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ASTIN BULLETIN

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ASTIN BULLETIN started in 1958 as a journal providing an outlet for actuarial studies in non-life insurance Since then a well-established non-life methodology did result, which is also applicable to other fields of insurance For that reason ASTIN BULLETIN will publish papers written from any quantitative point of view — whether actuarial, econometric, engineering, mathematical, statistical, etc — attacking theoretic and applied problems in any field, faced with elements of insurance and risk

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ASTIN MEMOIRS

By Paul Johansen

Nous avons pensé que les assurances contre l'incendie pouvaient aussi s'étayer du calcul. (T. J. Barrois, 1835)

To a mathematician 25 is a very uninteresting number: Just the square of a small prime. For a historian it means an epoch. If 25 years bring us back to one of the cataclysms of recent history, then it is worth looking back at them. (S. Vajda, 1964)

1

A few years ago at the ASTIN colloquium in Washington Robert Beard presented a paper about the early history of ASTIN. The paper is published in volume 9 of the *Astin Bulletin*. 25 years ago Beard was in the epicenter of the events or rather he was the epicenter himself. Therefore, it could be regarded as superfluous if so soon after I give my own version of the same events.

In the home of my grandparents I had much fun from an old-fashioned stereoscope. Looking into two glasses at two photographs taken of the same object from slightly different positions, your view gained a new dimension. In other words you could look behind the nearest objects and feel a certain depth in the picture. It is my hope that the reader being familiar with Beard's story and now reading a new version seen from a slightly different personal point of observation, might gain a more detailed view of what happened.

But let us start from the beginning. Several old friends have supported and supplemented my memory, and I am very grateful to them.

2.1

The early actuarial activities in the field of life assurance are well known. The investigations of the statistics of human mortality, the mathematical model of the mortality table, and the invention of commutation symbols made possible the mathematical foundation some 200 years ago of scientific life assurance. Each step of the evolution was smoothly communicated to specialists in various countries, and thus the aspect of the actuary as a life assurance mathematician developed during the last century. The Institute of Actuaries was founded in London in 1848, and several other national associations were born during the next generation. An international Permanent Committee was established in 1897 in order to promote actuarial collaboration across borders.

2.2

Life assurance was not the only field of mathematical activity in insurance. One phenomenon which from of old attracted the attention was how the fire risk varies with the size of the exposed objects. This forms a striking parallel to the variation of mortality risk with age. Tariffs based upon fire risk premium rates increasing with the sums insured were introduced in Denmark in 1857. Even before that—in 1835—a French industrialist, Théodore Joseph Barrois had worked out a mathematical model describing this phenomenon. The same model was independently published by a Russian, Sergius von Sawitsch in 1909. Much later similar results are presented by various other authors, including P. O. Berge and myself. In Italy, d'Addario studied the simultaneous variation of outbreaks of fires and average damage degrees. His results were published in 1940. It is characteristic that in the beginning each author was acting alone. Each knowing little of the works of his predecessors had to start practically from scratch. As a result the total effect of all the efforts was not much greater than that of each individually. The multiplicative effect of collaborating in the research did not come to work.

2.3

In the United States quite a different type of risk was studied by a large group of mathematicians and statisticians working in the field of workmen's compensation insurance. According to the law at that time in the States, this insurance line was written exclusively by specialized companies, and for that reason the group in question—organized in 1914 in the Casualty Actuarial Society—had not much contact with actuaries working in life assurance or other fields.

Workmen's Compensation includes policies each covering a substantial number of persons and with many indemnity payments each year. As the risk varies from one policy to the other according to variations of protection, experience rating is useful. So the system of Credibility was developed to decide what weight can be put upon the actual loss experience of the individual risk covered by the policy when assessing next year's premium rate.

2.4

About 1920 a group of Italian actuaries had studied the mathematical and statistical bases for evaluating loss reserves in certain non-life fields, and on the agenda of the 1934 congress in Rome we find the theme of assessing premiums and loss reserves in accident and third party liability insurance. Italian speakers proposed formation of an international group of actuaries to promote studies of this subject. Such a group, however, did not materialize.

In 1937 at the Paris congress, several themes on the agenda concerned non-life insurance questions. The subject of reinsurance explicitly covered both life and non-life. One subject considered a possible international statistic of employers' liability insurance and one was devoted to the mathematics of fire and general insurance. In this section also problems from marine insurance were discussed and an international working group was formed to follow up this discussion. Herman Wold and Paul Qvale were among its members and Paul Riebesell was appointed Chairman of the group. As the international tension increased during the next few years, this group, however, never really started working. In other circumstances it might have developed into an ASTIN group.

2.5

Between the two World Wars two monographs of general insurance theory were published—N. Sergovskij: Einführung in die Theorie der Feuerversicherung, Prague 1931, and Paul Riebesell: Einführung in die Sachversicherungsmathematik, Berlin 1936.

Both authors regard for instance the fire premium rate as a product of an ignition probability and an average loss degree. These two factors are by definition regarded as independent. Reinsurance techniques were described and these as well as general risk theory were based upon the classical individual aspect of risk, and statistical independence of the individual claims was presumed.

These books were quickly outdated as a result of the progress of the statistical science.

2.6

In 1909 Filip Lundberg had given the first version of his revolutionary general risk theory. Lundberg's first works are extremely difficult reading. Only many years later the collective risk theory was made more generally accessible by Harald Cramér and his school in the new language of stochastic processes. Cramér has told me that when he was a young actuary of the Swedish Supervision Board, he was asked to study Lundberg's works with the words: Try to explain the meaning of this, it is an attempt to bring some mathematical order into fire insurance.

The collective risk theory does not limit itself to one specific insurance line. It considers the total flow only of premium payments and loss payments of a portfolio, and this portfolio can be a mixture of insurance lines and even of life and non-life. The study of the risk theory is equally important for life and non-life actuaries.

3.1

Before the last turn of the century, Danish insurance companies were mostly specialized, each writing one line only: fire, life, or accident. Each company had a great number of part-time agents with maybe the strictly necessary knowledge of one insurance form. During and after the twenties this pattern totally changed. The companies got together into groups or took up new lines of insurance, and each larger unity built up a staff of professional exclusive agents to become insurance advisers covering all the insurance requirements of its clientele. Often a group includes a life company or life was taken up by a non-life company, or vice versa. Similar organizing changes took place in other Scandinavian countries and this evolution implied that quite a number of life actuaries became involved in non-life problems.

3.2

In 1934 I was working as an actuary in a newly founded life department of an old fire insurance company. By and by I became involved in fundamental questions concerning reinsurance of fire risks. Here in a humble way I tried to take into account a hint of the collective risk theory.

My first opportunity to meet foreign colleagues with similar problems was during the centenary celebrations of the Institute of Actuaries in 1947. In London I paid a visit to Ernest Eldridge, a friend of the company with which I worked, and he introduced me to Robert Beard who had quite recently converted to non-life work. We decided on the spot to keep in touch in order to form a link between the few British and Scandinavian actuaries interested in the mathematics of general insurance. On that day a lifelong companionship and friendship was founded.

3.3

At that time my daily work was still within life assurance. But in my spare hours I worked as a statistical adviser to a group of small mutual companies writing mainly fire insurance of rural buildings, an insurance form which according to law was managed by special companies only. In that capacity I had taken over from Gunnar Benktander who had established common risk statistics. Hereby I worked out a mathematical model describing the increase of the fire risk rates with the size of the buildings. Later, in 1957, I published a paper on that model at the congress of New York. By queer accident this model was a special case of the one that Barrois had described some 120 years before. Later, 1959, I was appointed actuary of the Nye Danske af 1864, being the first to occupy such a position with a Danish non-life company.

4.1

During the 1947 London centenary another valuable first contact was made when Franckx, Hillary Seal, Steven Vajda, and Boleslaw Monic met and discussed non-life actuarial problems. Monic, a manager of a reinsurance company was to play a leading role in the coming international organization. It was his firm view that something should be done and he suggested the arrangement of an international competition.

This was the origin of the competition arranged by Monic's company, Algemeene Herverzekering Maatschappij. It took place in 1951–1952, and the jury was composed by Monic, Franckx, Vajda, and Bruno de Finetti, and the theme was non-proportional reinsurance. The top prizes in the competition were awarded to Hans Ammeter and to Jean Sousselier.

By that initiative a strong continental group of actuaries was formed. This group which was mainly initiated by the efforts of Monic and Franckx, took an active part in the founding process of ASTIN. The abbreviation of Actuarial STudies In Non-life insurance into ASTIN is due to Franckx.

4.2

The first post-war international actuarial congress was held in 1951 in Scheveningen. Here Eldridge had arranged a special meeting on excess of loss reinsurance. The meeting became a great success with 4 papers submitted and 59 persons present. Among the speakers were Monic, Beard, and de Jongh. I must confess that although I took part in the congress I did miss that meeting which was the first roll-call of future ASTIN members.

Monic had persuaded de Jongh to take the Chair although de Jongh was not convinced that the congresses were the right forum for excess of loss studies. In 1950 he had tried to found a special excess of loss center with Riebesell, but this was interrupted by Riebesell's death.

4.3

In July, 1953 a Scandinavian initiative was taken to found an actuarial study group on non-life matters. H. Hellemann and H. Colding-Jørgensen were working as actuaries in the tariff organizations of general insurance in Norway and Denmark, respectively, and they sent an invitation to their colleagues in similar positions in Sweden and Finland to join them. This resulted in the formation of the Nordic Tariff-organizations' Actuaries (NTA) and already in November, 1953 this group had its first meeting in Ehrendal, Sweden. Present were 14 actuaries working in the tariff organizations of the four countries or the attached companies. The meeting took two full days and the main subject was the scope of the future co-operation on these matters in Scandinavia, but also various specific mathematical models and applications of risk theory were dealt with and a possible extension of the co-operation to other parts of the world.

During the next few years similar meetings were held in Norway, Denmark, and Finland. At the time when a second meeting was to be organized in Sweden, it was decided to convert the NTA meeting into the ASTIN colloquium of Rättvik and so the specific NTA activities ended.

The NTA had a letter club where stencilled papers on recent research were circulated. Through me a close contact to ASTIN was established, and at one NTA meeting a new draft of the ASTIN rules was worked out which became the lay-out for the actual ASTIN rules.

During the NTA period the interest for non-life problems was growing in Scandinavian actuarial associations. Several papers on non-life problems were read at the meetings. In Denmark where I was President of the Society from 1953 to 1959 we had normally one non-life subject each year. In Sweden the Board of the Society was supplemented with a member having special insight in general insurance. In 1954 the first candidate for that seat was Ingvar Sternberg.

5.1

The Council of the Permanent Committee for International Actuarial Congresses—now the International Actuarial Association—held its annual meeting in Brussels on September 26th, 1953. The President was A. Théate and the General Secretary, Charles Boels, by tradition chosen among the members of the Belgian delegation. Present were 21 members from 8 countries. Among those present for the first time were Edouard Franckx and myself. For us this should be the beginning of a close collaboration and of a very long service on this Committee.

Towards the end of the meeting Prof. Engelfriet of Holland informed the Council that he had heard of the recent establishment of a new international association of actuaries outside the Permanent Committee, and he asked if any member could give information about this group and its aims.

Franckx replied that some actuaries would like to form an international group interested in the application of actuarial techniques to problems of non-life branches. The group was not to be considered as dissident from the Permanent Committee but quite on the contrary hoped to have the support and the patronage of the Committee.

The President thanked Franckx and expressed the interest of the meeting. He promised all possible help on the part of the Committee towards the achievement of the object which the promoters of this new association had in view.

5.2

In a circular letter of November, 1953 the group presented itself like this:

"A number of persons interested in insurance have noticed that whereas the study of life assurance, thanks to the contributions of actuaries of all countries, has attained a remarkable degree of scientific development, the theory of non-life branches has hardly been touched.

Judging by the response from various countries, it seems that it would be worth while to make a common effort in order to try to fill this gap in the scientific field.

For this purpose the undersigned have formed a Preparatory Committee and one of them, Monsieur Franckx, has had the opportunity to submit a summary of its aims to the Council of the Comité Permanent des Congrès Internationaux d'Actuaires at a meeting held on 26th September, last. Through its President the Comité Permanent promised to lend all possible support to the project.

If, as we hope, you share our interests in the matter, the Preparatory Committee would be pleased if you will join them by becoming a founder member at the proposed Association for Actuarial Studies in Non-life Insurance (ASTIN).

If you think that our efforts deserve to be more widely known, we shall be grateful if you will mention them to other colleagues and let us have their names and qualifications.

We attach suggested Articles of Association, but this is only a first draft and we shall welcome any criticism or alternative proposals.

It is hoped that it will be possible, on the occasion of the forthcoming International Congress in Madrid, to arrange a private meeting of all founder members in order to proceed with the formation of the Association."

5.3

This letter was signed by Ammeter, de Finetti, Franckx, de Jongh, Monic, Sousselier, and Vajda. With the letter followed the draft rules of the association. The content of these rules was in short that the aim was co-operation and exchange of scientific information between persons interested in the actuarial aspects of insurance, mainly connected with branches other than life. The association should publish papers on topics related to its aims and a bulletin containing notes of general interest to members. The association should hold discussion meetings as and when convenient and conferences not less frequently than every three years.

The association should establish contact both locally and internationally with other bodies of similar interest. Nothing specifically was mentioned about the contact with the Permanent Committee.

The wording of these rules caused some doubt among the members of the Council. Several were in doubt whether this association merited the support of the Permanent Committee. In a letter to Franckx the Bureau stated that the question of support to ASTIN would be restudied at the next committee meeting in Madrid. Until then, the ASTIN Preparatory Committee should not talk too much about this support.

5.4

On June 2nd, 1954 the Council of the Permanent Committee met in Madrid before the opening of the 14th international congress. The agenda had a special item: Possibility of forming a new international association of actuaries (ASTIN).

The President, Théate recalled the circumstances in which this question came before the Permanent Committee at the last meeting, and read the letter which he had sent to ASTIN in order to avoid any misinterpretation of the minutes of the meeting.

Several members drew attention to the fact that the actuarial associations of the different countries had often taken an interest in actuarial matters other than those proper to the life branch, various questions relating to non-life business having in fact been dealt with at previous congresses. As both the Permanent Committee and the national actuarial associations had shown their interest in actuarial development in all fields, the technical study of non-life matters could easily be conducted within the framework of the existing associations without it being necessary to form an independent international organization.

Sir George Maddex suggested the formation—within the framework of the Permanent Committee—of sub-committees for the technical study of non-life problems.

Albert Linton said that in the United States the Casualty Actuarial Society was dealing with questions other than life, but up to now that Society had not developed to any great extent. I stated that in Scandinavia a number of members of the national actuarial societies had founded a group for studies in non-life actuarial problems and that this group was eager to get in contact with colleagues in other countries.

Franckx reaffirmed that the creation of ASTIN was not in conflict with the Permanent Committee. The statutes of ASTIN could be altered to avoid being interpreted as in conflict with those of the Permanent Committee. He did not refuse to contemplate the setting up of a special non-life sub-committee under the Permanent Committee.

It was pointed out that the 14th congress would discuss questions which did in effect come into the scope of ASTIN.

Concluding, it was decided that the Permanent Committee should reestablish contact with the ASTIN officials and at Sir George Maddex' suggestion, Théate and Boels were asked to look into the matter again with Franckx.

5.5

The ASTIN meeting during the congress developed as planned and among the speakers were Bertil Almer, Ammeter, and Sternberg. A preliminary ASTIN committee was elected: Beard, Franckx, Monic, and myself.

The audience of the meeting, however, was very small. It had proved impossible to announce the meeting through the official channels of the congress, so only a handful of directly addressed people had some kind of clandestine gathering. But there was Spring in the air when the founding fathers closed this first ASTIN meeting.

5.6

The following meeting of the Permanent Committee was held in Brussels on September 24th, 1955. Here, Professor Marchand submitted a plan for modifying the regulations of the Permanent Committee to allow for the co-operation of this body with ASTIN. In short his plan was that sections formed by a number of members for studies of special problems might be recognized. Each such section should be represented on the Council of the Permanent Committee by at least one member.

In general the proposal of Marchand was supported in the following discussion. It was stated that all members of the Permanent Committee might participate in the work of such sections. It was decided that the Permanent Committee should be represented on the Board of a section by at least one member, and not vice versa.

The question was raised, if persons who were not actuaries could become members of these sections. Here, Franckx stated that indeed it would be so. These persons, nevertheless, would not be ordinary members of the section, but only invited to participate in the work of the section on account of their special knowledge.

It was approved that the Bureau should prepare a new wording of the regulations to embody the various considerations put forward, to be submitted to the next general meeting in September, 1956.

5.7

During the following year a new proposal was worked out in complete agreement with the founder members of ASTIN, and after approval by the national associations the modifications were submitted to the Council. These modifications allowed for sections formed by a number of members for studies of special problems. Each section should have its own regulations, previously approved by the Council and should elect its committee, except for the member appointed by the Permanent Committee.

The wording was accepted by the Council to be put before the coming 15th congress in New York and Toronto. I could confirm that any member of the Permanent Committee had the right to participate in the work of the section and that the members of the ASTIN section, in my view, should be full members of the Permanent Committee.

5.8

Next time the Council held its meeting in New York before the opening of the congress. The draft rules for the ASTIN section were approved by the Council and Sir George Maddex was nominated the first delegate of the Permanent Committee to the ASTIN committee.

Before the next annual meeting Charles Boels, the General Secretary of the Permanent Committee suddenly died. As his successor was appointed Edouard Franckx. In September, 1958 at the Council meeting in Brussels, I gave a report on the first activities of ASTIN. I said that it had not been easy to constitute an effectively working international group, but it was my conviction that those involved were on the right way. I promised that in future more should be heard about ASTIN activities.

6

The recognition process as we have seen was long and tedious. What was wrong or what went wrong? Maybe the first attempt was regarded a little suspicious because Monic represented a young reinsurance company with new ideas of reinsurance treaties.

An original strong support came from Scandinavia and Marchand's initiative was important.

The opposition at a certain point was predominant in Britain where actuaries were interested in life only and general insurance was dealt with by the Chartered Institute. There was a fear that outsiders should flood the actuarial profession and therefore the question of admission of non-actuaries to ASTIN could be a crucial point. It was only the whole-hearted efforts of Beard, a generally estimated former life actuary and the support of George Maddex who as the Government's Actuary had an immensely recognized position that made possible a British opening to ASTIN.

In the United States up to the congress of 1957 there was an approach between life and non-life actuaries. Francis Perryman who has held in great esteem by both sides played a decisive role.

In France, a small number of actuaries were very much occupied with bonus in motor insurance and at an early stage, Marcel Henry issued an invitation to Franckx to a meeting which should be the first ASTIN colloquium in la Baule.

7

Finally the great day had come, the day for which we had worked hard and looked forward to through ten long years. The formation of ASTIN was formally accepted, and one of the themes of the congress papers had been chosen by the Preparatory Committee. A cocktail party was given in our honour by the Casualty Actuarial Society, and our activities during the congress were officially recognized and properly announced. Everything went smoothly.

