs in so many commercial activities, it was by no means a surprise to witness the independent development, in numerous areas, of notions very close to game theory. So the disruptive nucleolus is called (in its 3-player version) the "separable costs remaining benefits method", the Gately method, the Louderback method, the Glaeser method, or furthermore the "alternate cost avoided method", depending on the kind of literature one consults.

This enormous duplication of scientific work fortunately seems to come to an end; the contacts between researchers of different areas are improving, the authors more and more explicitly refer to game theory (Hamlen, Hamlen and Tschirhart (1977, 1980), Jensen (1977)) to propose cost allocations. We may now have come full circle, since game theory begins to be applied to the kind of problems that created it.

Many practitioners (and actuaries) still consider game theory as a mathematical toy without any possibility of practical implementation. Let us undeceive them by mentioning several effective applications of solution concepts of game theory:

- —tax allocation among the divisions of McDonnell–Douglas Corporation (Verrechia, 1982)
- —repartition of the renting costs of WATS telephone lines at Cornell University (Billera, Heath and Raanan, 1978)
- —allocation of tree logs after transportation between the Finnish pulp and paper companies (Sääksjärvi, 1976, 1982)
- —maintenance costs of the Houston medical library (new books, periodicals, furniture) shared between the participating hospitals (Bres *et al.*, 1979)
- —financing of large water resource development projects in Tennessee (Straffin and Heaney, 1981)
- —construction costs of multipurpose reservoirs in the U.S. (Inter-Agency Committee on Water Resources (1958))

as well as several domains where a concept of game theory has been proposed

—depreciation problems in financial analysis (Callen, 1978)
—construction of an 80-kilometer water supply tunnel in Sweden (Young *et al.*, 1980)
—building of a power plant in India (Gately, 1974)
—subsidization of public transportation in Bogota (Diaz and Owen, 1979)
—landing fees at Birmingham airport (Littlechild and Thompson, 1977)
—allotment of water between agricultural communities in Japan (Suzuki and Nakayama, 1976)
—construction of a waste treatment center in the U.S. (Heaney, 1979)
—building of a water-filtering plant, financed by three "polluting" factories (Loehmann *et al.*, 1979, Bogardi and Sziderovski, 1976)

. . .

Also in insurance, the possibilities of application are numerous:

—allocation between companies of the costs
- of a professional union (like U.P.E.A. in Belgium)
- of a statistical bureau (like A.G.S.A.A. in France or Försakringstekniska Forskningsnämnden in Sweden)
- of risks supervision and claims appraisal in case of coinsurance;

—allocation between the different classes of a company of most operating costs (see Section 1).

Allocations based on game theoretical considerations have the only disadvantage of requiring more information, since it is necessary to obtain $2^n - 1$ costs $c(S)$, one for each non-void coalition of N.

8. A PROBLEM OF INTEREST ALLOCATION

EXAMPLE 3. The treasurer of ASTIN (player 1) wishes to invest the amount of 1 800 000 Belgian Francs on a short term (3 months) basis. In Belgium, the yield of such an investment is a function of the sum deposited.

Deposit	Annual interest rate
0–1 000 000	7 75%
1 000 000–3 000 000	10 25%
3 000 000–5 000 000	12%

Player 1 contacts the I.A.A. (player 2) and A.A.Br.* (Player 3) treasurers in order to make a group investment. I.A.A. deposits 900 000 fr in the commun fund, A.A.Br. 300 000 fr. How should the interests be split among the 3 players?

* Association des Actuaires issus de l'Université Libre de Bruxelles.

The solution always adopted in practice amounts to award the same yield (12%) to everyone. This allotment is acceptable, since it belongs to the core; it however implies perfect solidarity between the players, who all accept not to use their various threat possibilities. As this allocation is not the only acceptable one, it is interesting to compare the different methods. It is easy to check that

$$v(1)=46\,125$$
$$v(2)=17\,437.5$$
$$v(3)=5812.5$$
$$v(12)=69\,187.5$$
$$v(13)=53\,812.5$$
$$v(23)=30\,750$$
$$v(123)=90\,000.$$

Core:

$$46\,125 \leqslant y_1 \leqslant 59\,250$$
$$17\,437.5 \leqslant y_2 \leqslant 36\,187.5$$
$$5812.5 \leqslant y_3 \leqslant 20\,812.5.$$

Proportional repartition: 54 000 (12%), 27 000 (12%), 9000 (12%)

Shapley value: 51 750 (11.5%), 25 875 (11.5%), 12 375 (16.5%).

According to the Shapley value, the third player takes a great advantage from the fact that he is essential to reach the 3-million mark; his admission value is very high when he comes in last.

Nucleolus: 52 687.5 (11.71%), 24 937.5 (11.08%), 12 375 (16.5%).

The nucleolus, as generous towards A.A.Br. as the Shapley value, also takes into account the fact that ASTIN is in a better situation than I.A.A., since it can achieve a yield of 10.25% by playing alone, while I.A.A. would only make 7.75% in that case. Note that ASTIN and I.A.A. receive the same amount, in francs, over what they would have earned by playing alone:

$$y_1 - v(1) = y_2 - v(2) = \delta = 6562.5$$

Proportional nucleolus: 54 000 (12%), 27 000 (12%), 9000 (12%).

We obtain in this case the "intuitive" proportional repartition. We shall see later on that this is not always the case.

Disruptive nucleolus: 51 900 (11.53%), 25 687.5 (11.42%), 12 412.5 (16.55%)

The strategic possibilities of the players depend on the amounts they provide. Let us consider two variations of example 3.

EXAMPLE 3′

ASTIN:	1 700 000 fr
I.A.A.:	1 100 000 fr
A.A.Br.:	300 000 fr

Proportional repartition: 51 000 (12%), 33 000 (12%), 9000 (12%).

Shapley value: 48 395.83 (11.39%), 33 020.83 (12.01%), 11 583.33 (15.44%).

Nucleolus: 48 708.33 (11.46%), 33 333.33 (12.12%), 10 958.33 (14.61%).

Proportional nucleolus: 51 000 (12%), 33 000 (12%), 9000 (12%).

Disruptive nucleolus: 48 481.65 (11.41%), 33 106.65 (12.04%), 11 411.7 (15.22%).

Notice the effects of the more favourable situation of I.A.A., who owns more than a million and can achieve alone a yield of 10.25%: this improves its bargaining power.

EXAMPLE 3″

ASTIN:	1 700 000 fr
I.A.A.:	1 400 000 fr
A.A.Br.:	300 000 fr.

Proportional repartition: 51 000 (12%), 42 000 (12%), 9000 (12%).

Shapley value: 51 093.75 (12.02%), 43 406.25 (12.4%), 7500 (10%).

Nucleolus: 51 140.625 (12.03%), 43 453.125 (12.41%), 7406.25 (9.875%).

Proportional nucleolus: 52 378.37 (12.32%), 43 621 63 (12.46%), 6000 (8%).