At 2.30 p.m. on October 16th 1957 the inaugural meeting took place at the Hotel Commodore. The President of the Casualty Actuarial Society, Doc Masterson presided and 46 persons were attending. You can read about all that happened in volume 1 of the *Astin Bulletin*.

Franckx gave a report on behalf of the Preparatory Committee and papers on former non-life actuarial activities were presented by Lars Wilhelmsen, Longley-Cook, Ammeter, Depoid, and de Finetti.

The first ASTIN committee as proposed by Doc Masterson was unanimously accepted and also the draft of the rules. Besides George Maddex, who was appointed by the Permanent Committee, the committee was composed of the four members of the Preparatory Committee, and in addition Francis Perryman and Carl Philipson. At the first meeting of the new committee with George Maddex in the Chair these appointments were made

Chairman Johansen
Vice-Chairman Perryman
Editor Franckx
Secretary Beard
Treasurer Monic

8.1

A question with high priority was the organizing of the *Bulletin*. Besides the editor an editorial panel was formed consisting of Ammeter, de Finetti, Depoid, Engelfriet, Longley-Cook, and C.-O. Segerdahl.

The heavy commitments falling on Franckx in connection with the 16th congress to be held in Brussels made it necessary that he should be relieved of his editorial work and so Beard took over provisionally besides his secretarial job. He had a tremendous load of work and did it well.

8.2

It was decided that the office of Chairman should rotate rather frequently so that many nations successively can have that honour. Therefore, a long series of actuaries from many countries have already served their two years' term and given their devotion and inspiration to ASTIN.

From an early stage the treasury has been handled from Brussels, this being practical as the fees are paid together with those of the IAA. The secretarial work has always been done in London. They both speak and write English so well, these Britons.

During the first 25 years quite a number of ASTIN members have served on its committee. We owe them thanks but they will not be named here. Some of them have passed away. We miss them and mourn them.

8.3

The 15th ASTIN colloquium is organized in Loen, Norway, and the 16th in 1982 in Brussels will coincide with ASTIN's 25 years.

Much work has been done in order to prepare and organize these meetings. I can speak about it because I was involved with the colloquium of Randers. Each has had its specific melody and each has introduced new ideas and added to the success of ASTIN. Again, I can speak about it being the only one who has participated in all these colloquia.

From the beginning we tried to create some efficient tools for international collaboration. A few days of intensive work, often in a place remote from the temptations of the big cities and without too much social activity, can bring your research a long step forward. With a much smaller audience than the big congresses it is possible to create an intimate atmosphere of discussion and the dialogue can sometimes become a wonderful mathematical ping-pong.

For practical reasons the languages have always been restricted to English and French only. Participants in the colloquia must have knowledge of both languages in order to get the full result of the dialogue. On the other hand, any speaker should do his utmost to speak slowly and clearly. Whichever language he uses, it will be a foreign language to the majority of the audience. With this linguistic restriction we can avoid too much costly—and not always effective—translation, which can be a hindrance to the spontaneity of the exchange of views.

9

Strange things have happened which can enlighten the memoirs of an old ASTIN fan, like the story of a certain Lady Evelyn who never turned up. I had suggested an informal meeting in Brussels with Beard and Maddex and received a cryptic answer by telegram:

MEETING EVELYN FRIDAY SATISFACTORY SIR GEORGE AND MYSELF STOP BEARD.

The mysterious Lady was an error of interpretation, Evelyn instead of evening.

In some places a Secretary is even more equal than others. During the colloquium of Sopot, Poland the imperial suite of the fine, old hotel was offered to our Secretary, Peter Green, while the Chairman had to be content with a more modest room like those of the other participants.

At the same colloquium a committee was formed to thoroughly formulate the themes of the coming meeting in Randers to attract the interest of Eastern European actuaries. In that committee were participants from Finland, Poland, and Czechoslovakia, and in a moment of inspiration I asked them to finish the text, polish the words, and check the result.

10

At intervals ASTIN's eternal crisis has been discussed. As probably in all similar associations there are some members who want to know a little about everything and others who know practically everything about very little, and this little is often regarded as of still less practical use by the first.

By this feud lowbrows and highbrows may still influence each other. Some find this fighting awful. I do not—I find it reciprocally much inspiring. ASTIN must have room for both theorists and practicians in order to thrive and until now we have had many fine representatives of both camps. ASTIN must always keep the nose in the sky and the feet planted in solid soil.

11

During the first 25 years ASTIN has grown from a handful of members to well over 1000. 12 volumes of the *Bulletin* contain most of the papers presented at the colloquia, the subject varying from specific mathematical models or statistics from various non-life insurance fields to fundamental basic research.

What will happen in the future? The general technological, economic, and social development will certainly continue. This will inevitably imply that new risks will arise and grow, which will call for new kinds of insurance coverage. Enormous risk accumulation will call for more insurance coverage and new thoughts in reinsurance.

In this process technical specialists will be needed with the background of a qualified actuarial education. This will be a challenge to ASTIN and to its members.

THE FUTURE OF ASTIN*

By Hans Buhlmann

My assistants at ETH have a wall calendar—not with the usual pictures of Swiss mountains, hills and lakes, but with "quotations for intelligent people". Recently, the quotation for the week read as follows: "Even the future is no longer what it used to be in the past".

Observe that also in this supposedly intelligent approach it seems impossible to speak about the future without referring to the past. I shall not deviate from this rule. Of course, my task is greatly simplified by the fact that Paul Johansen has just entertained you in a charming way about the past 25 years of ASTIN and the earlier endeavors leading to the foundation of ASTIN.

In the year 1693, Edmond Halley constructed the first mortality table based on mortality data from Breslau which he had obtained through the intervention of Leibniz. This can be regarded as the starting point of actuarial *science*. In my opinion it can however not be considered as the starting point of the actuarial *profession*. Why? Yes, Halley's table was used for some eighty years because subsequent information coincided with his estimate of mortality. Yes, De Moivre, in his classic textbook of 1725, performed ingenious calculations of annuities, based on the same table. Yes, Süssmilch published the first basic and substantial work of demography in 1741 but—here comes the big but—no government (and nobody else sold annuity insurance at that time) made use of the available scientific method to calculate annuities. Perhaps the first statistical results to be taken seriously were the Northampton tables of 1780, devised by Richard Price. Incidentally, this date coincides reasonably well with the first valuation by William Morgan in 1786. Hence, I think that either of these dates may be taken, at the earliest, as the start of the actuarial profession, a profession being by definition a dedicated group of people *accepted by society* for the performance of a particular skill. Let me make my point explicit: We have historical evidence of the existence of actuarial *science* about 90 years prior to the emergence of the actuarial *profession*. Had I gone back to Johan de Witt and Johannes Hudde instead of Edmond Halley, this span would even exceed 100 years!

Of course, ASTIN is still within these first 100 years of endeavor. If for the sake of time comparison you agree that I identify Halley with Filip Lundberg, ASTIN's chance to create a profession within 100 years extends until 2009 or approximately to its fiftieth anniversary in 2007. With this outlook we have touched upon one of the essential purposes of ASTIN: to create a profession, the profession of the non-life actuary, according to the definition just given: "a dedicated group accepted by society for the performance of a particular skill". Has this possibly been achieved already? The answer varies from country to country. It is a clear "yes" for countries where the non-life actuary has a function by law or where common practice is such as if the function were stipulated by

* Presented at the 25th Anniversary celebration of ASTIN, September 27th, 1982, Liège, Belgium.

law. In how many countries has this point already been reached? In some, but undoubtedly the professional standing of the life actuary is still far ahead of the non-life actuary's. In the historical perspective the non-life side is, however, not doing too badly. Observe that by the time shift agreed upon we are still 14 years prior to the Northampton tables and 20 years prior to Morgan's reserve calculations, hence 82 years before the foundation of the Institute of Actuaries!

Being still in the moulding period of the profession, it might be appropriate to look at this moulding process in more detail. What made this process start? The background for it must be seen in a development of the thinking in this century. This development—philosophically speaking—is characterized by a change of attitude against determinism. According to your personal taste, you may trace this back to the theoretical developments in quantum mechanics or to the social conditions of a crowded depersonalized world or simply to a philosophical reaction against the exaggerated determinism of the 19th century.

Mathematics as a highly specialized language for the scientist has organized this new philosophical attitude in an axiomatically based discipline called probability theory. Statisticians have become aware that the probabilistic view would highly increase the scientific value of their conclusions when interpreting data. This led to the creation of a new science called mathematical statistics. Economists have incorporated risk and uncertainty into their theories. Operation researchers have designed strategic decision schemes for a non-deterministic environment and engineers have started to review their traditional pragmatic safety concepts on the basis of probabilistic models.

The interaction of this new attitude in science has also had its impact on the actuarial community. This interaction took place and is still taking place in two directions:

1. It allowed a new understanding of the concepts underlying the already existing actuarial activity.
2. It opened new fields of activity for the actuary, especially—of course—in the non-life branches.

1. We have probably forgotten that in the last century mortality tables were considered as laws of nature, and some time earlier, e.g., by Süssmilch, even as an example of divine order. It was left to our century to reinterpret this basic tool of the life actuary as a table of probabilities. With this understanding, it was now possible not only to calculate mean value premiums and mean value reserves but also fluctuation loadings, contingency reserves, retentions and solvency margins. Life assurance has become probabilistic. Looking backwards, it seems extremely astonishing that it had not always been that way. The answer to this puzzle might be found in the fact that in the traditional forms of life assurance the savings component predominates heavily over the risk component. In spite of this side remark it is clear that the techniques of the life actuary have been substantially refined since the advent of probabilistic methods and that these methods have opened new possibilities in life assurance. There is, of course, room for further refinement.

2. The first new fields which have opened up to actuarial activity in the non-life area are sickness insurance and workmen's compensation. These branches stand somewhere between the long-term nature of life assurance and the short-term nature of fire. For this reason, actuaries had always been in close connection with developments in these fields, and quite successfully so. But then actuaries were asked to put their skills to work in motor insurance. As you know, in this line of business we have seen a trmendous improvement in both thinking and practice over the last 25 years. It is fair to say that many actuaries have substantially contributed to this progress. Then came fire insurance, aviation, marine. In all these areas one can find pieces of actuarial work which have deeply influenced practical development.

This *process of interaction* of modern thought with our actuarial profession and with the whole insurance industry is what ASTIN is all about. What keeps the process going?

It is, first of all, *people* who keep it going. And the diversity of people makes it a fascinating group indeed. Here are influential managers who by their decisions can set the style and tone in the industry. Whether another branch of insurance should go scientific or not depends very much on their judgment. Here are the practical actuaries who bring the knowledge of the problems. They are faced daily with risk selection, risk rating and possibly reserving. Their participation in the process is motivated by their longing for a better understanding of their problems and for good solutions to them. And here are the theoretically minded researchers. Carrying the torch of modern methods, they hope to demonstrate the power of these methods, being sometimes possibly more motivated by scientific recognition than by the aim to solve all of the industry's problems. If you now imagine all the possible mixtures of the types just described, you have a more or less realistic picture of the group.

The fact that ASTIN comprises all these people is a *necessary* condition for the interaction process to go on. Without the support of the managers, the work of the non-life actuary would have little chance to be accepted by the industry; without the practicing actuaries the process would end in an ivory tower, and without the theoreticians solutions would remain *ad hoc*.

The fact that this diversified group gets together is—on the other hand—not *sufficient* for the functioning of the interaction process. The key to interaction and, let me add, to a successful future of ASTIN, is communication. This is a commonplace remark, but I still make it because we must realize that this may be our severest problem in the next twenty-five years. Paul Johansen has told us that ASTIN has grown from a handful of members to well over 1000, say by a factor of 50. Assuming that communication possibilities and their consequences are proportional to the number of pairs of members, the communication problems must have increased in the proportion of 1:2500.

This implies a completely different interaction style. It must also mean that not all channels of communications can function any longer, simply because the number of possibilities is getting too high. This has to be accepted because it is unavoidable.

But remember that historically we are still in the pre-Northampton table period. Hence, many things still have to be done. Can they be achieved? How? We must improve communication! Here are some concrete proposals:

1. Encourage the formation of formal or informal small interaction groups by vertical splitting (leading to national ASTIN groups) as well as by horizontal splitting (grouping according to special interests like e.g., RESTIN, Oberwolfach, etc.). These smaller groups are the places where the spark must catch and we must see to it that we create as many of these occasions as possible.
2. The most important turntable of communication is the ASTIN colloquia. How can they fulfil this function as the number of participants gets bigger and bigger? We must realize that in the past the glamor of these colloquia has greatly derived from the spontaneity of the discussions and from the fact that contributions were not necessarily presented in a perfectly polished form. This ideal setting for small meetings is not necessarily optimal for larger ones. I submit that we should try to have more survey talks on both theoretical and practical progress. The average colloquium participant profits more from such lectures than from so-called discussions where, for the most part, authors speak about their own papers.
3. The *Astin Bulletin* should be used more frequently for publication of practical work. When have we seen the last publication illustrating the application of a useful method with real data? (I can assure you that the lack of such papers is not due to the policy of the editorial board.)
4. Communication is finally a matter of personal style and commitment. We all must take our communication partners seriously and put more effort into understanding what the other person has to say to us than into what we want to say to him.

I hope that these proposals sound reasonable to you. Of course, it is easy to make them, but here comes another crucial point. Interaction not only requires people and communication facilities; it also needs *time*. The interaction process simply cannot take place if nobody has time to interact. Unfortunately, this time problem is rather unevenly distributed in our group. Using my classification of ASTIN members, let us count out the managers, because they don't have time by definition. Academics seem to be more fortunate as far as time allowance is concerned. But it is most deplorable that, according to my experience, the practical actuaries are given too little time by their employers to work seriously on fundamental problems. In the daily routine of the practical actuary, urgency is constantly superseding importance. Of course, there are exceptions, but not enough of them. May I add at this juncture that also ASTIN as an organization has this time problem. Without a permanent secretariat, and with a committee spread all over the world, typically meeting three times in four years, we are trying to keep together an organization of more than a thousand members and to publish a journal appearing twice a year. It sometimes seems to me that the functioning of ASTIN is a miracle.

Well having now reached the level of miracles why not express some wishful projections for the next 25 years:

1. The techniques of the life and the non-life actuary should move closer to each other. Both the life actuary exploring the behavior of homogeneous risks over time and the non-life actuary modelling heterogeneous risks in a short-term period can contribute something to each other.

2. The non-life actuary incorporating time more naturally into his models should develop a clear methodology for loss reserving. At the next jubilee of ASTIN it might be commonly accepted that—with the exception of case reserves for extraordinary claims—claims reserving clearly lies within actuarial responsibility.

3. The life actuary might use risk analytic methods for the analysis of the asset part of the balance sheet. This seems, indeed, the part of his business most vulnerable to chance fluctuations. Hence, by the year 2000, like his probabilistically-minded colleagues in engineering, he will argue on the solid basis of a stochastic model to explain his solvency safety factors, and he will advise the insurance company on investment strategies geared to a model of fluctuating assets.

Let me stop with these three wishes. We must—I repeat this in view of the historical perspective—have patience. But history teaches us even more: Could de Witt and Halley, De Moivre and Süssmilch ever have foreseen the economic and social consequences of their intellectual endeavours? I believe that the ultimate effect of the interaction process between scientific thought and professional practice can neither be forecast nor planned. The interaction process itself is—as I said before—a miracle. Let me then wish for ASTIN on its 25th anniversary that the miracle will go on!

LARGE CLAIMS IN INSURANCE MATHEMATICS*

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LADIES, GENTLEMEN,

I take great pleasure in addressing this audience. As you might know I'm a mathematician with a deep interest in insurance mathematics. As such, it is my sincere opinion that the gap between practising actuaries and theoretical researchers can be made substantially smaller. If my contribution can help in bridging the gap, I will feel fully compensated for the effort it took to prepare this lecture and the results contained therein.

The simple fact that we meet on the occasion of the sixteenth ASTIN Colloquium gives me a challenging opportunity to help in creating a platform on which both theoreticians and practitioners can meet.

The subject of my lecture stems from a long interest in large claims: What are they? Are they really dangerous? Is there a way to get them under control? Can one recognize them in practical situations?

I like to express in simple mathematical terms some results that might help in acquiring better insight on the impact of large claims in insurance mathematics. Perhaps, no result will be of immediate applicability as reality is too complicated to be described by the simplicity of the results to follow. Nevertheless the latter can be considered as building blocks of a real world in which one has to tackle large claims in theory as well as in practice.

1. WHAT ARE LARGE CLAIMS?

On other occasions I have tried to set up mathematical definitions of what one might call a large claim. None of these approaches seemed to satisfy the practitioners. What could we do better than inquire with people in practice, what they meant by large claims? Here is an anthology of the main answers to the question stated above.

ANSWER 1. *Large claims are the upper* 10% *largest claims.*

It is not quite clear why 10 is used? I see two main reasons why this answer is put forward. The lay out of claim statistics very often has extremely broad intervals for the highest claims; secondly, many reinsurance treaties use proportional reinsurance.

ANSWER 2. *Every claim that consumes at least* 5% *of the sum of claims, or at least* 5% *of the net premiums.*

* Invited Lecture at the 16th ASTIN Colloquium, 27–30th September, 1982, Liège, Belgium.

This description might be appropriate for small portfolios although it is again not very precise.

ANSWER 3. *Every claim for which the actuary has to go and see one of the chief members of the company.*

Alternatively every claim overshooting a preassigned quantity. Needless to say that stop-loss reinsurance treaties trigger off this reply.

ANSWER 4. *Hidden in the following exchange of thoughts.*

T: Mister actuary, what do you mean by large claims?
A: They don't exist.
T: Sorry, I don't quite understand.
A: Well, large claims don't exist since we reinsure dangerous portfolios.
T: But how do you find out whether or not a portfolio is dangerous?
A: This is clear: we watch for large claims.
T: But, what do you mean by large claims?

From the above answers we can draw some conclusions:

— practitioners believe in large claims;
— they don't precisely know how to define them;
— the reinsurance treaties used in their company give guidelines on how to deal with large claims.

Almost all respondents gave some explicit examples of what they consider to be large claims. There is of course the classical set: earthquakes, tornados, air crashes, floods, etc. At least two other samples were illustrated with actual data.

EXAMPLE 1. Portfolio of fire insurance for wooden houses in Scandinavia. apart from small fires, sometimes a burning house sets the surrounding forest on fire and threatens other houses in the immediate vicinity.

EXAMPLE 2. Portfolio of schoolbus insurance. The typical course of life of a bus looks as follows: the first 8 to 10 years the bus is used on long distance trips; then the bus is employed on one day excursions; the bus ends its career as a schoolbus. It is not hard to forecast that lack of good maintenance makes schoolbuses accident prone.

How does one transform the above vague quotations into hard mathematical terms? Scanning the existing literature dealing with large claims we find that there is often agreement on the claim size distribution of large claims. The non-existence of certain moments or the use of so-called "shadow claims" suggest that fat-tailed distributions like the Pareto-distributions are appropriate models for dealing with large claims.

2 HOW DANGEROUS ARE LARGE CLAIMS?

To get some feeling for the danger resulting from large claims we deal with the classical ruinproblem. Assume that consecutive claims occur, according to a Poisson process, $\{N(t);\ t \geq 0\}$ with parameter $\lambda > 0$. Call these successive claim amounts $B_1, B_2, \ldots$ and assume that they are independent with common distribution of B, say $B(x) = P\{B \leq x\}$ where $B(0) = 0$. We assume moreover that the claims sizes are independent of the Poisson process. The risk reserve accumulated up to time t is given by

$$Y(t) = u + ct - \sum_{i=1}^{N(t)} B_i$$

where u is the initial reserve and c is the loading corresponding to premium payments.

We are interested in the distribution of the time of ruin, i.e., $T_u = \inf\{t\colon Y(t) < 0\}(= +\infty$ if no such t exists), where u refers to the initial reserve. Our basic assumptions are the following.

A(i): $c = 1$, this is established by a proper choice of the time scale;
A(ii): $\rho \equiv \lambda EB < 1$, on the average the income per unit time exceeds the expenses;
A(iii): $1 - B(x) \sim x^{-\alpha} L(x)$ where $\alpha \geq 1$ and L a slowly varying function (B is of Pareto-type).

We write

$$P\{T_u \leq t\} = P\{T_u < \infty\} - P\{t < T_u < \infty\}. \tag{1}$$

It is well-known (see: P. Embrechts and N. Veraverbeke (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math. Econom.* **1**, 55–72) that under A(i), (ii), (iii) ruin in finite time satisfies the asymptotic equality

$$P\{T_u < \infty\} \sim \frac{\rho}{1-\rho} [1 - \bar{B}(u)] \quad \text{for } x \to \infty \tag{2}$$

where $\bar{B}(u) = (EB)^{-1} \int_0^u [1 - B(y)]\, dy$. (This specific result can also be found in B. von Bahr (1975). Asymptotic ruin probabilities when exponential moments do not exist. *Scand. Actuarial J.* 6–10.) A few trial calculations show that even for very large u, this probability may be considerable if α is small or (and) if ρ is close to 1.

Looking back at (1) one might hope that the term $P\{t < T_u < \infty\}$ will lower the above probability considerably. However A(i), (ii), (iii) imply that for all $u \geq 0$ and $t \to \infty$ (a full proof will be published later)

$$P[t < T_u < \infty] \sim \rho(1-\rho)^{1-\alpha}[1 - \bar{B}(t)]\left\{1 + \sum_1^{\infty} \rho^n \bar{B}^{(n)}(u)\right\} \tag{3}$$

which for u large reads

$$P[t<T_u<\infty]\sim\rho(1-\rho)^{-\alpha}[1-\bar{B}(t)]$$

independently of u. Hence some of the probability of getting ruined in $[0, t]$ can be shifted to $[t, \infty)$ by increasing u; however ruin remains highly probable. It is obvious from the above considerations that something has to be done. Listening to the practitioner reinsurance might be appropriate.

3. IS REINSURANCE HELPFUL?

Let us rephrase the question somewhat to get a result with broader applicability. Instead of A(iii) we would like to impose a condition that mainly allows small claims. Although the condition B(iv) is somewhat technical one might vaguely interpret it as meaning that

$$1-B(x)<Ke^{-\delta x}$$

for some $\delta>0$ so that high claims are very improbable. Let $\Lambda(s)=s-\lambda[1-b(s)]$ where $b(s)=E[\exp-sB]$ is the Laplace transform of the claim size B. We assume

B(i): $c=1$;
B(ii); $\rho<1$;
B(iii): $B(x)$ is a non-lattice distribution;
B(iv): there exist a value $\kappa>0$ such that $\Lambda(-\kappa)=0$.

The results corresponding to (2) and (3) are now: Under B(i), (ii) and (iv), there exists a constant C_1 such that (see the above mentioned paper by Embrechts-Veraverbeke)

$$P\{T_u<\infty\}\simeq C_1e^{-\kappa u}\quad\text{as } u\to\infty. \tag{4}$$

Also (see J. L. Teugels (1982). Estimation of ruin probabilities. *Insurance Math. Econom.* **1**, 163–175) for a constant C_2 and u and t large

$$P\{t<T_u<\infty\}\sim C_2e^{-vu}u\,e^{-\theta t}t^{-3/2} \tag{5}$$

for a constant $v\in(0,\kappa)$ and a constant $\theta>0$. Actually v is defined by $\Lambda'(-v)=0$ while $\theta=-\Lambda(-v)$.

The interpretation of (4) and (5) is that if the initial reserve u is large enough, ruin in $[0, T]$ and in $[T,\infty]$ is highly improbable. Alternatively one might say that under the B-conditions no reinsurance is anymore necessary.

Turning back to large claims, stop-loss reinsurance is based on a retention M, the corresponding truncated distribution has no tail and hence for constants C_M, v_M and θ_M by (5)

$$P\{t<T_u<\infty\}\sim C_Mu\,e^{-v_Mu}e^{-\theta_Mt}t^{-3/2}. \tag{6}$$

A basic problem is how to determine M in such a way that after one year ruin is only possible with a small probability, starting with initial reserve u. Now one can get some rough estimates on v_M and θ_M.

$$(7) \qquad v_M \begin{cases} \geqslant \dfrac{1}{M} \log\left\{1+\dfrac{1-\rho}{\rho}\varphi\right\} \\ \leqslant \dfrac{1}{M}\dfrac{\varphi}{\varphi-1}\log 1+\left\{1+\dfrac{1-\rho}{\rho}\dfrac{1}{A}\right\} \end{cases}$$

where $\varphi=\alpha^{-1}\mu M$, $\mu=EY$, $\alpha=EY^2$ and $A=\mu^{-1}M\{1-B(M)\}$. Also

$$(8) \qquad \theta_M \begin{cases} \geqslant v_M(1-\rho)+\dfrac{\alpha\rho v_M}{2M\mu}(1-e^{v_M M}) \\ \leqslant v_M(1-\rho)+v_M\rho\left(1-\exp\dfrac{\alpha v_M}{2\mu}\right). \end{cases}$$

These formulas are interesting as first approximation in that the dependence on M and ρ is emphasized. The larger M, the smaller v_M and θ_M; the closer ρ is to 1 the smaller v_M and θ_M.

The more information one has about the claim size distribution the more precision can be obtained for v_M and θ_M. We will return to these results in a forthcoming publication.

To get a quick overview of the differences between the situations described in 2 and 3 I propose to introduce *isoruines*, i.e., curves in the (t, u) plane that give the same probability of ruin. Three typical cases are depicted in fig. 1. Here $\varepsilon=P\{T_u\leqslant t\}$. The full lines correspond to an exponential distribution with $\lambda=\frac{1}{2}$, $\mu=1$ while the dotted lines come from $\bar{B}(x)=2/\pi$ arctan x and $\rho=\frac{1}{2}$, $\alpha=2$.

For fixed time the initial reserves for the Pareto-type case are much larger than for the exponential case. Also the infinite horizon values are very different. For example for the exponential case $u=7.9$ gives $\varepsilon=0.01$ while for the Pareto-type case $u=69.4$.

Let me point out that there are two types of isoruins i.e., $P[T_u\leqslant t]$ and $P[T_u>t]$ with quite different characteristics.

4. HOW CAN ONE DETECT LARGE CLAIMS?

We like to formulate an approach which might be useful in practice if suitably adapted. A characteristic of existing reinsurance procedures is that estimated retentions and premiums are based on past year's data. Perhaps one realizes an overall loss in the portfolio; in other cases like largest claims reinsurance and ECOMOR, estimates are based on the largest claims registered during the year. Unfortunately the ordering of claims in increasing order is a time consuming undertaken even for a computer. More important is that the ruin disaster is only discovered at the end of the book-keeping year.

The following procedure tries to do better. We assume that claims are Pareto distributed so that $1-B(x)=x^{-\alpha}$ $(x\geqslant 1)$ with unknown α. We look at the claims as they come in: any time a claim is reported bigger than all previous claims we get a *warning*. More precisely we look at the sequence of so-called *record times*

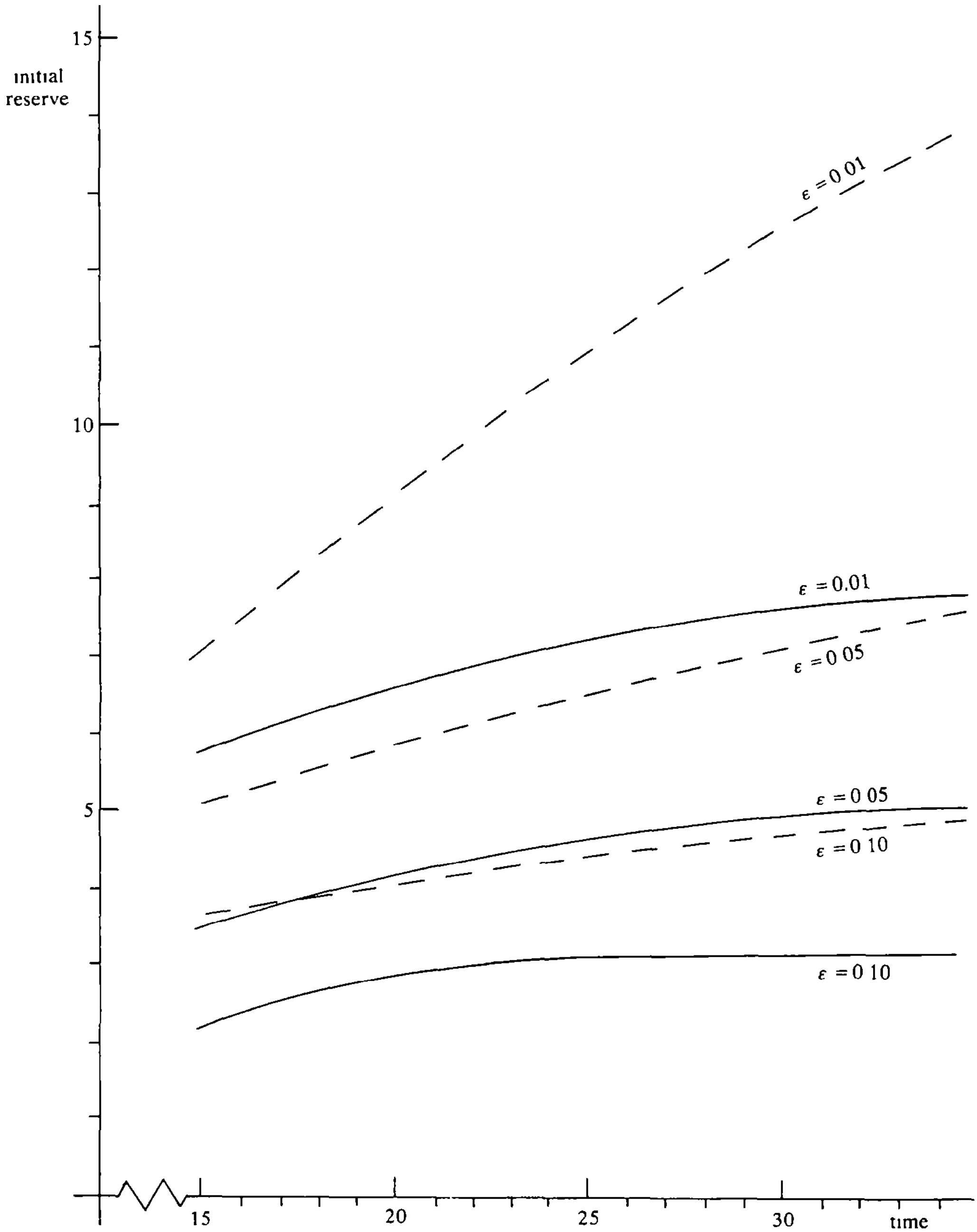

FIGURE 1. Isoruins defined by $\varepsilon = P\{T_u \leq t\}$

and *record values*.

$$\begin{aligned} N_1 &= 1, & X_1 &= B_1 \\ N_2 &= \inf\{l: B_l > X_1\}, & X_2 &= B_{N_2} \\ N_3 &= \inf\{l: B_l > X_2\}, & X_3 &= B_{N_3} \\ &\dots & &\dots . \end{aligned}$$

It is easy to show that $1/\alpha$ can be excellently estimated by the sequence of warning values. Indeed, for any k, $1/k \log X_k$ is an unbiased, efficient estimator for $1/\alpha$. (Alternatively the sample mean of the $\log B_i$ could be used to estimate $1/\alpha$).

If we are afraid for example that the average claim EB should be infinite we can construct confidence intervals for $1/\alpha$ and check whether or not the value 1 belongs to it. In this sense consecutive warnings could lead to an alarmingly low value of α.

To be more precise let us denote by $p_n^{(\alpha)}(\beta)$ the probability that all estimates of $1/\alpha$ based on the first n warning values suggest that $\alpha < \beta^{-1}$, i.e.,

$$p_n^{(\alpha)}(\beta) = P\left\{\bigcap_{k=1}^{n}\left[\frac{\log X_k}{k} > \beta\right]\right\}.$$

For example we give a short table for $\beta = 1$ and $n = 1, 2, 3, 4, 5$.

$n \rightarrow$

$\beta = 1$	1	2	3	4	5
4	0 01832	0.00168	0 00020	0 00001	4 10^{-6}
3	0.04979	0 00991	0.00253	0 00072	0.00022
2	0 13533	0.05495	0 02727	0.01487	0.00858
1.5	0.22313	0 12447	0.08193	0 05825	0.04334

For example, assume $\alpha = 4$. Any time a warnıng value suggest that $\alpha < 1$ we sound the alarm. The first warning results in a fake alarm with probability 0.01832. A second fake alarm is so improbable that we better drop the hypothesis that $\alpha = 4$ (or even $\alpha \geqslant 4$).

Similarly, 5 consecutive alarms make $\alpha \geqslant 1.5$ already quite unlikely.

Any time a warning leads to an alarm, the company might ponder to take reinsurance and that while the claims are still coming in.

Although this and allied procedures look promising refinements are necessary since in a sample of size n there are on the average only $\log n$ warning or record values.

5. CONCLUSIONS

We have only indicated some major items where recent mathematical developments can help the practitioner to get a better understanding of reality.

As I hope to continue research in the area of primary reinsurance I hope to get inspiring suggestions from you. Let me mention a few topics on which the practitioner has acquired insight, indispensable for the theoretician:

- what are the rules of thumb used in practice to decide about reinsurance?
- how are the retentions chosen?
- what premium principles are used?
- who makes the final decision?
- can you provide examples (or even data) on portfolios where large claims do occur?

Let me finally draw a parallel between insurance mathematics and statistics. In both fields there are researchers and practitioners; in both areas a gap is felt in between theory and practice; fortunately in both domains practitioners and theoreticians meet in fruitful conferences.

There is one more parallel that nicely applies to myself: not too many people in actuarial sciences have to worry about large claims. On the statistical side, few statisticians are involved in the study of the corresponding area of statistics, namely that of so-called outliers. If you feel that my interest in large claims is unsound, then do with me as with outliers in statistics: get them out. But if you feel that large claims are important, help me in getting a better understanding of what they are and what you would really like to do with them.

Thank you for you kind attention.

ACKNOWLEDGMENT

The author would like to thank a large number of actuaries that commented on his lecture, by suggesting alterations, by giving more insight in reinsurance treaties, by providing data, etc. Also the referee of the paper made some general comments that will help the author to accomplish the task lead out in his own introduction.

ASYMPTOTIC BEHAVIOUR OF COMPOUND DISTRIBUTIONS AND STOP-LOSS PREMIUMS

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ABSTRACT

The paper gives some asymptotic results for the compound distribution of aggregate claims when the claim number distribution is negative binomial. The case when the claim numbers are geometrically distributed, is treated separately.

1. INTRODUCTION

1A. Let $\tilde{x}_1, \tilde{x}_2, \ldots$ be independent identically distributed random variables (the independent severities) on $(0, \infty)$ with cumulative distribution F. Let $\tilde{n}$ be a random variable (the claim number), independent of the $\tilde{x}_i$'s with distribution on the non-negative integers defined by

$$p_n = \Pr(\tilde{n} = n) = \binom{\alpha + n - 1}{n} p^n (1-p)^\alpha, \qquad (\alpha > 0, 0 < p < 1). \tag{1}$$

Let

$$\tilde{s} = \begin{cases} \sum_{i=1}^{\tilde{n}} \tilde{x}_i & (\tilde{n} > 0) \\ 0 & (\tilde{n} = 0) \end{cases}$$

(the total aggregate claim amount). Then the cumulative distribution of $\tilde{s}$ is

$$G(s) = \sum_{n=0}^{\infty} p_n F^{n*}(s). \tag{2}$$

The idea of the present paper is to develop asymptotic expressions when $s \uparrow \infty$ for

(i) the tail

$$H(s) = 1 - G(s);$$

(ii) the stop-loss premium

$$K(s) = \mathscr{E}(\max(\tilde{s} - s, 0)) = \int_s^\infty H(x)\, dx;$$

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(iii) the density

$$g(s)=\frac{d}{ds}G(s)$$

or the point probability

$$g_s=\Pr(\tilde{s}=s).$$

1B. We are going to use some notation and results from Feller (1971):

If A and B are functions, by the notation $A(s)\sim B(s)$ as s tends to, say, a, we shall mean that the ratio $A(s)/B(s)$ tends to 1 as s tends to a.

We shall call the severity distribution F arithmetic if it is concentrated on the set $\{\lambda, 2\lambda, 3\lambda, \dots\}$ for some λ, and we shall call the largest such λ the span of the distribution. When we treat arithmetic distributions, we shall for convenience assume that the span is equal to one; general span-length is obtained by rescaling.

We shall say that a function A is ultimately monotone if there exists a y such that $A(x)$ is monotone for all $x>y$.

It is assumed that there exists a κ satisfying

$$\frac{1}{p}=\int_{(0,\infty)} e^{\kappa x}\, dF(x), \tag{3}$$

and that

$$\nu=p\int_{(0,\infty)} x\, e^{\kappa x}\, dF(x) \tag{4}$$

is finite.

2. GEOMETRICALLY DISTRIBUTED CLAIM NUMBER

2A. In the present section we are going to assume that p_n satisfies (1) with $\alpha=1$, that is,

$$p_n=p^n(1-p).$$

Then the distribution G satisfies the identity

$$G(s)=1-p+p\int_{(0,s]} G(s-x)\, dF(x), \qquad (s>0) \tag{5}$$

as is seen by rewriting (2) as

$$\begin{aligned} G(s)&=\sum_{n=0}^{\infty} p^n(1-p)F^{n^*}(s)\\ &=1-p+p\sum_{n=0}^{\infty} p^n(1-p)(F^{n^*}*F)(x)\\ &=1-p+p(G*F)(s). \end{aligned}$$

We see that (5) has the form of a renewal equation with defective distribution pF. This means that we can apply results from renewal theory. In subsections 2B–C we do this for the non-arithmetic case, in subsection 2D for the arithmetic case.