Disruptive nucleolus: 51 127.01 (12.03%), 43 439.52 (12.41%), 7 433.47 (9.91%).

Notice the deep change: the share of A.A.Br., which is not necessary any more to reach 3 millions, is considerably reduced, even in the case of the proportional nucleolus.

The Shapley value and the nucleolus do not seem to be good solution concepts to this problem; in both cases the reasoning is performed in an additive way while the spirit of the problem is multiplicative. When two players form a coalition, the Shapley value simply shares the benefits of cooperation in two equal parts, and equal amounts do not lead to equal percentages. In addition to its theoretical properties, the proportional nucleolus proceeds in a multiplicative way, and seems more adapted to this specific problem.

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Finland is one of the only countries in which the solvency control of the non-life insurance companies is based on risk theory. The Finnish solvency legislation and the rules governing the equalization reserve are an example of how theory and practice may be combined in an outstanding way. The Finnish solvency system was introduced in 1953 where in general the solvency of the Finnish insurers was low. The introduction of the equalization reserve allowed the Finnish companies, free of tax, to equalize profit and loss in good and bad years by transference to and from the equalization reserve. Since the reserve was free of tax, it was necessary to introduce a specific transfer rule and to stipulate certain limits for the reserve. The equalization reserve was both regarded as a technical reserve and as part of the total solvency margin, which also includes the equity capital and underestimation of assets. The equalization reserve deals with the stochastic character of the insurance business and is used to equalize profit and loss in different years, whereas the total solvency margin has to be sufficient to safeguard the consumers' interest and must exceed a certain minimum solvency margin.

Since the introduction the system has functioned very satisfactory. The solvency of the insurers has improved, and the Finnish companies have been able to reduce reinsurance costs and to participate much more actively in the international insurance business to the benefit of the Finnish society and the Finnish consumers. The system has now functioned in almost thirty years, and since the previous revision was performed in 1965, it was decided in 1980 to review the entire system. For this purpose the Ministry of Social Affairs and Health (the Finnish supervising authority) appointed a project group to study the solvency problems in a broad sense and in particular to suggest new rules for the regulation of the equalization reserve. The chairman of the group was Teivo Pentikäinen, and the two volumes contain the extensive reporting from the project. Part I is designed to the general solvency aspects, whereas Part II contains the mathematical results.

The project group applied both empirical and theoretical methods in their work. Figures comprising loss ratios and the relative amount of the equalization reserve (relative to the earned premium) were collected for the Finnish non-life companies for the period 1962–1978. Similarly, loss ratios and the number of claims per insurance class were investigated. These figures all showed yearly variations, but an observation of great importance was the existence of cycles in the insurance result and the influence on the solvency. If the loss ratios are unfavorable in several consecutive years, the solvency margin may decrease tremendously since the solvency is affected by the accumulated bad results. This is illustrated by a major drop of more than 50% in the relative amount of the equalization reserve for the largest general companies in 1968–1974. These results highlight the impact on solvency of the cycles, and they should, therefore, explicitly be taken into consideration when the solvency problems should be discussed in details.

To do this and to make a realistic solvency study, a comprehensive theoretical model of a standard insurer has been constructed. The model takes several background factors into account, and it is stochastic in the way that the yearly claim amounts X are assumed to vary stochastically. In order to illustrate the different kinds of stochasticity which were observed in the empirical data, 4 levels of variation were introduced. The number of claims were assumed to be Poisson distributed, the claim size distributions were empirical, delivered by the Statistical Center of the Finnish Insurance Companies, short term variations in the basic parameters were introduced by allowing the expected number of claims (i.e. the Poisson parameter) to fluctuate from year to year. Finally, business cycles were introduced in a deterministic way by allowing the expected number of claims to vary along a sine curve with a wave length equal to 12 years and an amplitude equal to 10–15%. The other components in the model were the size of the insurer, the portfolio mix, the claim and premium inflation i_x and i_p, the interest rate earned on the reserves i_{tot}, the real growth rate i_g, the safety loading λ and the net retention.

Many of the basic parameters were estimated from the empirical data. Since the final results depend heavily on these values, it is worth mentioning some of them. The interest rate i_{tot} were 8.5%, claim inflation $i_x = 9\%$, portfolio growth $i_g = 6.1\%$, safety loading $\lambda = 4.1\%$ and the amplitude of the cycles in the loss ratios were estimated to 10%. Also the standard deviation and skewness of the short term variation of the Poisson parameter were estimated, but since the exposure (number of policies and information about different risk groups) was not included, the estimates may not be very reliable. And as a peculiarity, it was decided in the final recommendation to the Ministry to use standard deviations estimated from loss ratios, although they should describe the fluctuating Poisson parameter. This is, of course, unsatisfactory; it illustrates, however, the problems which arise when practically manageable systems have to be developed from limited empirical experience.

If one then in the model equals the premium earned and the investment income with the claims, the expenses and the change in the relative solvency margin/equalisation reserve one may obtain the following fundamental transition equation

$$u_1 = ru_0 + (\bar{f} + \lambda - f), \tag{1}$$

where $u = U/B$ denotes the relative solvency margin/equalization reserve and where U and B are the actual margin/reserve and premium earned, respectively. The other quantities are the actual loss ratio $f = X/B$, its mean $\bar{f} = E(X)/B$, and the safety loading λ r is the inflation and growth adjusted interest factor $r = (1+i_{tot})/\{(1+i_g)(1+i_p)\}$, and it is typical less than 1. This basic relation (1) is valid both when u denotes the relative solvency margin and the relative amount of the equalization reserve. But in the latter case λ and i_{tot} should be substituted by a loading coefficient a, and a nominal interest rate i_n, which both should be approved by the Ministry. In that case (1) becomes the transfer rule which regulates the flow of the equalization reserve. The equation (1) describes how the solvency ratio or the relative amount of the equalization reserve changes from year to year. It illustrates how it is increased by the investment income and by the safety loading, but reduced by inflation and real growth. The fluctuations are caused by the stochastic deviation of the actual loss ratio f from its mean $\bar{f}$.

From this relation (1) numerous simulation studies have been performed in order to evaluate the range of the fluctuations in the future solvency margin. Also analytical methods have been applied. The purpose of the study has not been to develop accurate forecasting models, but to study the consequences of an adverse development in the loss ratios whenever such a situation occurs. In the simulation, the yearly claim amount X were generated by a random number generator taking the different background factors and the different levels of stochasticity into account. Since the transition equation (1) depends on the actual loss ratio $f = X/B$, it is important to note that the calculation of the premium only takes portfolio growth and inflation into account. This means that the cycles are not taken into consideration, not even with a time lag. It implies that the premiums are not adjusted during a bad cycle period where the claim amounts may increase with up to 10–15% during a 6 year period. This assumption gives rise to an enormous increase in the minimum solvency margin and the increase may be of more than 50%. To illustrate some of these results it was found for the standard insurer that a minimum solvency margin equal to 42% of the premium was sufficient to ensure survival with 99% for a 10 year time span if the ruin barrier was 10% of premiums and if cycles were disregarded. The introduction of the cycles increased this minimum solvency margin from 42% to 94%. If the time span was reduced to 1 year, the figures were 25% and 39%, respectively. These figures illustrate the importance of the cycles, but they do also show the very high solvency requirements which are necessary to ensure the long term survival of the company. However, one would expect management to change policy if the solvency drops dramatically and the company shows a deficit in several years. Therefore, also a lower solvency margin ought to be sufficient to ensure the long term survival, but the study illustrates how business cycles may affect solvency in a severe way.