2B. Assume that F is non-arithmetic. Then by formulae (6.7) and (6.16) of Chapter XI in Feller (1971) we get

$$H(s) \sim \frac{1-p}{\kappa\nu} e^{-\kappa s}, \qquad (s \uparrow \infty).$$

From this we can also easily obtain an asymptotic expression for the stop-loss premium $K(s)$.

THEOREM 1. *The stop-loss premium $K(s)$ satisfies*

$$K(s) \sim \frac{1-p}{\kappa^2\nu} e^{-\kappa s}, \qquad (s \uparrow \infty).$$

PROOF. By using L'Hôpital's rule we get

$$\lim_{s\uparrow\infty} \frac{K(s)}{e^{-\kappa s}} = \lim_{s\uparrow\infty} \frac{-H(s)}{-\kappa e^{-\kappa s}} = \frac{1-p}{\kappa^2\nu},$$

which proves the theorem. Q.E.D.

2C. If F has a density f, then G has an atom

(6) $$G(0) = p_0$$

at zero, and for $s > 0$ a density

(7) $$g(s) = \sum_{n=1}^{\infty} p_n f^{n*}(s).$$

THEOREM 2. *The density $g(s)$ satisfies*

$$g(s) \sim \frac{1-p}{\nu} e^{-\kappa s}, \qquad (s \uparrow \infty).$$

PROOF. We use L'Hôpital's rule:

$$\frac{1-p}{\kappa\nu} = \lim_{s\uparrow\infty} \frac{H(s)}{e^{-\kappa s}} = \lim_{s\uparrow\infty} \frac{-g(s)}{-\kappa\, e^{-\kappa s}} = \frac{1}{\kappa} \lim_{s\uparrow\infty} \frac{g(s)}{e^{-\kappa s}}.$$

From this follows the theorem. Q.E.D.

2D. In this subsection we shall assume that the distribution F is arithmetic with unit span, and we introduce

$$f_k = \Pr(\tilde{x}_i = k), \qquad (k = 1, 2, \ldots).$$

Then

$$F(x)=\sum_{i=1}^{[x]} f_i, \qquad (x\geqslant 0).$$

THEOREM 3. *When s goes to infinity through the integers, we have*

(8) $$\text{(i)}\quad K(s)\sim\frac{(1-p)e^{\kappa}}{\nu(e^{\kappa}-1)^2}e^{-\kappa s}$$

(9) $$\text{(ii)}\quad H(s)\sim\frac{1-p}{\nu(e^{\kappa}-1)}e^{-\kappa s}$$

$$\text{(iii)}\quad g_s\sim\frac{1-p}{\nu e^{\kappa}}e^{-\kappa s}.$$

PROOF. For the whole proof s will always denote a non-negative integer.
(i) In the present case (5) becomes

$$G(s)=1-p+p\sum_{i=1}^{s} G(s-i)f_i.$$

As $G(s-i)=0$ for $i>s$, we may extend the sum to infinity,

$$G(s)=1-p+p\sum_{i=1}^{\infty} G(s-i)f_i.$$

Introduction of $H(s)=1-G(s)$ gives

$$H(s)=p\sum_{i=1}^{\infty} H(s-i)f_i.$$

We get

$$K(s)=\sum_{x=s}^{\infty} H(x)=\sum_{x=s}^{\infty} p\sum_{i=1}^{\infty} H(x-i)f_i$$

$$=p\sum_{i=1}^{\infty}\left(\sum_{y=s-i}^{\infty} H(y)\right)f_i,$$

and thus

$$K(s)=p\sum_{i=1}^{\infty} K(s-i)f_i.$$

By using

$$K(s-i)=\mathscr{E}(\tilde{s})+i-s, \qquad (i=s+1,s+2,\dots)$$

we obtain

$$K(s)=p\sum_{i=s+1}^{\infty}(\mathscr{E}(\tilde{s})+i-s)f_i+\sum_{i=1}^{s} K(s-i)pf_i.$$

Multiplying this equation by $e^{\kappa s}$ and introducing

$$f_i^* = p f_i e^{\kappa i}$$

and

$$K^*(s) = K(s)\, e^{\kappa s}$$

give

$$K^*(s) = p\, e^{\kappa s} \sum_{i=s+1}^{\infty} (\mathscr{E}(\tilde{s}) + i - s) f_i + \sum_{i=1}^{s} K^*(s-i) f_i^*. \tag{10}$$

Considered as point probabilities $f_1^*, f_2^*, \ldots$ defines a proper probability distribution because of (3). Hence (10) is a proper renewal equation, and the renewal theorem (Karlin and Taylor, 1975, p. 81) gives

$$\lim_{s\uparrow\infty} K^*(s) = \frac{1}{\nu} \sum_{j=0}^{\infty} p\, e^{\kappa j} \sum_{i=j+1}^{\infty} (\mathscr{E}(\tilde{s}) + i - j) f_i.$$

In the following development we use that

$$\mathscr{E}(\tilde{s}) = \mathscr{E}(\tilde{n}) \mathscr{E}(\tilde{x}_1) = \frac{p}{1-p} \mathscr{E}(\tilde{x}_1).$$

We have

$$\begin{aligned}
\nu \lim_{s\uparrow\infty} K^*(s) &= \sum_{j=0}^{\infty} p\, e^{\kappa j} \sum_{i=j}^{\infty} (\mathscr{E}(\tilde{s}) + i + 1 - j) f_{i+1} \\
&= p \sum_{i=0}^{\infty} f_{i+1} \sum_{j=0}^{i} (\mathscr{E}(\tilde{s}) + i + 1 - j)\, e^{\kappa j} \\
&= p \sum_{i=0}^{\infty} f_{i+1} \left[\frac{e^{\kappa(i+1)} - 1}{e^{\kappa} - 1} \mathscr{E}(\tilde{s}) + \sum_{j=0}^{i} \sum_{k=j}^{i} e^{\kappa j} \right] \\
&= p \left[\frac{\mathscr{E}(\tilde{s})}{e^{\kappa} - 1} \left(\frac{1}{p} - 1 \right) + \sum_{i=0}^{\infty} f_{i+1} \sum_{k=0}^{i} \sum_{j=0}^{k} e^{\kappa j} \right] \\
&= p \left[\frac{\mathscr{E}(\tilde{x})}{e^{\kappa} - 1} + \sum_{i=0}^{\infty} f_{i+1} \sum_{k=0}^{i} \frac{e^{\kappa(k+1)} - 1}{e^{\kappa} - 1} \right] \\
&= \frac{p}{e^{\kappa} - 1} \left[\mathscr{E}(\tilde{x}) + \sum_{i=0}^{\infty} f_{i+1} \sum_{k=0}^{i} (e^{\kappa(k+1)} - 1) \right] \\
&= \frac{p}{e^{\kappa} - 1} \left[\mathscr{E}(\tilde{x}) + \sum_{i=0}^{\infty} f_{i+1} \left(\frac{e^{\kappa(i+2)} - e^{\kappa}}{e^{\kappa} - 1} - (i+1) \right) \right] \\
&= \frac{p}{e^{\kappa} - 1} \left[\mathscr{E}(\tilde{x}) + \frac{e^{\kappa}}{e^{\kappa} - 1} \left(\frac{1}{p} - 1 \right) - \mathscr{E}(\tilde{x}) \right] = \frac{(1-p)\, e^{\kappa}}{(e^{\kappa} - 1)^2}.
\end{aligned}$$

From this follows (i).

(ii) As

$$H(s)=K(s)-K(s+1),$$

we get

$$\lim_{s\uparrow\infty} H(s)\, e^{\kappa s}=\lim_{s\uparrow\infty} K(s)\, e^{\kappa s}-e^{-\kappa} \lim_{s\uparrow\infty} K(s+1)\, e^{\kappa(s+1)}$$

$$=(1-e^{-\kappa}) \lim_{s\uparrow\infty} K(s)\, e^{\kappa s},$$

which proves (ii).

(iii) As

$$g_s=H(s)-H(s+1),$$

the proof of (iii) goes as the proof of (ii).

This completes the proof of Theorem 3. Q.E.D.

3. NEGATIVE BINOMIALLY DISTRIBUTED CLAIM NUMBER

3A. We shall now drop the restriction $\alpha=1$ in (1). Then we have the following theorems:

THEOREM 4. *If*

$$R(s)=e^{\kappa s}H(s)$$

is ultimately monotone, then

$$H(s)\sim\frac{1}{\kappa\Gamma(\alpha)}\left(\frac{1-p}{\nu}\right)^{\alpha} s^{\alpha-1}\, e^{-\kappa s}, \qquad (s\uparrow\infty). \tag{11}$$

PROOF. Let

$$\phi(t)=\int_{(0,\infty)} e^{-tx}\, dF(x), \qquad (t\geqslant-\kappa)$$

$$\sigma(t)=\int_{[0,\infty)} e^{-ts}\, dG(s)=\left(\frac{1-p}{1-p\phi(t)}\right)^{\alpha}, \qquad (t>-\kappa) \tag{12}$$

$$\omega(t)=\int_{[0,\infty)} e^{-ts}R(s)\, ds=\frac{1-\sigma(t-\kappa)}{t-\kappa}, \qquad (0<t<\kappa).$$

We want to show that

$$R(s)\sim\frac{1}{\kappa\Gamma(\alpha)}\left(\frac{1-p}{\nu}\right)^{\alpha} s^{\alpha-1}. \qquad (s\uparrow\infty)$$

By Theorem 4 on p. 446 in Feller (1971) this is equivalent to

$$\omega(t)\sim\frac{1}{\kappa}\left(\frac{1-p}{\nu t}\right)^{\alpha}, \qquad (t\downarrow 0).$$

Let

$$\psi(t) = \kappa\left(\frac{\nu t}{1-p}\right)^{\alpha} \omega(t).$$

We have to show that

(13) $$\lim_{t\downarrow 0} \psi(t) = 1.$$

We have

$$\lim_{t\downarrow 0} \psi(t) = \lim_{t\downarrow 0} \frac{1-\sigma(t-\kappa)}{t-\kappa}\left(\frac{\nu t}{1-p}\right)^{\alpha}\kappa$$

$$= \lim_{t\downarrow 0} \sigma(t-\kappa)\left(\frac{\nu t}{1-p}\right)^{\alpha}$$

$$= \lim_{t\downarrow 0} \left(\frac{\nu t}{1-p\phi(t-\kappa)}\right)^{\alpha}.$$

As, by L'Hôpital's rule,

$$\lim_{t\downarrow 0} \frac{\nu t}{1-p\phi(t-\kappa)} = \lim_{t\downarrow 0} \frac{\nu}{-p\phi'(t-\kappa)} = \frac{\nu}{-p\phi'(-\kappa)} = 1,$$

(13) holds, and hence the theorem is proved. Q.E.D.

THEOREM 5. *If* (11) *holds, then*

(14) $$K(s) \sim \frac{1}{\kappa^2\Gamma(\alpha)}\left(\frac{1-p}{\nu}\right)^{\alpha} s^{\alpha-1} e^{-\kappa s}, \qquad (s\uparrow\infty).$$

PROOF. L'Hôpital's rule gives

$$\lim_{s\uparrow\infty} \frac{K(s)}{s^{\alpha-1}e^{-\kappa s}} = \lim_{s\uparrow\infty} \frac{-H(s)}{-\kappa s^{\alpha-1}e^{-\kappa s} + (\alpha-1)s^{\alpha-2}e^{-\kappa s}}$$

$$= \lim_{s\uparrow\infty} \frac{\kappa s}{\kappa s-\alpha+1}\,\frac{H(s)}{\kappa s^{\alpha-1}e^{-\kappa s}}$$

$$= \lim_{s\uparrow\infty} \frac{H(s)}{\kappa s^{\alpha-1}e^{-\kappa s}} = \frac{1}{\kappa^2\Gamma(\alpha)}\left(\frac{1-p}{\nu}\right)^{\alpha},$$

which proves the theorem. Q.E.D.

3B. If F has a density f, then G has an atom p_0 at zero, and for $s>0$ a density $g(s)$ given by (7). Then we have the following theorem.

THEOREM 6. *If* (11) *holds, the density* $g(s)$ *satisfies*

$$g(s) \sim \frac{1}{\Gamma(\alpha)}\left(\frac{1-p}{\nu}\right)^{\alpha} s^{\alpha-1} e^{-\kappa s}, \qquad (s\uparrow\infty).$$

PROOF. L'Hôpital's rule gives

$$\frac{1}{\kappa\Gamma(\alpha)}\left(\frac{1-p}{\nu}\right)^{\alpha}=\lim_{s\uparrow\infty}\frac{H(s)}{s^{\alpha-1}e^{-\kappa s}}=\lim_{s\uparrow\infty}\frac{-g(s)}{-\kappa s^{\alpha-1}e^{-\kappa s}+(\alpha-1)s^{\alpha-2}e^{-\kappa s}}$$

$$=\lim_{s\uparrow\infty}\frac{\kappa s}{\kappa s-\alpha+1}\frac{g(s)}{\kappa s^{\alpha-1}e^{-\kappa s}}=\lim_{s\uparrow\infty}\frac{g(s)}{\kappa s^{\alpha-1}e^{-\kappa s}},$$

and this proves the theorem. Q.E.D.

3C. The assumption that $R(s)$ is ultimately monotone, is awkward, as it seems very difficult to show that it is satisfied. As, when the condition holds,

$$R(s)\sim R^*(s), \qquad (s\uparrow\infty)$$

with

$$R^*(s)=\frac{1}{\kappa\Gamma(\alpha)}\left(\frac{1-p}{\nu}\right)^{\alpha}s^{\alpha-1},$$

which is monotonely increasing to infinity for $\alpha>1$, constant for $\alpha=1$, and monotonely decreasing to zero for $\alpha<1$, the condition must mean that $R(s)$ is ultimately monotonely *increasing* for $\alpha>1$ and ultimately monotonely *decreasing* for $\alpha<1$; for $\alpha=1$ we cannot say whether the ultimate monotony is increasing or decreasing, but in that case it does not matter as we then have the theorems of Section 2.

In the arithmetic case the assumption of ultimate monotony of $R(s)$ does not hold as $R(s)$ then increases continuously when s is a non-integer and decreases in jumps at integers. But in this case Theorems 4 and 5 *cannot* hold as for $\alpha=1$ (9) and (8) contradict (11) and (14).

If F has a density f, the following lemma gives a condition equivalent to ultimate monotony of $R(s)$ when $\alpha\neq 1$.

LEMMA 1. *Assume that F has a density f and that $\alpha\neq 1$. Then $R(s)$ is ultimately monotone if and only if there exists an s_α such that for all $s>s_\alpha$*

$$\frac{g(s)}{H(s)}<\kappa, \qquad (\alpha>1)$$

$$\frac{g(s)}{H(s)}>\kappa, \qquad (\alpha<1).$$

PROOF. For $s>0$

$$\frac{dR}{ds}=e^{\kappa s}(\kappa H(s)-g(s)),$$

and hence

$$\frac{dR}{ds}\gtrless 0\Leftrightarrow\frac{g(s)}{H(s)}\lessgtr\kappa.$$

But because of the ultimate monotony of $R(s)$ there must exist an s_α such that for all $s > s_\alpha$, $dR/ds \gtreqless 0$ as $\alpha \gtreqless 1$. This proves the lemma. Q.E.D.

The author believes that the assumption of ultimate monotony of $R(s)$ in Theorem 4 may be replaced by the assumption that the distribution F is non-arithmetic, but has not been able to prove this result. An indication that the result holds, is that it holds in the special case $\alpha = 1$ as shown in subsection 2B. Another indication is given by the following example.

EXAMPLE. Let the severity distribution F be defined by the density

$$f(x) = \beta e^{-\beta x}, \qquad (x > 0, \beta > 0).$$

Then the Laplace transform of F is

$$\phi(t;\beta) = \frac{\beta}{\beta + t}, \tag{15}$$

and from (12) follows that the Laplace transform of G is

$$\sigma(t) = (1 - p + p\phi(t;\beta(1-p)))^\alpha.$$

By expanding we get

$$\sigma(t) = \sum_{i=0}^{\infty} \binom{\alpha}{i} (1-p)^{\alpha-i} p^i \phi(t;\beta(1-p))^i.$$

As $\phi(t;\beta(1-p))^i$ is the Laplace transform of the distribution $\Gamma(s; i, \beta(1-p))$ defined by

$$\Gamma(s; a, b) = \frac{b^a}{\Gamma(a)} \int_0^s r^{a-1} e^{-br}\, dr, \qquad (s, a, b > 0)$$

we get

$$H(s) = \sum_{i=0}^{\infty} \binom{\alpha}{i} (1-p)^{\alpha-i} p^i (1 - \Gamma(s; i, \beta(1-p)).$$

Assume that α is an integer. Then

$$H(s) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (1-p)^{\alpha-i} p^i (1 - \Gamma(s; i, \beta(1-p)).$$

As by L'Hôpital's rule

$$\lim_{s\uparrow\infty} \frac{1 - \Gamma(s; i, \beta(1-p))}{s^{\alpha-1} e^{-\beta(1-p)s}} = \begin{cases} 0, & (i < \alpha) \\ \dfrac{[\beta(1-p)]^{\alpha-1}}{\Gamma(\alpha)}, & (i = \alpha) \end{cases}$$

we get

$$H(s) \sim \frac{1}{\beta(1-p)\Gamma(\alpha)} (\beta(1-p)p)^\alpha s^{\alpha-1} e^{-\beta(1-p)s}, \qquad (s \uparrow \infty). \tag{16}$$

From (3) and (15) we get

$$\frac{1}{p}=\frac{\beta}{\beta-\kappa},$$

that is,

$$\kappa=\beta(1-p), \tag{17}$$

and inserting this in (4) gives

$$\nu=\frac{1}{\beta p} \tag{18}$$

By inserting (17) and (18) in (16) we arrive at (11), which then holds also in the case when α is an integer and F is the exponential distribution.

REFERENCES

FELLER, W (1971) *An Introduction to Probability Theory and its Applications* Vol II (2nd ed.) Wiley, New York

KARLIN, S. and TAYLOR, H M. (1975) *A First Course in Stochastic Processes.* Academic Press New York

SOME NUMERICAL ASPECTS IN TRANSIENT RISK THEORY*

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ABSTRACT

We give some actual possibilities for computing numerical values in the classical risk models both in transient and asymptotical cases by introducing the concept of normed model. Some recent approximations are tested on numerical examples.

We also emphasize the interest of these methods to compute waiting time distributions (transient and stationary cases) in queueing theory.

1. MODELS CONSIDERED

1.1. *Risk Model*

We will limit our attention to the classical Cramér–Lundberg model for which we have the following characteristics:

(i) The claim number process is a Poisson one with parameter λ. Let $(A_n)_{n\geq 1}$ be the sequence of interarrival times between claims so that

$$\mathbb{E}(A_n)=\lambda^{-1}. \tag{1.1}$$

Following the current notation, $N(t)$ $(t\geq 0)$ represents the total number of claim occurrences on $(0, t]$.

(ii) The process of successive claim amounts is a sequence of non negative i.i.d. random variables $(B_n)_{n\geq 1}$ with d.f. $B(\cdot)$ such that

$$\mathbb{E}(B_n)=\beta \tag{1.2}$$

and this process is independent of $(A_n)_{n\geq 1}$.