The cycles represent only one important element of the model; the books also contain an extensive study of how the solvency and the fluctuations in the equalization reserve are affected by changing for instance the portfolio mix, the net retention, the inflation, the growth rate, the safety loading and the time span of the study. All these factors influence the solvency more or less, and they are all important elements when an overall solvency policy has to be determined.

The ultimate goal of the solvency study was to revise the rules for the calculation of the minimum solvency margin and to design new limits for the equalization reserve. The new rule for the calculation of the minimum solvency margin is based on the same idea as the previous one, i.e., the minimum margin has to be so large that the company is able to pay the next years claims with a probability of 99%. Some of the constants in the formula for the minimum solvency margin have been changed slightly in order to take the new empirical experience into account. It is important to note that, compared with the current EEC-rules, the Finnish minimum solvency rule is often larger and that it explicitly takes into account the portfolio structure, reinsurance, and the stochastic character of the insurance business, whereas the EEC-rule is just a fixed percentage of premium income.

Concerning the equalization reserve the project group introduced the concept of a target zone. The upper limit of the target zone is dimensioned at a level which permits the equalization reserve to fluctuate between zero and the upper limit. In other words, the upper limit has been derived so that it represents the height of a 99% confidence region of the future flow of the equalization reserve. In more practical terms that means that in good years the companies by applying (1) are allowed to increase the equalization reserve to such an amount, that they are able to meet the liabilities during a bad period, where a cycle may deteriorate the solvency in several consecutive years. However, a situation may occur where the transfer rule (1) gives rise to an equalization reserve, which exceeds the upper limit. In that case the company is forced to reduce the transference, for instance by premium reductions, in order to keep the equalization reserve inside the target zone. A lower limit of the target zone has also been introduced, but it was made optional since the short term survival of the company is safeguarded by the minimum solvency requirements.

The new rules for the minimum solvency margin and the equalization reserve were introduced in 1981. As a technical reserve, the equalization reserve was before the revision not shown explicitly in the yearly accounts since it was regarded as part of the claim reserve. This situation has now changed, and it is explicitly shown together with a solvency indicator, which is the reserve in percentage of the upper limit of the target zone. This solvency measure is of course only a very rough measure, but it makes comparisons between companies possible, and it has (of course) attracted great public interest.

The reader will understand from this review that the two books contain numerous elements of interest. The Finnish solvency legislation deserves special attention since it is one of the most advanced in the world, and the recent solvency investigation is a fine example of how an extensive theoretical model may be used to study practical problems, and how the results may be implemented in practice.

H. RAMLAU-HANSEN

J. van Eeghen (1981): *Loss Reserving Methods*. Surveys of Actuarial Studies No. 1. Nationale-Nederlanden N.V., Rotterdam. 114 pages

A non-life insurance company receives premiums in advance of the risk period insured. At the end of that period it must have reserves to cover unsettled claims. The loss reserve at a given time is the expected present value of all future payments for claims which have arisen by that time and which may not even have been reported as of the assessment date.

Traditionally insurance companies estimate loss reserves by the case by case method where the claim files of all the outstanding claims are investigated once a year and a subjective assessment of the claim cost is made for each claim. It is obvious that this solution is expensive and time consuming. Therefore, there has been an ever increasing demand for actuarial models where you obtain a collective estimate of the loss reserve for all or most of the claims.

During the recent years many papers have appeared in the actuarial journals on loss reserving. It has become complicated and time consuming to get an overview over this important field of insurance mathematics. To lighten this task J. van Eeghen has assembled and arranged all the most important contributions to loss reserving as of April 1981. The book "Loss Reserving Methods" which is published by the Dutch insurance company Nationale-Nederlanden contains summaries of some 13 papers. The author has not intended to develop new ideas but here and there he puts forward critical comments. As a general rule, the order of presentation of the different methods obeys the following pattern: (a) Model and assumptions, (b) Comments on the assumption, (c) Comments on the data, (d) Computations, (e) Numerical example. The numerical example serves as a check for people who want to write a computer program for the estimation procedures. Unfortunately, it is not mentioned which portfolios the data originate from.

Some very simple methods such as the average value method use data which can be produced by any accounting department. However, it is a common feature of the actuarial methods that they use the so called run-off triangle as a starting point, and these methods also make some assumptions about the structure of the run-off pattern. A run-off triangle is a natural way to represent the payment history of several consecutive accident years. The figures in the cells of the triangle may represent different quantities such as claim numbers, total payments, average payments etc. Some of the actuarial methods are easy to apply and have been used with good results in practice. In this category we find the chain ladder method and Taylor's separation methods. The former method is based on the assumption that the columns in the run-off triangle are proportional apart from random fluctuations where the triangle is filled with cumulative loss figures. Taylor's separation methods is based on the assumption that the entries in the triangle are only influenced by a column effect and diagonal effect apart from random fluctuation. Here the run-off triangle is filled with average payments per claim. This method gives an estimate of the past insurance inflation.

The more complicated models which are summarized in the book are not widely applied in practice to-day. However, as there are still many unsolved problems in loss reserving, the researcher may here find ideas and inspiration to build upon.

The presentation of the various methods is clear and systematic, and the book represents, therefore, a valuable guide to the actuary who wants a survey of an important field of insurance mathematics.

P. LINNEMANN

Astin-groep Nederland (1982). *New Motor Rating Structure in the Netherlands Actuarial, Statistical and Market Aspects*. 128 pages

At the end of 1981, a new motor rating structure was introduced in the Netherlands, after an extensive research performed by a working group of Dutch non-life actuaries. The Dutch ASTIN Group had the excellent idea to make the results of this study available to the actuarial community.

In the introductory paper (*The Motor Insurance Market in the Netherlands*), G. W. DE WIT provides some statistical, commercial and economic background to motor car third party liability and accidental damage. In spite of the fact that the companies are free to set up their own premiums and conditions, tariffication in the Netherlands was based, industry-wide, on the following classification criteria: the catalogue value of the vehicle, the number of kilometers driven per year, the claims experience of the driver (with maximal discounts reaching 40%, or even 60%), and certain occupations. The companies gathered extensive statistical data (700 000 policies observed during one year, 80 000 claims), and appointed a study group to propose a new rating structure; one of the requirements was premium neutrality on a large scale: the premium volume had to be the same before and after the introduction of the new structure (of course on the policy-holder's level large modifications were to be expected; therefore the new structure had to be approved by the Government Insurance Board before implementation, like any premium increase).

The second paper (*Development of the Study*, by F. K. GREGORIUS) is the more important of the booklet, since it summarizes all the important steps of the study: collection of the statistical material, presentation of the methodology and of the main statistical results, construction and presentation of the new structure, modifications induced by market forces. The reading of this paper should be a "must" for any actuary interested in motorcar insurance, whether from a theoretical or a practical point of view. Indeed all compromises that had to be made between theory and practice are thoroughly explained; for instance the author is fully aware that the study group did not develop a "perfect" rating system, scientifically based in every respect. The main objective was rather to achieve an improved rating structure in the shortest possible term. First, all the possible rating factors are listed, and critically examined with respect to measurability, reliability, and usefulness For instance, common sense and some statistical studies show that the driver's carefulness or driving skill, his nationality and his annual mileage are among the most significant discrimination factors; however it is very difficult to introduce them in a rating structure, since (i) driving skill can hardly be measured, (ii) nationality is unlikely to be accepted in practice, as it will be considered as unfair discrimination, (iii) annual mileage is not reliable, since it is subject to fraud. Therefore one of the first tasks of the research is to replace those awkward variables by proxies, or strongly correlated variables For example, "weight of the car" is quite a good proxy of mileage, "age of the driver" a weak proxy of driving skill.

Once the variables have been defined and the data collected, the researcher faces the important step of choosing a statistical method that selects the significant variables and combines them into a tariff. A wide range of methods is available, from the sophisticated non parametric distribution-free methods (that are the least subject to criticism, but require extensive programming) to the simple uni-dimensional approach (very elementary, but statistically unsound since it does not fully take into account the interrelationships between the explaining variables). The Dutch research group has devised its own method, a heuristic approach based mainly on a one-dimensional approach (and therefore open to some criticisms, although in the tariff construction phase, efforts were made in order to consider the interdependence of the selected variables).

The selected variables are

- the weight of the car (an original idea, since most countries use "cylinder volume" or "engine power". The three variables were considered to be of more or less equal predictive power, so weight was selected for practical purposes).
- In accidental damage the weight of the car is replaced by the catalogue value.
- Territory (3 classes, with moderate surcharges and discounts (15% and 12%))
- Use (15% surcharge if more than 20 000 km/year)
- Claims experience: a rather sophisticated bonus-malus system was devised (again, using a very crude heuristic approach); it consists of 20 classes, with premiums ranging from 40 to 160; the penalization for one claim varies from 1 to 8 classes; the starting class depends on both the age of the driver and its occupation.

The proposed tariff was however considered to be too complicated to pass the commercial test. Hence modifications were proposed, affecting principally the bonus-malus system, whose number of classes was reduced to 14.

The remaining five papers of the study each develop a special topic. H. PRINS (*Collection and Processing of Research Data*) provides insight into the way the data of the participating companies were collected, and mentions the different problems that had to be solved (homogeneity and reliability of data from different sources, I.B.N.R., corrections for knock-for-knock agreements, cost allocation between third party and accidental damage, very large claims, . . .).

In *Vehicle Dependent Rating Factors*, F. RUYGT focuses on the selection of the best car related variable, among the following ones: weight, engine power, catalogue value, year of construction, cylinder volume. Considerations of practical nature, and an analysis of loss figures by means of regression analysis justify the decision to replace catalogue value by weight in third party liability. First it is shown that an important part of the variance of the observations remains unexplained when one uses the variables "year of construction" and "catalogue value". The regression models using "engine power" and "weight" conclude that (i) only one of those variables should be used (this is intuitively obvious), (ii) the multiple correlation coefficient is slightly better when weight is used. This result is quite surprising, since usually "engine power" is selected; although weight is unquestionably highly correlated to the claim frequency, one would have expected engine power to have the best discriminating power; indeed the prospective buyer of a car usually has several options concerning the engine; once the type and make of the car have been selected, a more powerful engine increases the car weight by only a few percentage points, but greatly influences the speed, hence the destructive power and the risk premium. One should however note that the result obtained by our Dutch colleagues relies on several assumptions:

• of course the usual assumptions of regression analysis: normality, homoskedasticity, and, above all, linearity (the analysis of the marginal means shows that this assumption could be criticized; maybe a quadratic model would provide a better fit);

• the regression analyses were performed on the averages per cell, and not on the individual observations. This certainly influences the results. In particular, one should not be surprised to observe very high multiple correlation coefficients (around 0.95! The same analysis, performed with the individual observations, would most certainly have presented correlations well below 0.10);

• the data was split into 132 cells (12 weight classes, 11 engine power classes). This completely arbitrary separation may have influenced the result.

So in order to definitely solve the question "engine power, or weight?", one should ideally apply other techniques than regression analysis; one could use non parametric models, that incorporate the split into cells in the selection procedure.

The new bonus-malus system is thoroughly analyzed by H. PRINS and F. ROOZENBOOM (*Bonus-Malus*). First, the importance of a posteriori rating is again stressed, and the past No Claim Bonus system is described. The very rich data collected by the companies allowed for a very detailed analysis of the claim frequencies. The authors have rightly considered that those frequencies could be used to build a system more appropriate to the Dutch situation than to apply one of the existing models of the actuarial literature. Indeed, those models use assumptions that are not quite realistic (time independent densities per insured, for instance); they do not provide any way to derive the transition rules and the number of classes; finally they do not make use of the very detailed information available to the study group. Therefore the authors have devised their own heuristic procedure, based on a simple comparison of the claim frequencies of various sub-groups.

An originality of the system is the special treatment of beginning drivers. In most countries the technically necessary higher premiums for young drivers are obtained by a constant surcharge. Here, a much more elegant solution was found: to insert the beginners on a less advantageous step on the bonus-malus scale.

The same remark applies to the risk factor "Profession". Differences between claim frequencies of different professions were translated into differences in starting classes in the bonus-malus system. This approach is by far more satisfying than simply to introduce fixed surcharges or discounts, since everybody will be treated equitably in the long run (there are farmers who are bad risks, and professional users that provoke few accidents; the only way to treat them fairly seems to introduce different starting classes, and let the bonus-malus system do the discrimination).

In the second part of the paper the efficiency, the discriminatory power and the minimum variance bonus scale are computed for various Dutch systems (some existing ones, and the proposed one). It is shown that the proposed system is by far the best, out of all the system tested, with an efficiency around 0.3 for the most common values of the claim frequency. By computing the stationary probability distribution (i.e., the asymptotic occupational frequencies in the classes), it is shown that the proposed system should lead to a far better spread of the policies in the classes (it is well known that a major disadvantage of most existing bonus-malus systems is that after a few years most policies tend to concentrate in the best classes).