(iii) The premium income process has a constant rate per unit of time: c. To avoid certain ruin on $[0, \infty)$, we must have:

$$\frac{\lambda\beta}{c}<1. \tag{1.3}$$

So, we can define η, the security loading by

$$c=\lambda\beta(1+\eta). \tag{1.4}$$

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Every risk model is thus characterized by a triple $(\lambda, B(x), \eta)$. Now define,

$$S(t)=\sum_{n=1}^{N(t)} B_n \tag{1.5}$$

with the usual convention that a summation over a void indice set is 0, and

$$R(t)=u+c\cdot t-S(t) \tag{1.6}$$

where u, supposed to be positive, is the initial reserve. Of course, if $F(x, t)$ is the d.f. of $S(t)$, we have

$$F(x,t)=\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}B^{n*}(x) \tag{1.7}$$

where B^{n*} represents the n-fold convolution of B.

If T is the random variable, possibly defective, defined by

$$T=\inf\{t: R(t)<0\} \tag{1.8}$$

we have for the probabilities of non-ruin the following definitions:

(a) on a finite horizon time $[0, t]$

$$\phi(u,t)=\mathbb{P}[T>t] \tag{1.9}$$

(b) on a finite horizon time $[0, \infty)$

$$\phi(u)=\lim_{t\to\infty}\phi(u,t). \tag{1.10}$$

For the ruin probabilities, we have, of course

$$\psi(u,t)=1-\phi(u,t) \tag{1.11}$$

$$\psi(u)=1-\phi(u). \tag{1.12}$$

1.2. *Normed Risk Models*

1.2.1. *First Semi-Normed Relation*

Let R_0 and R_1 be two risk models characterized respectively by $(1, B(\cdot), \eta)$ and $(\lambda, B(\cdot), \eta)$.

If $\phi_0(u, t)$ and $\phi_1(u, t)$ are corresponding non-ruin probabilities, we want to find a relation between ϕ_0 and ϕ_1. To do so; let us remark that from (1.7)

$$F_0(x,t)=\sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}B(x) \tag{1.13}$$

$$F_1(x,t)=\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}B(x) \tag{1.14}$$

so that

$$F_1(x,t)=F_0(x,\lambda t) \tag{1.15}$$

or $S_1(t)$ has the same distribution as $S_0(\lambda t)$.

Now, from (1.9)

$$\phi_1(u,t) = \mathbb{P}[S_1(t') \leqslant u + c_1 \cdot t', t' \in [0,t]] \tag{1.16}$$

with $c_1 = \lambda \cdot \beta \cdot (1+\eta)$ by (1.4). For R_0, we have $c_0 = (1+\eta) \cdot 1 \cdot \beta$. Using (1.15), we get

$$\begin{aligned}\phi_1(u,t) &= \mathbb{P}[S_0(\lambda t') \leqslant u + \lambda \cdot c_0 \cdot t', t' \in [0,t]] \\ &= \mathbb{P}[S_0(t'') \leqslant u + c_0 t'', t'' \in [0, \lambda t]]\end{aligned}$$

and finally

$$\phi_1(u,t) = \phi_0(u, \lambda t). \tag{1.17}$$

1.2.2. *Second Semi-Normed Relation*

Following Pfenninger (1974), we can also normalize the claim size distribution. Let us consider the risk model R_1 and R_2, where R_2 is characterized by $(\lambda, B'(\cdot), \eta)$ with

$$B'(x) = B(\beta x) \tag{1.18}$$

i.e., $B'(x)$ is the d.f. of the random variables B_n/β.

We have

$$\begin{aligned}\phi_1(u,t) &= \mathbb{P}[S_1(t') \leqslant u + \lambda \cdot \beta \cdot (1+\eta)t', t' \in [0,t]] \\ &= \mathbb{P}\left[\sum_{n=0}^{N(t')} B_n \leqslant u + \lambda \cdot \beta \cdot (1+\eta)t', t' \in [0,t]\right] \\ &= \mathbb{P}\left[\sum_{n=0}^{N(t')} \frac{B_n}{\beta} \leqslant \frac{u}{\beta} + \lambda(1+\eta)t', t' \in [0,t]\right] \\ &= \mathbb{P}\left[S_2(t') \leqslant \frac{u}{\beta} + \lambda(1+\eta)t', t' \in [0,t]\right]\end{aligned}$$

and finally

$$\phi_1(u,t) = \phi_2\left(\frac{u}{\beta}, t\right). \tag{1.19}$$

1.2.3. *Normed Relation*

Combining the two preceding steps, we get the so-called normed relation for the risk models R_1 and R_3 respectively characterized by $(\lambda, B(\cdot), \eta)$ and $(1, B'(\cdot), \eta)$:

$$\phi_1(u,t) = \phi_3\left(\frac{u}{\beta}, \lambda \cdot t\right) \tag{1.20}$$

R_3 is called the *normed model.*

This relation gives some simplification for numerical computation, especially for tabulation purposes. For example, in the M/M/1 model for which

$$B(x)=\begin{cases}0 & x<0\\ 1-e^{-\mu x} & x\geq 0\end{cases}$$

characterized shortly by the triple (λ, μ, n), the normed model R_3 is given by the triple $(1, 1, \eta)$ so that we have only one parameter, the security loading.

From the numerical point of view, it suffices to treat this model to obtain results for any model with triple (λ, μ, η).

2. THE QUEUEING MODEL

We will only consider the classical M/G/1 model for which $\bar{\lambda}$ is the rate of arrivals and $\bar{B}(\cdot)$ the d.f. of the service time, with mean $\bar{\beta}$. If $\bar{N}(t)$ $(t\geq 0)$ represents the total number of arrivals on $[0, t]$ and W_n the waiting time of customer number n (we suppose that $W_0=0$, i.e., a time 0, a service is just beginning) it can be shown (Janssen (1977)) that

$$\mathbb{P}[W_{\bar{N}(t)}\leq u]=\bar{\phi}(u, t) \tag{2.1}$$

$$\lim_{t\to\infty}[W_{\bar{N}(t)}\leq u]=\bar{\phi}(u) \tag{2.2}$$

where $\bar{\phi}(u, t)$ and $\bar{\phi}(u)$ are the non-ruin probabilities of a risk model characterized by $\bar{\lambda}$ as claim number process parameter, by $\bar{B}(x)$ as claim size distribution and by $c=1$ as premium rate. The security loading of this corresponding risk process is, of course, given by

$$c=(1+\eta)\bar{\lambda}\cdot\bar{\beta} \quad \text{or} \quad \eta=\frac{1}{\bar{\lambda}\cdot\bar{\beta}}-1. \tag{2.3}$$

Consequently, to every M/G/1 queueing model, characterized by $\bar{\lambda}$ and $\bar{\beta}(x)$, corresponds a risk process with parameters $(\bar{\lambda}, \bar{\beta}(x), (1/\bar{\lambda}\bar{\beta})-1)$. Inversely every result for the Cramér–Lundberg model $(\lambda, B(x), \eta)$ can be transposed for a M/G/1 queueing model with parameters

$$\bar{\lambda}=\frac{1}{(1+\eta)\beta}$$

$$\bar{B}(x)=B(x).$$

For a fixed η and a given $B(x)$, we can see the relation between the normed-model non-ruin probability $\phi_3(u, t)$ and the waiting time distribution. We have:

$$\mathbb{P}[W_{\bar{N}(t)}\leq u]=\phi_3\left(\frac{u}{\beta}, \frac{t}{(1+\eta)\beta}\right). \tag{2.4}$$

3. NON-RUIN PROBABILITY IN THE TRANSIENT CASE FOR THE M/G/1 MODEL

Theoretically, two principal methods are used to solve this problem: the first method is based upon the double Laplace transform of $\phi(u, t)$ and the second one upon the previous determination of $\phi(0, t)$.

3.1. *Cramér–Arfwedson–Thorin*

The equation of Thorin (1968), valid in the general case GI/G/1 is:

$$\phi(u, t) = \int_0^t dK(v) \int_{-\infty}^{u+cv} \phi(u+cv-x, t-v)\, dB(x) + 1 - K(t) \tag{3.1}$$

where $K(t) = 1 - e^{-\lambda t}$. It gives the double Laplace–Stieltjes transform of $\phi(u, t) =$

$$\bar{\bar{\phi}}(s, z) = -z(1 - s/s_1(z))/(1 - cs - z - \bar{B}(s)) \tag{3.2}$$

where $s_1(z)$ is the only root with a negative real part in the Lundberg equation:

$$1 - z + c \cdot s - \bar{B}(s) = 0, \tag{3.3}$$

$\bar{B}(s)$ being the Laplace–Stieltjes transform of $B(x)$.

For the M/G/1 model, Cramér (1955) and Arfwedson (1950) obtained this result by using the integro-differential equation

$$c \frac{\partial}{\partial u} \phi(u, t) = \frac{\partial}{\partial t} \phi(u, t) + \phi(u, t) - \int_0^u \phi(u-y, t)\, dB(y). \tag{3.4}$$

This was also found by Beekman (1966) using results of Donsker and Baxter (1957) about processes with stationary independent increments.

Theoretically, thus, the problem is worked out, but we have to use twice the Laplace inversion. However, we dispose of fiable algorithms for this inversion (Piessens (1969), Stroud and Secrest (1966)), but this needs some care: the Laplace inversion of a good approximation of a given function is not surely a good approximation of the Laplace inversion of this function. Some precautions are thus required if we want to compute $\phi(u, t)$ by means of a double inversion of $\bar{\bar{\phi}}(s, z)$; probably for this reason, there are few results needing such double transformation in the risk theory literature.

However, if $B(x)$ is an exponential polynomial, i.e., if

$$B(x) = 1 - \sum_{v=1}^{m} b_v e^{-\beta_v x}, \qquad b_v > 0, \beta_v > 0 \quad v = 1, 2, \ldots m, \quad \sum b_v = 1 \tag{3.5}$$

then the problem can be solved with only one inversion. In this case, $\bar{\phi}(u, z)$, the Laplace transform of $\phi(u, t)$, is given by

$$\bar{\phi}(u, z) = 1 - \sum_{v=1}^{m} g_v(z)\, e^{-u\, s_2(z)} \tag{3.6}$$

where $s_{2v}(z)$ are the m roots of the Lundberg equation with a positive real part.

Furthermore, in this case, this equation is a polynomial one and the roots are easily obtained by well-known algorithms (Bairstow, Newton–Raphson) (see e.g., Wikstad (1977), Stroeymeyt (1977)).

It is also possible to approximate a general claim size distribution by an exponential polynomial; this was tested by Thorin and Wikstad (1977) for a lognormal distribution.

3.2. *Prabhu–Beňes–Seal*

The well-known relations of Prabhu (1961) can be used here:

$$\phi(u,t)=F(u+ct,t)-c\int_0^t \phi(0,t-\theta)f(u+c\theta,\theta)\,d\theta \tag{3.7}$$

$$\phi(0,t)=\frac{1}{ct}\int_0^t F(y,t)\,dy, \tag{3.8}$$

where $f(x,t)=\partial/\partial x\,F(x,t)$.

Although the function $F(x,t)$ is very difficult to handle directly, the use of the Laplace transform and an integration give the non-ruin probability.

3.3. *Direct Results for* M/M/1 *and* M/D/1 *Models*

M/M/1 model, i.e., the model with the following claim size distribution:

$$B(x)=\begin{cases}0 & x<0,\\ 1-e^{-x} & x\geq 0\end{cases}$$

is the really well-known model in risk theory, it has a direct solution in terms of a modified Bessel function of first class; some subroutines give very accurate values of this function (see e.g. Stroeymeyt (1977)).

The M/D/1 model with a deterministic claim amount can also be directly solved (see e.g., Seal (1974)).

4. THE ASYMPTOTIC NON-RUIN PROBABILITY

For a general M/G/1 model, we have:

$$\phi(u)=\lim_{t\to\infty}\phi(u,t)=\bar{\phi}(u,0) \tag{4.1}$$

where

$$\bar{\phi}(u,z)=\int_{0^-}^{\infty} e^{-zt}\,d_t\phi(u,t).$$

Thus, only one inversion of a Laplace transform is needed and we avoid some problems raised by the double inversion. Furthermore, in some special cases, the value is explicitly given. If $B(x)$ is an exponential polynomial (3.5), Cramér

(1955) gives an explicit formula:

$$\phi(u) = 1 - \sum_{k=1}^{m} C_k e^{-R_k u} \tag{4.2}$$

where $R_k, k = 1, 2, \ldots m$, denote the m roots of the Lundberg equation, a polynomial one in this case; $C_k, k = 1, 2, \ldots m$, are simple functions of those roots. Especially, if $B(x)$ is an exponential, we have the following expression:

$$\phi(u) = 1 - \frac{1}{1+\eta} e^{-\eta/(1+\eta) u}. \tag{4.3}$$

For the M/D/1 model, a recursive formula exists to compute $\phi(u)$.

As pointed out by Bohmam (1971) the computation of asymptotic non-ruin probabilities is now easy to do even with a common desk computer.

5. SOME NUMERICAL RESULTS IN THE TRANSIENT CASE

We will restrict ourselves to three models already treated in the literature:

Model A or M/M/1 model (see e.g., Seal (1974), Stroeymeyt (1977))

$$\lambda = 1$$

$$B(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

$$\eta = 0.1.$$

Model B or M/D/1 model (see e.g., Seal (1974))

$$\lambda = 1$$

$$B(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}$$

$$\eta = 0.$$

Model C (see e.g., Stroeymeyt (1977))

$$\lambda = 2$$

$$B(x) = \begin{cases} 0 & x < 0 \\ 1 - 0.8 e^{-0.7x} - 0.2 e^{-x} & x \geq 0 \end{cases}$$

$$\eta = 0.037234.$$

These models do not give rise to special computational difficulties, they are useful to test some approximations and bounds, and to test different methods.

5.1. *The Accuracy of the Laplace Inversion Methods*

To test the precision of the Laplace inversion methods, we give in Table 1 the real values of the non-ruin probability computed by means of a Bessel modified function for the model A (Column 1.i). The same values are computed by the

TABLE 1

MODEL A VALUES OF $\phi(0, T)$

T	(1 1)	(1 2)	(1 3)
0 1	0.90965	0 90321	0.89887
0 2	0 83561	0 82978	0 82586
0 3	0 77429	0 76905	0 76547
0 4	0 72295	0 71817	0 71497
0.5	0 67952	0.67527	0.67230
0.6	0.64242	0.63589	0 63589
0 7	0 61043	0 60688	0 60453
0 8	0.58260	0.57942	0 57726
0 9	0.55819	0 55530	0 55335
1 0	0.53660	0 53400	0 53223
2 0	0 40714	0 40621	0.40554
3 0	0 34479	0 34442	0.34421
4 0	0 30669	0.30656	0.30649
5 0	0 28040	0 28035	0 28034
6 0	0 26088	0.26086	0.26086
7 0	0 24566	0 24566	0 24566
8.0	0 23337	0 23337	0 23337
9 0	0 22319	0 22319	0.22319
10 0	0 21457	0 21457	0 21457
100	0.11001	0 11001	0.11002
200	0.09902	0.09902	0 09897

(1.1) Direct computation
(1 2) Stroud and Secrest method
(1 3) Piessens method

Stroud and Secrest method (Column 1.2) and by the Piessens method (Column 1.3) for the model A, for different values of t and for $u = 0$.

To obtain those values, the Prabhu–Beňes–Seal relations (3.7) and (3.8), were used. It can be pointed out that the non-ruin probabilities obtained by Laplace inversion are quite similar to the non-ruin probabilities "directly" computed, except for small values of t.

In Table 2, we give the non-ruin probabilities for the model C obtained by the Stroud and Secrest method (2.1) and by the Piessens method (2.2), for $u = 0$. Here also, it can be remarked that those methods give nearly the same values except for small values of t.

5.2. *Approximations of $F(x, t)$ by Means of Normal Power Approximation and Γ-function*

The form of the Prabhu–Beňes–Seal relations suggests that an approximation of $F(x, t)$ can provide a good approximation of the non-ruin probability. But those approximations of $F(x, t)$ are only valid for large t, and thus they bring a lot of imprecision in the integral

$$\int_0^t f(c\theta + u, \theta)\, \phi(0, t-\theta) d\theta \quad \text{in (3.7).}$$

TABLE 2
MODEL C VALUES OF $\phi(0, t)$

t	(2 1)	(2 2)
0 1	0.82524	0.82914
0 2	0 71305	0 71638
0.3	0 63429	0 63511
0 4	0 57240	0 57458
0.5	0.52601	0.52779
0.6	0 48913	0 49057
0.7	0 45904	0 46020
0 8	0 43396	0.43489
0 9	0.41267	0.41347
1	0 39432	0 39496
2	0 29016	0.29023
3	0 24180	0 24180
4	0 21252	0 21251
5	0 19239	0 19239
6	0 17748	0 17748
7	0.16586	0 16586
8	0 15648	0.15648
9	0.14871	0 14871
10	0 14213	0 14213

(2 1) Stroud and Secrest method
(2 2) Piessens method

However, some of those methods will provide an acceptable approximation of $\phi(0, t)$, when t is not too small.

Bohman and Esscher (1963) and Cramér (1955) give approximations of $F(x, t)$ in terms of $\Phi(x)$. the reduced normal distribution function. Normal Power approximations are proposed by Pesonen (1975) and by Taylor (1978). A Γ-function was also proposed by Seal (1978).

In our examples, the best method to calculate $\phi(0, t)$ seems to be the Normal Power approximation from Taylor (1978).

Table 3 contains some values of $\phi(0, t)$ and of this approximation for the M/M/1 model. The method of Taylor consisting of an approach of $\phi(u, t)$ by means of $\phi(0, t)+(1-\phi(0, t))\cdot G(w, t)$ involves some numerical complications: for one certain value of the security loading, η, negative numbers are obtained for a variance. Furthermore, this method occasionally involves some surprising results: an approximation for $\phi(1, 10)$ is smaller than the approximation for $\phi(1, 100)$. Taylor thinks that the consideration of higher order moments could give more accuracy but, of course, this will lead to complications from the numerical point of view.