The paper by J. VAN EEGHEN, J. NIJSSEN and F. RUYGT focuses on the very important topic of *Interdependence of Risk Factors: Applications of Some Models*. 3 methods for determining rate relativities between sub-classes when a multi-dimensional classification system is used, are investigated: the well known Bailey and Simon method, and two new ones: the method of marginal totals (the premiums should exactly compensate the incurred losses in the marginal distributions), and a variant, called the direct method. Definite advantages of the last two methods are presented. Of course many other methods have been presented in the literature, and it would have been very interesting to test them all (but that would have been a formidable task).

The same three authors also provided the last contrbution: *Does a Bonus–Malus System Always Lead to a Premium Crash? A Markovian Analysis.* It is indeed well known that, in most of the existing bonus–malus systems, the concentration of the policies in the best classes after a few years of operation, produces a drastic decrease of the premium volume. The total of all bonuses is (by far) not offset by the maluses. For instance, out of a theoretical premium income of 2062 millions francs, a Belgian company has awarded (in 1981) 651 millions of bonuses, and collected only 3 millions as maluses, an implicit average premium discount of 31.4%! A premium decrease is unfortunately inevitable for commercial reasons. With average claim frequencies nowadays close to 0.1, nine claim-free years should be necessary to offset the premium increase of a single accident, if one wants a financially balanced bonus-malus system. So the penalization for a claim should be at least nine classes, and this, however justified from an actuarial point of view, is very difficult to enforce, politically and commercially (moreover, such severe transition rules would strongly modify the claims pattern, since an enormous hunger for bonus would develop).

The calculation of the equilibrium distribution of the proposed Dutch bonus-malus system, using Markov chain theory, shows that this premium crash should not have too drastic consequences if—hopefully—economic conditions (like average claim frequency, composition of the portfolio, . . .) do not change too much: around one third of the policy-holders should ultimately find themselves in the best class.

Some considerations about the transition from the old bonus-malus system to the new one conclude this extremely interesting book.

J. LEMAIRE

J. Lemaire (1982). *L'assurance automobile: modèles mathématiques et statistiques*. 178 pages, FB 690 Bruxelles: Fernand Nathan, Editions Labor

This book on third-party automobile insurance is divided in four parts. The first part, which is non-mathematical, gives a description of the automobile insurance system in Belgium. This is also performed by means of tables with real empirical data. Furthermore, the situation in other countries is used for comparison purposes. Clearly, this first part forms a colourful introduction for the remainder of the book.

The second part addresses itself to the a priori classification of risks. It makes use of some elementary mathematics and statistics. An important topic which is discussed here is the question whether to study the number or the amount of the claims. The dependence of the average claim size on the number of claims is clearly presented and illustrated with real data. The choice and selection of explanatory dummy variables to classify the risks is discussed. This results in a linear scoring rule. This result is more or less based on the traditional assumptions of the standard linear model. The appropriateness of these assumptions for analyzing risk statistics is correctly criticised. The possibility of using generalized linear models, which pay more attention to the stochastic specification of the model, is not mentioned, however.

The third part makes more heavy use of mathematics and statistics. This part is on bonus-malus systems: the a posteriori classification of risks.

First some models for claim frequency data are presented and compared with real data. After that, a construction of an optimal bonus-malus system is given. The choice of an *optimal* system needs the specification of a loss function, as used in statistical decision theory. Various loss functions are presented and the implied behaviour of the optimal bonus-malus systems is given. Clearly, if the "optimal" bonus-malus system does not behave the way we like, something must be wrong with the specification of the loss function.

A very interesting chapter is on the possibility to take into account the severity of the claims: bonus-malus systems only utilize the number of claims, not their severity. A simple model is derived which "translates" severity by recognizing claims to be material claims or bodily-injury claims. This results in a simple rule to value a bodily-injury claim as a multiple of material claims.

The efficiency of bonus-malus systems is also discussed. Perhaps a reference to Borgan, Hoem and Norberg (SAJ, 1981) would have been in order.

A very important topic is on the behaviour of the policyholder. A decision problem which the policyholder has to face—to claim or not to claim—is formulated and applied to the situation in Belgium.

The final part focusses on the adequate calculation of the provision for incurred losses, reported or not yet reported. The importance of the adequate calculation of this provision, especially in third-party automobile insurance, is clearly emphasized. A presentation of the chain-ladder method, the separation method and a least squares method is given. The author correctly recognizes that all these methods are deterministic in the sense that they do not consider a stochastic process, which generates the data. All methods, including two variations of the chain ladder method, are applied to the same empirical data set and compared with each other.

The appeal and virtue of this book is in its use of empirical data as well as mathematics and statistics which remains on an elementary level. High-brow procedures are avoided, emphasis is on exposition and presentation. This gives this book a problem solving oriented flavour.

I think that this book is a worthy addition of the literature on modelling in automobile insurance.

P. TER BERG

17TH ASTIN COLLOQUIUM, LINDAU (WEST GERMANY), 2–6 OCTOBER 1983

The scientific part of the Lindau meeting was made up by four working sessions, one of them reserved for the traditional "Speaker's Corner" and the other three for the main subjects of the Congress.

Subject 1: The influence of different risk sharing arrangements on the risk behaviour of the participants in the direct insurance and reinsurance markets.
Subject 2: Data problems, statistical methods and numerical procedures in non-life insurance.
Subject 3: Planning and forecasting the technical and non-technical results of an insurance company.

Planning and forecasting—the subject of the first working session—are certainly well established in the daily practice of insurance companies. Regarding the papers submitted to the Colloquium, it is interesting to see that most of them deal only with pure mathematical forecasting especially of technical results and necessary technical reserves. This indicates that actuaries are not so much involved in the planning process as a whole and in the forecasting of non-technical results of an insurance company. Only the paper by STRAUB deals with the question "what can the actuary do in corporate planning?" He gives several examples out of his practical experience in a reinsurance company and considers the special case of the so-called "Cat Fund", i.e. the determination of the necessary risk capital for limiting the risk of a portfolio in a reasonable way.

Other tasks which actuaries can tackle in the planning process are:

— the breakdown of overall risk capital into subportfolios,
— improving scarce statistical material by simulation methods;
— comparing actual to planned figures (judging the "credibility" of the profit centre planning);
— quantifying the change in IBNR.

This latter problem of estimating the claims reserves is certainly one of the most prominent actuarial problems, of today. In this working session, the subject "claims reserves" was dealt with in the papers by DE FERRA, SÖDERSTROM and HERTIG. In the paper by DE FERRA an idea of Hachemeister has been taken up to describe the evolution of a claim by a Markov stochastic process. From a theoretical point of view, this model is very appealing and first empirical tests have shown a remarkable stability of the transition probabilities. The approach will certainly be pursued.

The paper by SODERSTROM gives some practical calculation methods for the determination of reserves in group sickness insurance. It should be mentioned, however, that in this class of business with its extensive population and its homogeneous claims data estimation of reserves is relatively easy. The interesting paper by HERTIG deals with the estimation of reserves in marine reinsurance.

He uses a lognormal distribution for the logarithmic increments of the loss ratios of consecutive years of development. This paper is a further indication that more and more reserving methods giving confidence intervals for the estimation of the necessary reserve are used, the statistical model assumptions of which can be tested.