5.3. *The De Vylder Approximation*

De Vylder (1978) proposed to approach the asymptotic non-ruin probability of a M/G/1 model by non-ruin probability of a M/M/1 model with such parameters

TABLE 3

	Model A					Model C			
T	(1)	(2)	(3)	(4)	T	(1)	(2)	(3)	(4)
0.1	0.46003	1 55294	0 90136	0 90965	0.1	0 44025	1.20985	0 85731	0 82914
0 2	0.44370	1.19918	0 86913	0.83561	0 2	0 41618	0 93522	0 75184	0 71638
0 3	0.43134	1 03439	0 81417	0.77429	0.3	0.39816	0.80241	0.65527	0 63511
0 4	0 42104	0 93164	0 75267	0.72295	0 4	0 38333	0.71731	0 58645	0 57458
0 5	0 41208	0.85861	0 70029	0 67952	0.5	0 37057	0.65557	0.53537	0 52779
0 6	0 40408	0.80266	0 65686	0 64242	0.6	0.35930	0.60755	0 49576	0 49057
0 7	0 39681	0.75765	0 62056	0 61043	0.7	0.34918	0 56851	0 46396	0 46020
0.8	0 39013	0 72021	0 58981	0.58260	0 8	0 33998	0 53578	0.43772	0 43489
0.9	0 38394	0.68828	0 56340	0 55819	0 9	0 33154	0.50774	0 41561	0 41347
1 0	0 37814	0 66052	0.54043	0 53660	1.0	0.32373	0 48330	0 39665	0.39496
2 0	0 33420	0 49466	0.40750	0 40714	2 0	0.26745	0 33895	0.29046	0 29023
3 0	0 30422	0.41030	0.34484	0 34479	3.0	0 23245	0.27019	0 24185	0 24180
4 0	0 28162	0 35665	0 30668	0 30669	4 0	0 20804	0 22980	0.21253	0 21251
5 0	0 26372	0.31910	0 28038	0.28040	5 0	0 18993	0 20341	0 19239	0 19239
6 0	0 24910	0 29131	0.26086	0 26088	6.0	0.17590	0.18488	0 17748	0 17748
7.0	0.23690	0 26995	0 24564	0.24566	7 0	0 16470	0 17112	0 16586	0 16586
8.0	0 22655	0.25306	0 23335	0 23337	8 0	0 15553	0 16045	0 15648	0 15648
9 0	0 21764	0 23941	0 22317	0 22319	9.0	0.14785	0.15187	0.14871	0 14871
10 0	0 20989	0.22815	0 21455	0 21457	10.0	0.14133	0.14477	0.14213	0 14213
20.0	0 16577	0 17309	0 16815	0 16816	20 0	0 10583	0.10785	0 10648	0 10649
40 0	0 13453	0 13913	0 13621	0 13621	40 0	0 08092	0 08233	0 08143	0.08143

(1) Normal-Power Approximations of $\phi(0, t)$ (two terms)
(2) Normal-Power Approximations of $\phi(0, t)$ (one term)
(3) G. C Taylor Approximation of $\phi(0, t)$.
(4) $\phi(0, t)$.

that the two reserve processes $R(t)$ have the same first moments. De Vylder emphasized the fact that the initial reserve must be large and supposed that this approximation can also be used for transient probabilities. In Table 4, we compare some results of this approximation for the model B and the model C. If this approximation is not good for small values of u, this very simple method gives acceptable values for important values of u ($u = 10$).

5.4. *Some Easily Computable Bounds in Transient Case*

We found it interesting to examine some easily computable bounds, to test approximations or calculations by means of Laplace inversion and to eliminate some aberrant results.

(1) *Gerber Minoration*: Gerber (1973) gives a minoration based upon martingales. It can be improved for the M/M/1 model. This minoration cannot be used with a null initial reserve except for the M/M/1 model. For the M/M/1 normed model, the Gerber minoration takes the following form:

$$\phi(u, t) \leq 1 - \min_{(c-1)/c < r < 1} (1-r) \exp\left(-ru - crt + t\frac{r}{1-r}\right).$$

TABLE 4

Model B

$u=0$			$u=1$			$u=2$		
t	(1)	(2)	t	(1)	(2)	t	(1)	(2)
1	0.73576	0.26218	1	0.91970	0.75288	1	0.98101	0.93381
2	0.60901	0.18674	2	0.83457	0.61680	2	0.94171	0.84879
3	0.53106	0.15284	3	0.76548	0.53420	3	0.89866	0.77767
4	0.47697	0.13252	4	0.70988	0.47763	4	0.85758	0.72010
5	0.43662	0.11861	5	0.66437	0.43584	5	0.82000	0.67297
6	0.40503	0.10833	6	0.62638	0.40337	6	0.78607	0.63369
7	0.37944	0.10032	7	0.59411	0.37721	7	0.75552	0.60040
8	0.35815	0.09387	8	0.56630	0.35554	8	0.72796	0.57176
9	0.34008	0.08852	9	0.54201	0.33722	9	0.70300	0.54680
10	0.32450	0.08399	10	0.52057	0.32147	10	0.68031	0.52481

$u=3$			$u=4$			$u=5$			$u=6$		
t	(1)	(2)	t	(1)	(2)	t	(1)	(2)	t	(1)	(2)
1	0.99634	0.98491	1	0.99941	0.99695	1	0.99992	0.99944	1	0.99999	0.99991
2	0.98231	0.94840	2	0.99528	0.98438	2	0.99888	0.99573	2	0.99976	0.99893
3	0.96124	0.90623	3	0.98669	0.96449	3	0.99586	0.98776	3	0.99883	0.99612
4	0.93698	0.86513	4	0.97461	0.94094	4	0.99063	0.97627	4	0.99681	0.99118
5	0.91181	0.82717	5	0.96024	0.91613	5	0.98342	0.96237	5	0.99358	0.98428
6	0.88695	0.79275	6	0.94455	0.89139	6	0.97466	0.94701	6	0.98917	0.97581
7	0.86298	0.76168	7	0.92822	0.86741	7	0.96475	0.93091	7	0.98372	0.96615
8	0.84016	0.73363	8	0.91171	0.84449	8	0.95405	0.91453	8	0.97741	0.95564
9	0.81859	0.70822	9	0.89533	0.82278	9	0.94283	0.89822	9	0.97039	0.94458
10	0.79827	0.68513	10	0.87925	0.80230	10	0.93132	0.88216	10	0.96283	0.93319

$u=7$			$u=8$			$u=9$			$u=10$		
t	(1)	(2)	t	(1)	(2)	t	(1)	(2)	t	(1)	(2)
1	1	0.99999	1	1	1	1	1	1	1	1	1
2	0.99995	0.99975	2	0.99999	0.99995	2	1	0.99999	2	1	1
3	0.99969	0.99886	3	0.99993	0.99968	3	0.99998	0.99992	3	1	0.99998
4	0.99899	0.99694	4	0.99970	0.99901	4	0.99992	0.99969	4	0.99998	0.99991
5	0.99768	0.99386	5	0.99921	0.99774	5	0.99975	0.99921	5	0.99993	0.99974
6	0.99566	0.98962	6	0.99836	0.99579	6	0.99942	0.99838	6	0.99980	0.99941
7	0.99291	0.98434	7	0.99708	0.99314	7	0.99886	0.99714	7	0.99958	0.99887
8	0.98948	0.97819	8	0.99535	0.98981	8	0.99805	0.99546	8	0.99922	0.99807
9	0.98543	0.97132	9	0.99317	0.98586	9	0.99695	0.99334	9	0.99870	0.99700
10	0.98082	0.96389	10	0.99055	0.98136	10	0.99555	0.99078	10	0.99799	0.99563

TABLE 4 (*continued*)

		Model C			
t	(1)	(2)	t	(1)	(2)
1	0 99164	0 99166	21	0 66608	0.66583
2	0.97316	0.97313	22	0.65748	0 65724
3	0.95019	0.95009	23	0.64930	0 64905
4	0.92596	0.92581	24	0 64150	0 64125
5	0 90206	0.90187	25	0.63405	0.63380
6	0.87917	0.87895	26	0.62693	0.62668
7	0 85758	0 85734	27	0.62012	0 61987
8	0.83735	0.83711	28	0 61359	0 61334
9	0 81847	0 81821	29	0 60733	0 60708
10	0 80084	0.80059	30	0 60132	0 60106
11	0 78440	0 78413	31	0 59554	0 59528
12	0 76902	0 76876	32	0 58998	0.58972
13	0 75463	0.75437	33	0 58463	0 58437
14	0 74115	0 74089	34	0 57947	0.57921
15	0.72848	0.72821	35	0 57450	0.57423
16	0 71655	0.71629	36	0 56970	0 56942
17	0 70531	0 70505	37	0.56506	0 56478
18	0 69469	0 69443	38	0.56058	0.56029
19	0.68464	0 68439	39	0 55624	0 55595
20	0 67512	0 67487	40	0.55204	0.55174

(1) $\phi(10, t)$.
(2) De Vylder approximation of $\phi(10, t)$

Taking the derivative, it can be easily proved that the minimum is attained for

$$\rho = 1 - \frac{\sqrt{1+4(u+ct)t}-1}{2(u+ct)}.$$

(2) *Gerber Majoration*: when the initial reserve is null, Gerber (1979) gives a majoration of $\phi(0, t)$

$$\phi(0, t) \leqslant \left(1 - \frac{\lambda\beta}{c}\right) + \frac{1}{ct}\,\frac{\lambda\sigma^2}{c - \lambda\beta}.$$

(3) *Beekman–Bowers Minoration*: Beekman and Bowers (1972) proposed a very simple minoration of $\phi(u, t)$

$$1 - \frac{\alpha_2 t}{u^2} \leqslant \phi(u, t).$$

Of course, for large values of t, this minoration becomes negative.

(4) Bounds based upon the asymptotic non-ruin probability: The asymptotic non-ruin probability is generally easy to compute: either explicit formula exist or only one Laplace inversion provides it. With these probabilities, it is possible to construct bounds for small values of t, bearing in mind that, especially in this case, different values were observed for Laplace inversion (see Delfosse 1980).
(4a) Minoration:

$$\frac{\phi(u)}{\phi(u+ct)} \leqslant \phi(u, t).$$

TABLE 5

BOUNDS AND APPROXIMATIONS DESCRIBED IN 5 4

		Model A $u=0$			
t	(1)	(4a)	$\phi(0, t)$	(4b)	(2)
0 1	0 77724	0.90950	0.90965	0 90992	1
0 2	0 62707	0 83472	0 83561	0 83727	↑
0 3	0 52256	0 77188	0 77429	0 77857	
0.4	0.44734	0 71834	0.72295	0 73076	
0 5	0.39140	0 67218	0.67952	0 69139	
0 6	0 34857	0 63197	0.64242	0 65855	
0 7	0.31491	0.59664	0 61043	0.63079	
0 8	0 28788	0.56534	0.58260	0.60699	
0 9	0 26576	0 53744	0 55819	0 58635	
1 0	0 24736	0 51239	0 53660	0 56822	
2.0	0 15722	0.35553	0.40714	0.45858	
3.0	0 12549	0.27841	0 34479	0 40224	
4 0	0 11015	0 23273	0 30669	0 36567	
5 0	0.10166	0.20265	0 28040	0.33926	
6 0	0 09666	0 18143	0.26088	0.31894	
7 0	0.09369	0.16572	0 24566	0 30263	
8 0	0 09199	0.15369	0.23337	0.28915	
9.0	0.09115	0 14421	0 22319	0 27775	↓
10	0 09091	0 13659	0.21457	0.26794	1
100	0.09091	0 09091	0 11001	0.13061	0.18181
200	0 09091	0 09091	0.09902	0 11145	0 13636

		Model A· $u=10$			
t	(4a)	(3)	(1)	$\phi(10, t)$	(4b)
0.1	0.99423	0 99800	0.99998	0 99999	0 99999
0 2	0 98869	0 99600	0 99994	0 99998	0.99998
0 3	0.98321	0 99400	0.99988	0.99997	0 99997
0 4	0 97784	0 99200	0 99980	0 99995	0 99995
0 5	0 97259	0 99000	0.99969	0 99992	0.99993
0.6	0.96744	0 98800	0 99956	0.99989	0.99990
0 7	0 96420	0 98600	0.99941	0.99985	0.99987
0.8	0 95746	0 98400	0.99923	0 99980	0 99983
0 9	0.95261	0 98200	0 99902	0.99975	0.99979
1	0.94787	0 98000	0 99879	0 99969	0.99974
2	0 90517	0 96000	0.99488	0.99865	0 99895
3	0.86972	0 94000	0 98833	0 99677	0.99757
4	0.83996	0 92000	0.97965	0 99410	0 99566
5	0 81473	0.90000	0 96942	0 99077	0 99332
6	0.79317	0 88000	0.95816	0 98689	0 99061
7	0 77463	0 86000	0 94629	0.98258	0.98761
8	0 75858	0 84000	0 93413	0.97796	0.98440
9	0 74462	0 82000	0 92190	0 97311	0.98103
10	0 73242	0.80000	0 90978	0 96810	0 97754
100	0 63375	0	0 63374	0.73947	0 78760
200	0.63374	0	0 67334	0 68217	0 72116

(4b) Majoration.

$$\phi(u,t) \leq \phi(u) \cdot F(u+ct,t) \cdot \frac{1}{(\phi * F(x,t))|_{x=u+ct}} \quad \text{(when } t \leq u/c\text{)}.$$

In Table 5, we present these bounds for the model A for $u = 0$ and for $u = 10$; in Table 6, the same bounds for the model C, for $u = 10$.

TABLE 6

BOUNDS AND APPROXIMATIONS DESCRIBED IN 5 4

		Model C	$u = 10$		
t	(4a)	(3)	(2)	$\phi(10, t)$	(4b)
1	0 83201	0 89871	0 93398	0 99164	0 99346
2	0 71966	0.79741	0 85465	0 97316	0 98010
3	0.63941	0 69612	0 78205	0 95019	0 96382
4	0 57937	0.59482	0 71969	0 92596	0 94665
5	0 53287	0 49353	0 66688	0 90206	0.92956
6	0.49588	0 39223	0 62208	0 87917	0 91300
7	0 46582	0 29094	0 58387	0 85758	0 89715
8	0.44098	0.18964	0 55101	0 83735	0 88210
9	0 42016	0 08835	0 52254	0 81847	0.86784
10	0 40250	0	0 49767	0 80084	0 85436
20	0.31268	0	0 35744	0 67512	0 75213
30	0 28239	0	0.29932	0 60132	0 68634
40	0 26982	0	0 26935	0 55204	0 63950

COMMENTS

These bounds are rather crude for certain values, but a package of these majorations and minorations does not take much computer time and allows to elıminate some inexact values. Our minoration $\phi(u)/\phi(u+ct)$ shows that $\phi(0, 0.1)$ and $\phi(0, 0.2)$ are too small in Model A and in Model C; for those values, the obtaıned non-ruin probabilities were the most different.

These minorations and majorations are also interesting to limit the use of precise but time-consumıng methods: those bounds can be used to restrain the area of possible computations, if we allow some parameters of the model to vary. For example, the calculation of the bounds (4a), (4b) takes 16 times less calculation time than the computation of an exact value by the Laplace inversion.

REFERENCES

ARFWEDSON, G (1950) Some Problems ın the Collectıve Theory of Rısk *Skand Aktuarııdskr* 1–38

ARFWEDSON, G (1954) Research ın Collectıve Rısk Theory Part I *Skand Aktuarııdskr* 191–223

BAXTER, G. and DONSKER, M. (1957) On the Dıstrıbutıon of the Supremum Functıonal for Process wıth Independent Increments. *Transactıons of the Amerıcan Mathematıcal Socıety* **85**, 73–87.

BEEKMAN, JOHN A. (1966) Research on the Collective Rısk Stochastıc Process *Skand. Aktuarııdskr* 65–77

BEEKMAN, JOHN A and NEWTON L BOWERS JR (1972) An Approximation to the Finite Time Ruin Function, Part II *Skand. Aktuartidskr* 128–137.

BOHMAN, M. (1971) The Ruin Probability in a Special Case *ASTIN Bull* **6**, 66–68

BOHMAN, M. and ESSCHER, F. (1963) Studies in Risk Theory with Numerical Illustrations concerning Distribution Functions and Stop-less Premiums *Skand Aktuartidskr.* 1963 (3–4) and 1964 (1–2).

CRAMÉR, M (1955) Collective Risk Theory *Jubilee Volume of Forsakringsaktiebogalet Skandia.*

DELFOSSE, PH (1980) Calcul de la Probabilité de Non-ruine en Théorie Collective du Risque dans le cas transitoire pour le modèle M/G/1, Mémoire de Licence en Sciences Actuarielles, Université Libre de Bruxelles

DE VYLDER, F (1978) A Practical Solution to the Problem of Ultimate Ruin Probability *Scand Actuarial Journal* **78-2**, 114–119

GERBER, H (1973) Martingales in Risk Theory, *Mit Schw. Vers. Mat* **73** (2), 205–215

GERBER, H (1979) *An Introduction to Mathematical Risk Theory.* Wharton School, University of Pennsylvania, Philadelphia

JANSSEN, J. (1977) The Semi-Markov Model in Risk Theory In *Advances in Operational Research*, edited by M Roubens North-Holland Amsterdam

PESONEN, E (1975) N·P-Technique as a Tool in Decision Making. *ASTIN Bull* **8**, 359–363

PFENNINGER, F. (1974) Eine Neue Methode zur Berechnung der Ruinwahrscheinlichkeit mittels Laguerre–Entwicklung. *DGVM-Blatter* Band XI Heft 4

PIESSENS, R (1969) New Quadrature Formulas for the Numerical Invasion of the Laplace Transform *Nordisk Tidskrift for Informations behandeling* (*BIT*) **9**, 351–361.

PRABHU, N. U (1961) On the Ruin Problem of Collective Risk Theory. *Ann. Math Statist.* **32**, 757–764

SEAL, H L (1969) *Stochastic Theory of a Risk Business* J Wiley New York.

SEAL, H. L (1974) The Numerical Calculation of $U(w, t)$ the Probability of Non-Ruin in an Interval $(0, t)$ *Scand. Actuarial Journal* 121–129

SEAL, H L (1978) *Survival Probabilities—The Goal of Risk Theory.* J. Wiley Chichester-New-York-Brisbane-Toronto.

STROEYMEYT, PH (1977) *Calcul numérique de la probabilité de non-ruine dans les cas transitoire et asymptotique* Mémoire de Licence en Sciences Actuarielles, Université Libre de Bruxelles

STROUD, A H and SECREST, D (1966) *Gaussian Quadrature Formules* Prentice Hall Inc Englewood Cliffs, New Jersey

TAYLOR, G C (1978) Representation and Explicit Calculation of Finite Time Ruin Probabilities *Scand Actuarial Journal* **78** (1), 1–18

THORIN, O. (1968) An Identity in the Collective Risk Theory with Some Applications. *Skand Aktuartidskr* 26–44

THORIN, O (1970) Some Remarks on the Ruin Problem in Case the Epochs of Claims Form a Renewal Process *Skand Aktuartidskr* 29–50

THORIN, O (1977) Ruin Probabilities Prepared for Numerical Calculation *Scand. Actuarial Journal. Suppl* 7–17

THORIN, O and WIKSTAD, N (1977) Calculation of Ruin Probabilities When the Claim Distribution is Lognormal *ASTN Bull* **9**, 231–246.

WIKSTAD, N (1977) How to Catalogue Ruin Probabilities According to the Classical Risk Theory *Scand Actuarial Journal Suppl* 19–24

CALCUL DES PRIMES ET MARCHANDAGE*

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PART I

Two premium calculation principles by negotiation. Using, as main tools,
the classical risk exchange model by Borch and
the bargaining models of Nash and Kalai–Smorodinsky,
we define two new premium calculation principles, whose main goal is to take explicitly into account the attitude towards risk of the policy-holders. Those principles are neither additive nor iterative, but they nevertheless possess several important properties: the premium is translation-invariant, it does not depend neither on the reserves nor on the portfolio of the company; it takes into account all the moments of the claim distribution; it is independent of the policy-holder's wealth but increases with his risk aversion.