Looking at the different purposes for which forecasts are made in insurance business, one of the most important is the determination of solvency reserves. At the colloquium NORBERG and SUNDT reported on a proposed system for solvency control at present discussed in Norway. They emphasized the following aspects of solvency control:

— sufficiency of the technical reserves;
— rules for the valuation of the assets;
— regular control;
— priority to insurance claims in case of bankruptcy;
— unified system for the reporting of statistical data.

The paper gave rise to an interesting discussion on the different aspects of solvency control, the impact of fluctuations in the non-technical results, business cycles and the role of the supervisory authorities. Business cycles were also the subject of a short paper by BOHMAN which was included in the discussion. A sophisticated forecasting model concerning premium rating formulas was presented in the paper by RANTALA where the framework of Kalman-filter-techniques was used to derive premium rating formulas which minimize premium fluctuations under given constraints on the solvency margin. Concluding remark to this working session could be the statement that there is still a long way to go before the non-technical aspects of insurance will be incorporated in actuarial methods in such a way that they are helpful to solve practical problems of insurance economics.

The second working session of the Congress discussed Subject 1 "The influence of different risk sharing arrangements on the risk behaviour of the participants in the direct insurance and reinsurance markets."

Risk sharing arrangements are the daily practice of insurance and reinsurance and there are important questions to be answered in this context, for example:

Which forms of risk sharing are to be chosen?
What should be the retention of each party?
How should risk premiums (loss expectancy and risk loading) be calculated?

Recently, particular progress has been made in the calculation of the loss expectancy (recursive algorithms, stop-loss premiums) and a number of premium principles for calculating the risk loading has been proposed. Although quite a number of actuarial theorems is known concerning risk sharing arrangements (the results on the optimality of different risk sharing arrangements by Borch, Buhlmann, Arrow *et al.* under special assumptions should be mentioned), the results of risk theory in this field have still been unsatisfactory under a practical point of view:

— premium calculation models are unrealistic as they do not include investment income and administration costs;
— there is a considerable amount of uncertainty in estimating the loss distributions ignored in the models of risk sharing,
— there exist accumulations of risk by the risk-accepting party;
— aspects like negotiation power are neglected;
— mostly risk-sharing is only regarded for one period;
— the choice of the suitable optimality criterion is still open and not fully discussed.

Some of these shortcomings of the existing models had been incorporated in the more detailed description of this subject in the hope that investigations into more realistic models would be carried out.

It has to be said that the papers presented under Subject 1 do not deal so much with these shortcomings from a practical point of view as with aspects of the three questions asked at the beginning.

Three papers deal with the most important form of non-proportional risk sharing, the deductible. In the paper "A note on an aspect of dangerousness of deductibles..." ALBRECHT criticizes the application of the coefficient of variation as a measure of risk.

He proposes the evaluation of risk by methods of utility theory. This led to a controversy stated in the paper by MACK and in a second paper by ALBRECHT. The discussion which followed the presentation of these papers can be summarized in the way that there is only a contradiction between the evaluation by utility theory and by the coefficient of variation when they are used as a measure for the same definition of dangerousness of claims distributions.

The paper by BORCH discusses the question how the safety loading has to be calculated. Since none of the numerous premium principles developed has found general acceptance, he attempts to clarify whether under certain market conditions rational behaviour may lead to a premium principle that is valid for all insurance companies. While BORCH regards the situation of a symmetrical risk exchange pool, the model of GERBER examines the situation where a portfolio is passed on from one insurer to exactly one reinsurer and so on. A hierarchical chain of companies thus shares the risk whereby only proportional risk sharing is regarded. The amount ceded and the loading factor in the premium are determined by a bargaining process. The results are very informative and helpful for the further investigation of this risk sharing problem.

While Gerber's paper deals with optimality investigations for forms of proportional risk sharing, in practice there are often non-proportional forms of risk sharing for which even the calculation of expected claims causes great difficulties: These problems are the topic of the papers by KREMER and MACK. KREMER discusses the largest claim reinsurance and its generalizations. The special significance of his paper lies in its theoretical content as the results important for practical forms of reinsurance were already given in a former paper by Kremer.

Under various assumptions on the claims distribution, especially looking at the case of a log-normal distribution, MACK treats the case where in addition to the deductible an annual limit on the aggregate loss exists. He examines the influence of changes in the model parameters and arrives at rating curves which are useful for the underwriting practice.

A quite different approach to the question of risk-sharing is taken by HELTEN and BECK. They have tried to analyse the present risk-sharing behaviour of German direct insurers by means of a questionnaire. In their paper, they report on the answers given in respect of the objectives pursued by the companies when taking reinsurance.

Subject 2 was entitled "Data problems, statistical methods and numerical procedures" and its heterogeneity was reflected by the papers presented. It is especially noteworthy to mention that quite a number of papers analysed empirical data with rather advanced statistical methods. A rough classification can be achieved by grouping them into "Statistical methods and statistical analysis of empirical data" and "Numerical procedures". To the first group belong papers on the analysis of claim numbers, the analysis of motor insurance problems and on the analysis of fire claims.

In the paper by ALBRECHT "Credibility for claim numbers . . ." an evolutionary credibility model (the underlying risk parameter changes in time) for the successive claim numbers of a single risk is examined. Its central result is that—in the case of the sequence of risk parameters being an arbitrary weakly stationary process—it is possible to calculate the coefficients of the credibility estimator (not the estimator itself!) recursively, as well as the mean square error. The paper examines the problem of estimating the structural parameters from collective data and considers various special cases.

In their paper on the analysis of claim numbers, AJNE and ANDERSSON use a particular ARIMA-model to forecast future claim numbers. The basic data consist of 84 monthly claim number figures (1975–1981) of householders comprehensive and of motor hull insurance. The authors report on some performed forecastings and their a posteriori comparison of estimated and true values indicates a reasonable performance.

In his paper on motor premium rating, COUTTS deals with nearly every aspect relevant to motor premium rating (forecasting, constructing tariff classes, expense allocation, marketing aspects, surplus analysis). Much emphasis is given to the treatment of practical problems arising when analysing company portfolios, a problem of special importance in countries where the tariff structure is not determined by supervisory authorities or insurance associations. In his paper, ALTING VON GEUSAU describes a model for analysing the effects of different bonus-malus-systems. He uses data from the Italian BM-system to demonstrate the usefulness of his model for answering different questions on premium development in time, i.e. to investigate whether the premium income remains adequate while the insured move to higher bonus classes.

The paper by RAMLAU-HANSEN reports on an empirical analysis of fire claims for single family houses from a major Danish non-life insurance company. As individual claim amount distribution a log-gamma distribution with a Pareto-type tail is used. In addition, a kind of graduation is performed by assuming that the expected claim amount is linearly increasing with the size of the house. The claim number distribution used is a Poisson distribution with a particular form of the parameter taking into account policy years and reporting time. Results on the linear dependence of the net risk premium on the size of the house, on the standard deviation of the total claim amount, on the skewness of the distribution and on the necessary risk loading are given, which demonstrate very clearly the big risk of fire portfolios.

During the last years, risk theory has made considerable progress in the determination of the claims distribution by numerical procedures through the application of recursive methods. A computing method has been developed by BERTRAM using properties of characteristic functions and the tool of the fast Fourier transform for numerical performance. While the recursive methods have considerable problems with negative risk sums (a case relevant for pension insurance), BERTRAM's method exhibits no such problems and, in addition, consumes very little computer time. This effect will be especially useful, if the Poisson intensity is large.

The paper by ETTL proves an interesting theoretical feature in this context. Starting from the well-known relation between the Laplace transform of the claim amount distribution and the aggregate claim distribution, he arrives at an integral equation for the accumulated claims distribution, a discretization of which leads to a recursive formula. Interesting theoretical results are also derived in the paper by NETZEL where the influence of different factors on the probability of ruin for an infinite time horizon is investigated. An integro-differential equation for the general problem is presented and in addition a closed expression is obtained for the case of an exponential distribution and an infinite retention. The probability of ruin is also dealt with in the paper by GOOVAERTS and DE VYLDER. They develop a stable (there exists a bound for the rounding errors) recursive algorithm for the calculation of the probability of ruin for an infinite time horizon and a fixed initial capital. The approximation error for the true ruin probability can be made arbitrarily small.

Summarizing Subject 2 it can be said that not only risk theory has been developed further in the last years, also the application of risk theoretical models to practical problems has made significant progress.

Last, but not least there were quite a lot of different papers presented at "Speaker's corner" during the colloquium. These range from papers of more theoretical interest discussing extension of risk theoretical models like ALBRECHT "Laplace transform . . .", JANSSEN/REINHARD and REICH to papers discussing practical problems like premium calculation for bank robbery and spoliation insurance (PÉREZ). Of particular interest are the papers by LEMAIRE dealing with the problem of the cost loading included in a commercial premium rate. He demonstrates that the proportional loading—mostly used in practice—will often result in an unfair allocation of the expenses to the different risks. He shows that the problem of cost allocation can be dealt with in a game-theoretical framework. The problems of determining a reasonable cost allocation is equivalent to determining an imputation of the core of a cooperative N-person game. This correspondence allows to apply results of game theory to derive special cost allocations. Lemaire shows that only one of the allocation methods regarded by him satisfies a set of reasonable postulates.

Besides the working sessions there were two lectures held during the colloquium. The first was by Professor DANNER on "the structure of risk classification in motor own damage insurance and its influence on motor car construction". Professor DANNER was engaged in the construction of a tariff for motor own damage insurance in Germany and reported on the tariff class construction depending on the repair costs of the specific models and on the claims frequency caused by the drivers of these models. In particular he pointed out that this tariff caused car manufacturers to put more emphasis on lower repair costs when constructing new models.

The second lecture was given by Professor FEILMEIER on "Numerical methods in calculating the aggregate loss distribution". He summarized the significant progress made in this field during the last years and commented in particular on the recursive methods and on the method using the fast Fourier transform. He underlined that the problem of numerical calculation of the aggregate claim distribution should no longer prevent anyone from using risk theoretical models in practice. This summary seems to be typical for the whole colloquium as most speakers emphasized the necessity of incorporating the well-developed risk-theoretical methods into the solution of practical problems, a classical concern of ASTIN.

P. ALBRECHT, K. FLEMMING, E. KREMER and T. MACK

Subject 1: The influence of different risk-sharing arrangements on the risk behaviour of the participants of the direct and reinsurance markets

P. Albrecht, A note on an "aspect of dangerousness" of deductibles—a criticism of the coefficient of variation.

Increasing risk and deductibles.

K. Borch, Equilibrium premiums in an insurance market.

H.U. Gerber, Chains of reinsurance.

E. Helten and D. Beck, Optimal reinsurance—a scientific fiction?

E. Kremer, An asymptotic formula for the net premium of some reinsurance treaties.

T. Mack, Premium calculation for deductible policies with an aggregate limit.

The utility of deductibles from the insurer's point of view.

Subject 2: Data problems, statistical methods and numerical procedures in non-life insurance

B. Ajne and K. Andersson, A note on time series analysis of numbers of claims.

P. Albrecht, Credibility for claim numbers in the case of a time dependent structure function: an application of doubly stochastic Poisson sequencies.

B. Alting von Geusau, The matrix method to solve motor insurance problems.

J. Bertram, Calculation of aggregate claims distributions in case of negative risk sums.

S. Coutts, Motor premium rating.
W. Ettl, Recursive formulas for compound distributions by Laplace transformation methods.
M. Goovaerts and F. De Vylder, A stable recursive algorithm for evaluation of ultimate ruin probabilities.
C. Netzel, Numerical study concerning ruin probability.
H. Ramlau-Hansen, Fire claims for single family houses.

Subject 3: Planning and forecasting technical and non-technical results of an insurance company

H. Bohman, Business cycles.
C. de Ferra, A stochastic model for the analysis and evaluation of the claims reserve.
J. Hertig, A statistical approach to IBNR-reserves in marine reinsurance.
N. E. Masterson, Non-life insurance short term forecasting.
R. Norberg and B. Sundt, Draft of a system for solvency control in non-life insurance.
J. Rantala, Experience rating of claims processes with stochastic trends.
L. G. Söderström, A practical application of an IBNR process for an almost stationary business.
E. Straub, Actuarial remarks on planning and controlling in reinsurance.

Speaker's Corner

P. Albrecht, Laplace transforms, Mellin transforms and mixed Poisson processes.
J. Janssen and J. M. Reinhard, Formes explicites de probabilités de ruine pour une classe de modèles de risque semi-markoviens.
W. S. Jewell and R. Schnieper, Credibility theory for second moments.
J. Lemaire, An application of game theory: cost allocation.
The influence of expense loadings on the fairness of a tariff.
A. Martinez Vazquez, Le test d'adherence des fonctions de repartition à type discrete dans l'assurance non-vie.
E. Prieto Pérez, Analysis of bank robbery and spoliation insurance.
A. Reich, Premium principles and translation invariance.

ANNOUNCEMENTS AND NEWS NOTES

1. This issue of *Astin Bulletin* is the first one of Volume 14, which originally was scheduled for publication in 1983, whereas now it is associated with 1984. The reason for a redesign of the publication schedule of our journal originates in the fact that *Astin Bulletin* has been structurally half a year behind on publication schedule for the past years, which was due to a lack of papers of sufficient quality to be published in our journal.

   We like to publish only papers which are worth reading and rereading.

   The second issue of Volume 14 is planned for October 1984.

2. Recently, the *Index Astin Bulletin 1957–1982* has been published. We hope that this index will be of value in searching for earlier contributions in the field of non-life insurance and risk theory.