PART II

Coalition against an insurance company. While computing the core of this risk exchange, we show that it can be of the policy-holder's interest to coalize in order to obtain premium cuts.

PREMIÈRE PARTIE: DEUX PRINCIPES DE CALCUL DES PRIMES PAR NEGOCIATION

Le modèle d'échange de risques de Borch entre plusieurs compagnies d'assurances soucieuses d'améliorer leur situation en formant un pool de réassurance a fait l'objet de très nombreuses publications. Ce n'est que depuis quelques années cependant que l'on semble s'être aperçu que le même modèle pouvait être utilisé pour décrire toute économie d'échange, en particulier le contrat d'assurance simple entre un assuré et sa compagnie. Si l'on suppose que les préférences de l'assuré peuvent être décrites par une fonction d'utilité exponentielle, et que l'assureur est indifférent au risque en première approximation, les contrats Pareto-optimaux consistent en une couverture complète du risque, moyennant le paiement d'une prime que le critère de Pareto-optimalité ne permet pas de déterminer.

Parmi les différents modèles de marchandage présentés en théorie des jeux et en économie mathématique, les plus satisfaisants nous semblent être ceux de Nash et de Kalai–Smorodinsky. Nous allons appliquer ces deux modèles à la

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négociation que constitue la signature d'un contrat d'assurance. Cela nous permettra de déterminer un traité unique sur la courbe Pareto-optimale. En d'autres termes, les deux modèles de marchandage vont nous permettre de définir deux nouveaux principes de calcul des primes, principes dont nous étudierons les propriétés.

2. LE MODÈLE CLASSIQUE D'ÉCHANGE DE RISQUES

Soient $[J_1, \ldots, J_n]$ n agents économiques soumis à un risque. J_j est caractérisé par

1. sa fonction d'utilité $u_j(x)$, supposée à dérivée première positive et à dérivée seconde non-positive, et
2. sa situation initiale $[R_j, F_j(x_j)]$, où R_j est le montant dont il dispose, et $F_j(x_j)$ est la fonction de répartition des débours à venir.

Les agents vont chercher à améliorer leur situation en concluant un traité d'échange de risques $\bar{y} = [y_1(\bar{x}), \ldots, y_n(\bar{x})]$, avec $\sum_{j=1}^{n} y_j(\bar{x}) = \sum_{j=1}^{n} x_j$, où $y_j(\bar{x}) = y_j(x_1, \ldots, x_n)$ est le montant payé par l'agent J_j, après échange, si les débours pour les n participants s'élèvent respectivement à $x_1, \ldots, x_n$. La signature d'un tel traité modifie l'évaluation de la situation de J_j de

$$U_j(x_j) = U_j[R_j, F_j(x_j)] = \int_0^{\infty} u_j(R_j - x_j)\, dF_j(x_j)$$

en

$$U_j(\bar{y}) = \int_{\theta} u_j[R_j - y_j(\bar{x})]\, dF(\bar{x})$$

où θ est l'orthant positif de E^n et $F(\bar{x})$ la fonction de répartition de la distribution n-dimensionnelle des débours.

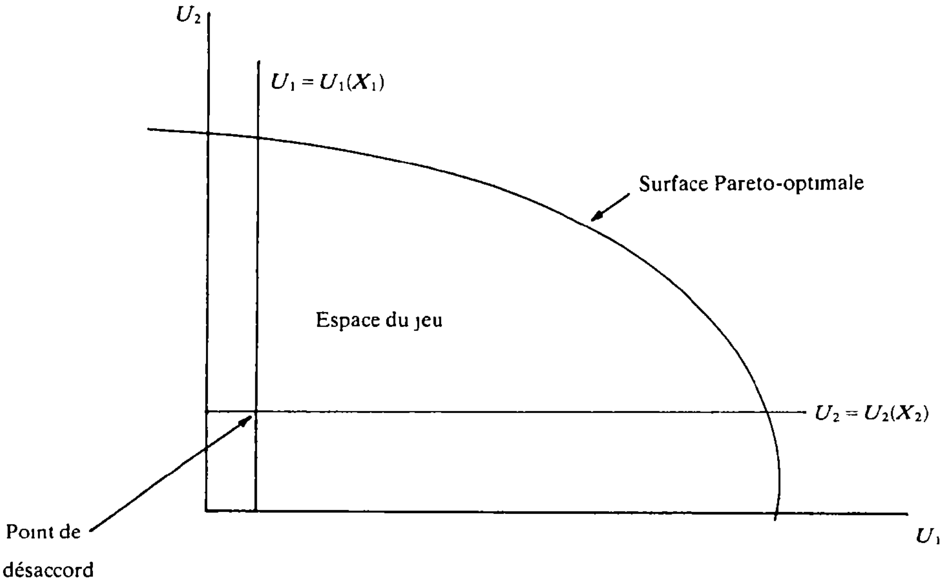
Un traité $\bar{y}$ est dit Pareto-optimal s'il n'existe aucun $\bar{y}'$ qui lui soit préférable, c'est-à-dire tel que $U_j(\bar{y}') \geq U_j(\bar{y})$ pour tout j, avec au moins une inégalité stricte. L'ensemble des échanges Pareto-optimaux a été caractérisé par Borch (1960):

THEOREME 1. *$\bar{y}$ est un traité Pareto-optimal ss'il existe n constantes positives $k_1, \ldots, k_n$, telles que, avec une probabilité* 1,

$$k_j u_j'[R_j - y_j(\bar{x})] = k_1 u_1'[R_1 - y_1(\bar{x})] \qquad j = 1, \ldots, n.$$

Un tel traité est unique lorsque les k_j sont fixés. Sauf cas de dégénérescence des $F_j(x_j)$, il existe une infinité de k_j conduisant à un traité Pareto-optimal, même lorsque l'on impose la condition de rationalité individuelle, qui exige qu'un agent n'accepte jamais un traité qui détériore sa situation initiale: $U_j(\bar{y}) \geq U_j(x_j)\ \forall j$.

La multiplicité des solutions provient du fait que dans la définition de la Pareto-optimalité ne figure aucun axiome de partage. La coopération permet d'accroître le bien-être global, mais rien n'est dit quant à la manière de répartir ce bénéfice entre les participants. Chacun a intérêt à obtenir une valeur de k_j la plus grande possible, de manière à payer le moins possible. Nous avons une situation où les intérêts des joueurs sont partiellement complémentaires (globalement, le marché a intérêt à signer un traité Pareto-optimal) et partiellement contradictoires (une fois la surface Pareto-optimale atteinte, tout gain d'utilité pour un agent ne peut se faire qu'au détriment d'un autre).



Ce modèle a fait l'objet de nombreuses études (Bühlmann et Jewell (1979), Gerber (1978), Lemaire (1977), Baton et Lemaire (1981), ...) dans le cas où les agents économiques sont des compagnies d'assurances désireuses de former un pool de réassurance. Ce n'est que fort récemment (Moffet (1979), Bardola (1981)) que l'on semble s'être aperçu que le même modèle pouvait s'appliquer au marché à deux agents formé par une compagnie d'assurances J_1 et un assuré potentiel J_2.

Gerber (1974*a*, *b*), Leepin (1975) et d'autres auteurs ayant mis en évidence les propriétés fort intéressantes des fonctions d'utilité exponentielles du type

$$u_i(x) = \frac{1}{a}(1 - e^{-ax}),$$

nous nous limiterons à ce cas particulier. De plus, comme il est évident que l'aversion au risque de la compagnie est infiniment plus faible que celle de l'assuré (J_2 raisonne en milliers de francs, J_1 en millions) nous pouvons, en première approximation, représenter le comportement de J_1 par une fonction d'utilité linéaire. Nous avons donc

J_1: assureur. Réserves initiales R_1
Portefeuille existant de fonction de répartition $F_1(x_1)$
Utilité $u_1(x) = x$.

J_2: assuré. Fortune initiale R_2
Risque à assurer de fonction de répartition $F_2(x_2)$
Utilité $u_2(x) = (1/a)(1 - e^{-ax})$.

En posant $k_1 = 1$ (les k_i ne sont définis qu'à une constante multiplicative près), l'application du théorème 1 donne

$$k_2 e^{-a[R_2 - y_2(x_1, x_2)]} = 1$$

ou

$$\boxed{y_2(x_1, x_2) = R_2 - \frac{1}{a}\log k_2 \quad \text{et} \quad y_1(x_1, x_2) = x_1 + x_2 - y_2(x_1, x_2).}$$

Par conséquent l'assureur prend à sa charge la totalité du risque moyennant le paiement d'une prime $P(x_2) = y_2(x_1, x_2)$, fixée dès que k_2 est déterminé. L'assuré cède à la compagnie tout son avoir R_2, moins une constante qui dépend de son aversion au risque a: plus celle-ci est grande, plus le montant que J_2 est disposé à céder est élevé. On calcule facilement

l'utilité initiale de la compagnie

$$U_1(x_1) = \int_0^\infty (R_1 - x_1)\, dF_1(x_1) = R_1 - E(x_1)$$

l'utilité initiale de l'assuré

$$U_2(x_2) = \int_0^\infty \frac{1}{a}(1 - e^{-a(R_2 - x_2)})\, dF_2(x_2) = \frac{1}{a}[1 - e^{-aR_2}M_2(a)]$$

où $M_2(a)$ est la fonction génératrice des moments du risque à assurer calculée au point a. En supposant l'indépendance entre les variables x_1 et x_2 on calcule

l'utilité finale de la compagnie

$$U_1(\bar{y}) = \int_0^\infty \int_0^\infty [R_1 - y_1(x_1, x_2)]\, dF_1(x)\, dF_2(x_2)$$

$$= R_1 + R_2 - E(x_1) - E(x_2) - \frac{1}{a}\log k_2$$

l'utilité finale de l'assuré

$$U_2(\bar{y}) = \int_0^\infty \int_0^\infty \frac{1}{a}[1 - e^{-a[R_2 - y_2(x_1, x_2)]}\, dF_1(x_1)\, dF_2(x_2) = \frac{1}{a}\left(1 - \frac{1}{k_2}\right)$$

le gain d'utilité pour la compagnie

$$U_1(\bar{y}) - U_1(x_1) = R_2 - E(x_2) - \frac{1}{a}\log k_2$$

le gain d'utilité pour l'assuré

$$U_2(\bar{y}) - U_2(x_2) = \frac{1}{a}\left[e^{-aR_2}M_2(a) - \frac{1}{k_2}\right].$$

Ceci démontre à nouveau l'opposition des intérêts des deux protagonistes: l'assureur va essayer d'obtenir le plus petit k_2 possible, alors que l'assuré poursuit le but inverse. Il va donc s'établir une négociation entre J_1 et J_2 dans le but de se mettre d'accord sur une valeur de k_2. En d'autres termes, les deux agents doivent négocier pour déterminer la prime.

3. MODÈLES DE MARCHANDAGE À DEUX JOUEURS

Un jeu de marchandage à deux joueurs est un couple $(S, \bar{d})$, où

S est un ensemble convexe compact représentant les paiements réalisables dans l'espace Euclidien à deux dimensions E^2 des utilités des joueurs; $\bar{d}$ est le point de désaccord formé par les utilités initiales des joueurs.

Notons B l'ensemble de tous les couples $(S, \bar{d})$.

Une solution est une règle qui associe à tout jeu de marchandage un paiement réalisable; il s'agit donc d'une fonction $f . B \to E^2$ telle que $f(S, \bar{d})$ est un élément $\bar{x} = (x_1, x_2)$ de S pour tout $(S, \bar{d})$ de B: $f_1(S, \bar{d}) = x_1$, $f_2(S, \bar{d}) = x_2$.

Il est évident qu'aucun joueur ne va accepter une solution qui n'est pas rationnelle individuellement. Nous pouvons donc limiter S à l'ensemble des points $\bar{x}$ tels que $x_i \geqslant d_i$, $i = 1, 2$.

Les concepts de solution qui semblent les plus satisfaisants au point de vue axiomatique sont dues, l'un à Nash (1950), l'autre à Kalai et Smorodinsky (1975).

3.1. *Le Modèle de Nash*

AXIOME 1. *Indépendance par rapport aux représentations équivalentes des utilités.*

La solution n'est pas affectée par une transformation linéaire positive effectuée sur les utilités des joueurs. Pour tout $(S, \bar{d})$ et tous nombres réels $a_i > 0$, b_i $(i = 1, 2)$, soit $(S', \bar{d}')$ le jeu défini par $S' = \{\bar{y} \in E^2 | \exists \bar{x} \in S \text{ tel que } y_i = a_i x_i + b_i\}$ et $d_i' = a_i d_i + b_i$, $i = 1, 2$. Alors $f_i(S', \bar{d}') = a_i f_i(S, \bar{d}) + b_i$, $i = 1, 2$. Cet axiome ne fait que refléter l'information contenue dans les fonctions d'utilité; puisque celles-ci ne sont définies qu'à une transformation linéaire près, il doit en être de même de la solution.

AXIOME 2. *Symétrie*

Tout jeu symétrique a une solution symétrique. Un jeu est symétrique si

(i) $d_1 = d_2$,

(ii) $(x_1, x_2) \in S \Rightarrow (x_2, x_1) \in S$

L'axiome exige que, dans ce cas, $f_1(S, \bar{d}) = f_2(S, \bar{d})$.

Comme la propriété précédente, cet axiome demande que la solution ne dépende que de l'information contenue dans le modèle; si les joueurs ne peuvent être différenciés par les règles du jeu, une permutation ne peut modifier la solution. Si les participants ont même fonction d'utilité, même fortune initiale et si l'espace du jeu est symétrique, la solution doit accorder le même gain en utilité à chacun.

AXIOME 3. *Pareto-optimalité*

$$\forall (S, \bar{d}) \in B, \text{ si } \bar{x} \text{ et } \bar{y} \in S \text{ sont tels que } y_i > x_i \quad i = 1, 2, \text{ alors } f(S, \bar{d}) \neq \bar{x}.$$

AXIOME 4. *Indépendance par rapport aux alternatives non pertinentes*

La solution ne change pas si l'on retire de l'espace du jeu tout point autre que le point de désaccord ou la solution elle-même.

Un énoncé équivalent de l'axiome est le suivant: soient $(S, \bar{d})$ et $(T, \bar{d})$ deux jeux tels que T contient S et que $f(T, \bar{d})$ est un élément de S. Alors $f_i(S, \bar{d}) = f_i(T, \bar{d})$, $i = 1, 2$.

Cet axiome exprime le genre de négociation que le modèle de Nash est censé représenter; il exprime une propriété de structure du processus de marchandage: pendant celui-ci, l'ensemble des points susceptibles d'être choisis se rétrécit progressivement, de telle sorte qu'à la fin du marchandage la solution n'est en compétition qu'avec des points extrêmement voisins, et non avec des alternatives plus éloignées éliminées pendant les premières phases de la négociation.

THEOREME 2. *Il existe une et une seule solution satisfaisant aux* 4 *axiomes; c'est le point maximisant le produit des gains d'utilité des joueurs. C'est donc la fonction* $f = F$ *définie par* $F(S, \bar{d}) = \bar{x}$, *telle que* $\bar{x} \geqslant \bar{d}$ *et que* $(x_1 - d_1)(x_2 - d_2) > (y_1 - d_1)(y_2 - d_2)$ $\forall y \neq x \in S$.

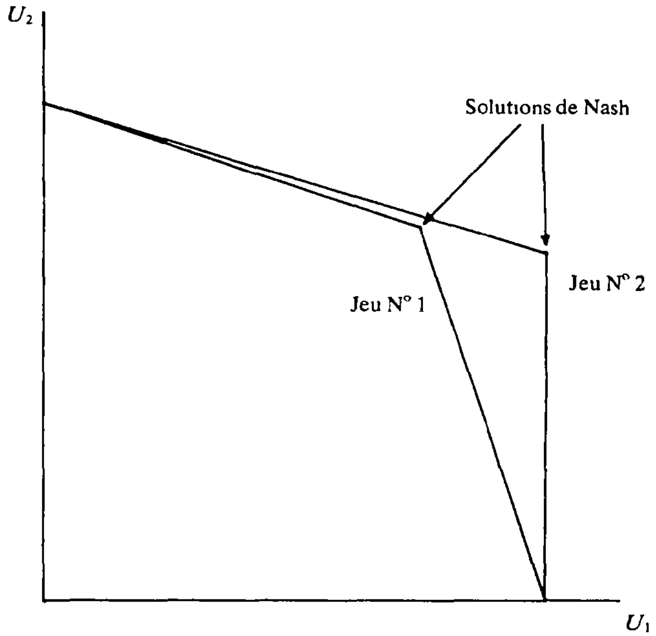
THEOREME 3 (Roth, 1980). *Le gain d'utilité que la solution de Nash accorde à un joueur croît lorsque l'aversion au risque de son adversaire augmente. En d'autres termes* $F_i(S', \bar{d}') > F_i(S, \bar{d})$, *où* $(S', \bar{d}')$ *est obtenu à partir de* $(S, \bar{d})$ *en remplaçant le joueur* $J_j \neq J_i$ *par quelqu'un qui a plus peur du risque.*

CRITIQUE. Si les trois premiers axiomes ont été épargnés par les critiques, il n'en va pas de même de l'axiome 4. Kalai et Smorodinsky ont résumé ces attaques par l'exemple suivant

Jeu $n°1$: l'espace de marchandage est limité par le polygone reliant les points (0, 0), (0, 1), (0,75, 0,75) et (1, 0).

Jeu $n°2$: l'espace de marchandage est limité par le polygone reliant les points (0, 0), (0, 1), (1, 0,7) et (1, 0).

On constate que, quel que soit le gain de J_1, J_2 peut obtenir plus dans le jeu $n°2$ que dans le jeu $n°1$. J_2 a donc de bonnes raisons de réclamer un montant plus élevé dans le jeu $n°2$. Or, la solution de Nash est (0,75, 0,75) dans le jeu $n°1$, et (1, 0,7) dans le jeu $n°2$, ce qui ne satisfait pas à la demande de J_2.



3.2. *Le modèle de Kalai–Smorodinsky*

Ces deux auteurs ont présenté un autre modèle, qui reprend les axiomes 1 à 3, mais remplace le 4ème axiome par le suivant.