   The new rules of ASTIN also are included in this index.

3. At the 17th Astin Colloquium in Lindau (West Germany) Charles A. Hachemeister (USA) was elected as a member of the Committee of Astin.

   Jan Jung (Sweden) and Ragnar Norberg (Norway) resigned as members of the editorial board of our journal. Jan Jung has been an active member of this board as from 1976 and the present status of Astin Bulletin owes a lot to him. Although Ragnar Norberg only served for three years as a member of this board, the level of his excellent referee reports will remain a model for the future.

   In order to avoid any attrition of the editorial board of *Astin Bulletin*, the Committee of Astin has appointed Björn Ajne (Sweden), Jukka Rantala (Finland) and Bjørn Sundt (Norway) as new members of this board.

P. TER BERG

OBITUARY

R. E. BEARD

11th January 1911–7th November 1983

On 7th November, 1983 the actuarial profession lost one of its brightest lights when Mr. Robert Eric Beard, "Bobbie" to all who knew him, died. He will be best remembered for his contributions to Non-Life Insurance, both in the wide variety of the papers which he wrote, and in his enthusiastic participation at meetings. He always gave the impression of a man with a thousand ideas all trying to tumble out at once and many of his friends had difficulty in keeping up with his thoughts, but always the smile and the twinkling eyes were there to tell you of the warmth of his personality.

Bobbie Beard became a fellow of the Institute in 1938. During the Second World War he was one of a select band of very clever people working in the Admiralty and was closely involved with the early development of techniques such as operational research which in those days was classified as secret. After the war he readily gave his services to the Institute of Actuaries and served as Examiner from 1945 to 1948, and was a member of Council for thirteen years between 1951 and 1965. During that time he was Honorary Secretary from 1959 to 1961 and Vice President from 1962 to 1965.

He presented two papers for discussion at the Institute, and the Journal of the Institute contains a number of his notes, mostly on mathematical subjects. He also contributed papers to International Congresses of Actuaries, to the Students' Society, to the Royal Statistical Society, to ASTIN, to the Assurance Medical Society and to other bodies. He took a leading part in the development of the mathematical theory of risk and in recognition of his contribution to a symposium on this subject in Stockholm in 1968 was awarded a bronze medal by the organizers, the Filip Lundberg Foundation for Scientific Research. He was joint author with Pentikainen and Pesonen of a textbook on Risk Theory published in 1969. He travelled widely and gave talks to actuarial bodies in Australia, Denmark, Finland, Holland, New Zealand, Norway, South Africa and Sweden. He was corresponding member of the Association Royale des Actuaires Belges. He was nominated for the Council of the Institute of Mathematics and its Applications and was a Vice-President of the British Cancer Council. For all these services he received the Institute of Actuaries Silver Medal in 1972.

Bobbie played a central role in the founding of ASTIN. From its beginnings until 1967, that is for ten years, he performed the very important task of the Secretary of the ASTIN Committee. In 1959 and 1960 he was, in addition, editor of the *Astin Bulletin*, and from 1962 to 1964, Chairman of ASTIN. He continued to place his wide experience and highly valued counsel at the disposal of the Committee until 1977.

It is not easy within the compass of a few remarks to pay adequate tribute to such prolific activity. Basically it derives from Bobbie's roving and enquiring mind, his enthusiasm for research and his interest in mathematics. His earlier contributions were largely mathematical in the field of graduation. Later, however, he applied his mathematical ability to problems of Non-Life Insurance. Development of ASTIN and of its Bulletin owes much to his enthusiasm. But it would do less than justice to Bobbie's services to the profession if the impression was left that they consisted largely of mathematical research. He was concerned, particularly in recent years, with the practical application of actuarial methods of Non-Life Insurance and was a most valuable ambassador for the profession in persuading those with practical experience in that field of the contribution that actuaries can make to the financial steering of Non-Life business.

Bobbie was an enthusiast in all he did and was always ready to help the profession both on his own initiative and in response to any request made to him. To go to him for advice and comment was always a stimulating experience because he always had something new and interesting to say about a problem.

He will be sadly missed by all his friends and the profession is the poorer for his loss.

F. E. GUASCHI

Photograph shows the presentation of Silver Medal to Mr. R. E. Beard, O.B.E. (centre), by Mr. R. S. Skerman (President) June 1972.

INSTRUCTIONS TO AUTHORS

1. Papers for publication should be sent in triplicate to:

   Peter TER BERG, Hildebrandhove 38,
   2726 AW Zoetermeer, Netherlands.

   Submission of a paper is held to imply that it contains original unpublished work and is not being submitted for publication elsewhere.
   Receipt of the paper will be confirmed and followed by a referee process, which will take about three months.

2. The first page of each paper should start with the title, the name(s) of the author(s), an abstract of the paper as well as some major keywords.
   An institutional affiliation can be placed between the name(s) of the author(s) and the abstract.
   Acknowledgements and grants received should be placed as a footnote, which is not included in the count of the other footnotes.
   Footnotes should be kept to a minimum

3. Manuscripts should be typewritten on one side of the paper, double-spaced with wide margins.

4. Tables should only be included if really essential Tables should be numbered consecutively. Do not use vertical lines!

5. Upon acceptance of a paper, any figures must be drawn in black ink on white paper in a form suitable for photographic reproduction with a lettering of uniform size and sufficiently large to be legible when reduced to final size.

6. Important formulae should be displayed and numbered on the right hand side of the page. In mathematical expressions, especially in the text, authors are requested to minimize unusual or expensive typographical requirements This may be achieved by using the solidus instead of built-up fractions and to write complicated exponentials in the form exp( ).
   Matrices (uppercase) and vectors (lowercase) will be printed boldface. Boldface should be indicated by wavy underlining.

7. References in the text are given by the author's name in capitals, followed by the year of publication between parentheses.
   Examples: ROTHENBERG (1973) has done.. ; . should be referred to FOLKS and CHHIKARA (1978);... as shown by DE FINETTI (1937).
   At the end of the paper the references should be grouped alphabetically and chronologically. For journal references give author(s), year, title, journal, volume and pages. For book references give author(s), year, title, city and publisher. Illustrated with the above-mentioned references, this works out as.

FINETTI, B. DE (1937). La prévision, ses lois logiques, ses sources subjectives, *Annales de l'Institut Henry Poincaré* **7**, 1–68. Reprinted as: Foresight its Logical Laws, its Subjective Sources, in H. E. Kyburg and H. G. Smokler (Eds) (1980). *Studies in Subjective Probability.* Huntington: Krieger Publishing Company, Inc.

FOLKS, J. L., and R. S. CHHIKARA (1978). The Inverse Gaussian Distribution and its Statistical Application—A Review (with Discussion), *Journal of the Royal Statistical Society B* **40**, 263–289.

ROTHENBERG, T J. (1973). *Efficient Estimation with A Priori Information.* New Haven: Yale University Press.

Observe that abbreviations should not be used!

COMMITTEE OF ASTIN

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