AXIOME 5. *Monotonie*

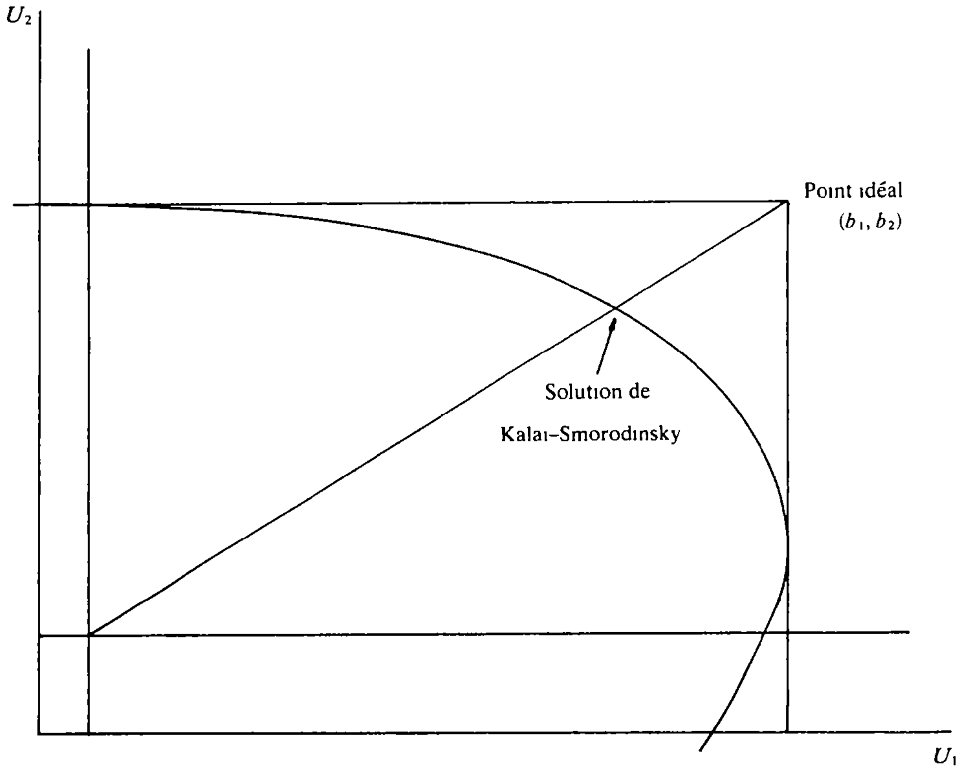
Si, quelle que soit la demande de son adversaire, les règles du jeu sont telles qu'un joueur reçoit plus dans un jeu que dans un autre, la solution lui accorde un gain supérieur dans le premier jeu.

Désignons par $\bar{b}(S)=(b_1, b_2)$ le point "idéal" formé par les demandes maximales des joueurs: $b_i = \max\{x_i|(x_1, x_2) \in S\}$. L'axiome peut alors s'énoncer sous la forme suivante: si $(T, \bar{d})$ et $(S, \bar{d})$ sont deux jeux tels que T contient S et $\bar{b}(S)=\bar{b}(T)$, alors $f(T, \bar{d}) \geqslant f(S, \bar{d})$.

Soit $G(S, \bar{d})$ la solution qui consiste à chercher le point d'intersection entre la courbe Pareto-optimale et la droite joignant $\bar{d}$ à $\bar{b}$. Donc $G(S, \bar{d}) = \bar{x}$ tel que $\bar{x} \in S$, $(x_1-d_1)/(x_2-d_2)=(b_1-d_1)/(b_2-d_2)$ et $\bar{x} \geqslant \bar{y}$ pour tout $\bar{y} \in S$ tel que $(y_1-d_1)/(y_2-d_2)=(b_1-d_1)/(b_2-d_2)$.

THEOREME 4. *Il existe une et une seule solution satisfaisant aux axiomes* 1, 2, 3 *et* 5: *c'est* $G(S, \bar{d})$.

THEOREME 5 (Roth, 1980). *Le gain d'utilité que la solution de Kalai–Smorodinsky accorde à un joueur croît lorsque l'aversion au risque de son adversaire augmente.*



La solution de Kalai–Smorodinsky ne satisfait pas bien sûr à l'axiome 4, puisque $G(S, \bar{d})$ dépend non seulement du point de désaccord mais aussi du point idéal. On peut cependant montrer que $G(S, \bar{d})$ est indépendant des alternatives qui ne déterminent pas le point idéal. Donc $G(S, \bar{d})$ (tout comme d'ailleurs $F(S, \bar{d})$) satisfait à l'axiome 6.

AXIOME 6. *Indépendance par rapport aux alternatives autres que le point de désaccord et le point idéal.*

Si $(S, \bar{d})$ et $(T, \bar{d})$ sont deux jeux tels que T contient S, $\bar{b}(S) = \bar{b}(T)$ et $f(T, \bar{d})$ est un élément de S, alors $f(S, \bar{d}) = f(T, \bar{d})$.

Si la solution de Kalai–Smorodinsky a l'avantage de satisfaire à l'axiome de monotonie, elle n'est cependant pas exempte de critiques. On peut par exemple montrer (Roth, 1980) qu'il est impossible de généraliser ce concept à un jeu à plus de deux joueurs, alors que le modèle de Nash a déjà fait l'objet de telles extensions (voir par exemple Lemaire (1973)).

THEOREME 6. *Pour un jeu de marchandage à 3 joueurs ou plus, il n'existe aucune solution Pareto-optimale, symétrique et monotone.*

On constate une propriété de dualité entre les deux solutions proposées: le point sélectionné par le modèle de Nash correspond au rectangle de plus grande surface à l'intérieur de S, tandis que le point isolé par Kalai–Smorodinsky correspond au rectangle de plus petite surface à l'extérieur de S.

On trouve d'autres concepts de solution en théorie des jeux. Ils ne nous paraissent cependant pas adaptés à notre problème. Par exemple les solutions qui consistent à sélectionner le traité le plus "proche" du point idéal, au sens d'une certaine norme, ne peuvent nous convenir car elles ne sont pas indépendantes d'une transformation linéaire effectuée sur les utilités des joueurs.

4. CALCUL DES PRIMES

Les deux modèles de marchandage décrits en Section 3 nous permettent d'isoler un point sur la courbe Pareto-optimale, c'est-à-dire de déterminer k_2 et la prime.

4.1. *Solution de Nash*

Nous devons maximiser le produit

$$\frac{1}{a}\left[R_2-E_2(x_2)-\frac{1}{a}\log k_2\right]\left[e^{-aR_2}M_2(a)-\frac{1}{k_2}\right].$$

Posons

$$A=R_2-E_2(x_2)$$

$$B=e^{-aR_2}M_2(a)$$

$$\psi=\left(A-\frac{1}{a}\log k_2\right)\left(B-\frac{1}{k_2}\right)$$

$$\frac{d\psi}{dk_2}=0\Leftrightarrow A-\frac{1}{a}\log k_2-\frac{B}{a}k_2+\frac{1}{a}=0.$$

Cette équation détermine k_2, et donc la prime $P(x_2)$, en fonction des constantes A, B et de l'aversion au risque a.

4.2. *Solution de Kalai–Smorodinsky*

On vérifie que

$$b_1=R_1-E_1(x_1)-E_2(x_2)+\frac{1}{a}\log M_2(a)$$

$$b_2=\frac{1}{a}[1-e^{-a[R_2-E_2(x_2)]}].$$

L'équation de la droite reliant le point de désaccord et le point (b_1, b_2) est

$$\frac{u_1-(R_1-E_1(x_1))}{\frac{1}{a}\log M_2(a)-E_2(x_2)}=\frac{u_2-\frac{1}{a}(1-e^{-aR_2}M_2(a))}{\frac{1}{a}e^{-aR_2}[M_2(a)-e^{aE_2(x_2)}]}.$$

Les équations paramétriques de la courbe Pareto-optimale sont

$$u_1=R_1-E_1(x_1)-E_2(x_2)-\frac{1}{a}\log k_2+R_2$$

$$u_2=\frac{1}{a}\left(1-\frac{1}{k_2}\right).$$

Après remplacement et calculs, on obtient

$$\left(A-\frac{1}{a}\log k_2\right)(B-e^{-aA})-\left(A+\frac{1}{a}\log B\right)\left(B-\frac{1}{k_2}\right)=0,$$

équation qui détermine implicitement k_2 et la prime $P(x_2)$.

5. PROPRIÉTÉS

En déterminant un traité d'assurance unique, nous avons en fait défini deux nouveaux principes de calcul des primes: le principe de Nash et le principe de Kalai–Smorodinsky. Quelles sont les propriétés de ces principes?

1. *La prime contient un chargement de sécurité.*
La prime pure vaut $E_2(x_2)$, la prime effective $P(x_2)=y_2(x_1, x_2)$.

$$P(x_2)-E_2(x_2)=R_2-\frac{1}{a}\log k_2-E_2(x_2).$$

Le second membre est le gain d'utilité que la compagnie obtient par le traité. Comme les deux modèles considérés conduisent à une solution individuellement rationnelle, $P(x_2)$ est bien une prime chargée.

2. *La prime ne peut dépasser le montant maximum des sinistres* (*no ripoff condition*)
Cette propriété résulte du fait que le contrat conduit nécessairement à un gain d'utilité pour l'assuré.

3. *La prime fait intervenir tous les moments de la distribution des sinistres* (de par le présence de la fonction génératrice des moments).

4. *La prime est indépendante des réserves R_1 et du portefeuille de la compagnie d'assurance.*

5. *La prime est indépendante de la fortune R_2 de l'assuré.*

Démontrons cette propriété pour le principe de Nash.

THEOREME 7. *La prime, calculée selon le principe de Nash, ne dépend pas de R_2.*

DÉMONSTRATION. La prime vaut $P(x_2) = R_2 - (1/a)\log k_2$, avec

$$R_2 - E_2(x_2) + \frac{1}{a} - \frac{k_2 e^{-aR_2} M_2(a)}{a} - \frac{1}{a}\log k_2 = 0. \tag{1}$$

Ajoutons à R_2 une quantité arbitraire α et vérifions que $P(x_2)$ ne change pas.

$$P'(x_2) = R_2 + \alpha - \frac{1}{a}\log k_2',$$

où

$$R_2 + \alpha - E_2(x_2) + \frac{1}{a} - \frac{k_2' e^{-a(R_2+\alpha)} M_2(a)}{a} - \frac{1}{a}\log k_2' = 0. \tag{2}$$

k_2 et k_2' sont liés par la relation $k_2' = k_2 e^{a\alpha}$ (il suffit pour s'en convaincre d'effectuer le remplacement dans (2), qui se réduit alors à (1). Par conséquent

$$\begin{aligned} P'(x_2) &= R_2 + \alpha - \frac{1}{a}\log(k_2 e^{a\alpha}) \\ &= R_2 + \alpha - \frac{1}{a}\log k_2 - \alpha \\ &= P(x_2). \end{aligned}$$

En fait les propriétés 3 et 4 résultent du fait qu'une modification de R_1 ou R_2 entraîne une transformation linéaire d'une fonction d'utilité exponentielle.

6. *La prime croît avec l'aversion au risque de l'assuré*

Cette propriété est une conséquence immédiate des théorèmes 3 et 5 et du fait que le chargement de sécurité de l'assureur n'est rien d'autre que son gain d'utilité.

7. *La prime est invariante par translation $P(x_2 + c) = P(x_2) + c$*

La démonstration de cette propriété est fort semblable à celle de la propriété 4, en posant cette fois $k_2' = k_2 e^{-ac}$.

Par contre de nombreux contre-exemples permettent de vérifier que les deux principes de calcul des primes ici définis ne satisfont ni à la propriété d'additivité ni à celle d'itérativité.

8. *Exemple*

Nous reprenons ici les données utilisées par Moffet (1979).

Distribution du coût des sinistres pour l'assuré

x_2	0	1	2	3	4	5	7	10	15	20
Prob.	0,3	0,05	0,06	0,08	0,1	0,13	0,15	0,07	0,04	0,02

La prime pure vaut 4,21.

Nous devons, pour obtenir la prime, résoudre les équations suivantes:

1. Principe de Nash: $35{,}79-\frac{1}{a}\log k_2-e^{-40a}M_2(a)+\frac{1}{a}=0.$

2. Principe de Kalai–Smorodinsky:

$$\left(35{,}79-\frac{1}{a}\log k_2\right)(e^{-40a}M_2(a)-e^{-35{,}79a})-(35{,}79-40\,M_2(a))$$
$$\times\left(e^{-40a}M_2(a)-\frac{1}{k_2}\right)=0.$$

La résolution de ces deux équations pour différentes valeurs de a conduit aux résultats repris ci-dessous. Le tableau indique la prime ainsi que les gains en utilité des deux joueurs.

	NASH			KALAI-SMORODINSKY		
a	P	u_1	u_2	P	u_1	u_2
0,01	4,259	0,05	$3{,}42\times10^{-2}$	4,259	0,05	$3{,}42\times10^{-2}$
0,05	4,482	0,27	$4{,}60\times10^{-2}$	4,482	0,27	$4{,}60\times10^{-2}$
0,1	4,824	0,61	$1{,}82\times10^{-2}$	4,824	0,61	$1{,}82\times10^{-2}$
0,25	6,425	2,21	$5{,}01\times10^{-4}$	6,442	2,23	$4{,}97\times10^{-4}$
0,5	9,866	5,66	$1{,}62\times10^{-6}$	10,142	5,93	$1{,}53\times10^{-6}$
1	13,747	9,54	$3{,}73\times10^{-11}$	14,230	10,02	$3{,}48\times10^{-11}$

Lorsque l'aversion au risque de l'assuré est faible, le jeu est presque symétrique; la prime ne dépasse que de peu la prime pure. Le principe de Kalai–Smorodinsky est plus avantageux pour l'assureur que le principe de Nash: la prime est légèrement plus élevée.

6. CONCLUSIONS

La science actuarielle a connu ces dernières années, avec l'étude de nombreux principes de calcul des primes, une évolution fort intéressante. Le pilier principal de l'actuariat traditionnel, le principe d'équivalence—que l'on appelle aujourd'hui le principe de l'espérance mathématique—a perdu un peu de son rôle central suite à l'introduction de principes de plus en plus sophistiqués. L'on a commencé par introduire des paramètres de la distribution des sinistres autres que la moyenne (variance, écart-type, coefficient d'asymétrie, semi-variance, . . .). Puis des concepts en provenance de l'économie mathématique ont fait irruption en actuariat et les principes les plus récents introduisent la situation du marché et les préférences de comportement des protagonistes. Les deux principes de calcul présentés ici vont encore plus loin: nous avons déterminé des primes par marchandage, en nous basant uniquement sur l'attitude des agents économiques en présence de risques, en faisant totalement abstraction du principe d'équivalence et de la prime pure.

Si l'on pensera sans doute à juste titre que nous sommes allés trop loin, il n'empêche que nos deux principes vérifient un ensemble de propriétés satisfaisant: la prime contient un chargement de sécurité, elle ne peut dépasser le montant maximum des sinistres, elle ne dépend ni des réserves de l'assureur, ni de la fortune de l'assuré, ni du portefeuille de la compagnie. Elle fait intervenir tous les moments de la distribution des sinistres et est invariante par translation. Bien sûr cet ensemble de propriétés ne peut soutenir la comparaison avec celles vérifiées par le principe de l'utilité nulle, par exemple. Néanmoins cela ne doit pas nous faire perdre de vue la qualité à notre sens fondamentale des deux principes ici présentés, à savoir le fait d'introduire explicitement dans le calcul des primes le comportement de l'assuré. Les principes de calcul des primes introduits précédemment semblent avoir fait la part un peu trop belle aux compagnies en oubliant que dans tout contrat d'assurance il y a une autre partie tout aussi importante: l'assuré. Nous avons tenté ici de rétablir l'équilibre en introduisant dans le raisonnement l'attitude vis-à-vis du risque de ce dernier.

DEUXIÈME PARTIE COALITION CONTRE UNE COMPAGNIE D'ASSURANCES

Dans ce qui précède, nous avons appliqué le théorème de Borch à un micro-marché formé par un assuré et un assureur. Rien ne nous empêche bien sûr d'ajouter des joueurs au raisonnement. Dans ce qui suit, nous allons introduire un troisième joueur, J_3, un assuré caractérisé par sa situation $[R_3, F_3(x_3)]$ et sa fonction d'utilité $u_3(x)$, que nous supposons également exponentielle, de paramètre b. Nous supposons l'indépendance entre les risques des 3 joueurs.

Par application du théorème de Borch, nous obtenons que l'assureur J_1 va reprendre la totalité des risques de ses partenaires, moyennant le paiement de primes $P(x_2)$ et $P(x_3)$.

$$y_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - P(x_2) - P(x_3)$$

$$y_2(x_1, x_2, x_3) = P(x_2) = R_2 - \frac{1}{a}\log k_2$$

$$y_3(x_1, x_2, x_3) = P(x_3) = R_3 - \frac{1}{b}\log k_3.$$

L'introduction d'un deuxième assuré permet d'ajouter une dimension au problème. En effet les assurés disposent maintenant d'une alternative à l'assurance pure et simple: ils peuvent s'échanger leurs risques sans nécessairement passer par l'assureur. Ils peuvent également utiliser cette possibilité supplémentaire comme menace dans le but d'obtenir des primes moins élevées. C'est ce que nous allons démontrer en calculant le coeur du marché.

DÉFINITIONS

1. Soit $N = \{J_1, \ldots, J_n\}$ les agents du marché d'échange de risques, et $S \subset N$ toute coalition de joueurs. Un traité $\bar{y}'$ domine $\bar{y}$ par raport à S si $\bar{y}'$ est réalisable pour S et si $u_j(\bar{y}') \geq u_j(\bar{y})$ $\forall j \in S$ (avec au moins une inégalité stricte).

2. $\bar{y}'$ domine $\bar{y}$ s'il existe une coalition S telle que $\bar{y}'$ domine $\bar{y}$ par rapport à cette coalition.

3. Le coeur du marché est l'ensemble des traités non dominés.

Le coeur est donc un ensemble stable, puisqu'aucune coalition n'a intérêt à quitter le marché.

Le coeur d'un marché d'échange de risques a été caractérisé récemment par Baton–Lemaire (1981) dans le cas particulier où tous les agents utilisent une fonction d'utilité exponentielle (de paramètre c_j pour J_j).

THEOREME 9. *Un traité appartient au coeur d'un marché d'échange de risques à utilités exponentielles si et seulement si*

$$y_j(x_1, \ldots, x_n) = q_j(x_1 + \cdots + x_n) + y_j(0),$$

avec

$$q_j = \frac{1/c_j}{\sum_{i=1}^{n} 1/c_i}$$

$$\sum_{j=1}^{n} y_j(0) = 0$$

$$\sum_{j \in S} y_j(0) \leq \sum_{j \in S} (P_j^S - P_j^N) \quad \forall S \subset N, \quad S \neq \varnothing,$$

où

$$P_j^S = \frac{1}{c_j} \sum_{k \in S} \log M_k \left(\frac{1}{\sum_{j \in S} 1/c_j} \right)$$

est la prime qu'appliquerait J_j dans la coalition S s'il utilisait le principe de l'utilité exponentielle.

Dans le cas de notre marché à 3 joueurs, le théorème est d'application car les fonctions d'utilité linéaires peuvent être considérées comme un cas particulier des fonctions exponentielles.

Nous avons

$$q_1 = 1, \qquad q_2 = q_3 = 0$$

$$y_1(0) = -P(x_2) - P(x_3), \qquad y_2(0) = P(x_2), \qquad y_3(0) = P(x_3).$$

Nous pouvons par conséquent calculer

Coalition	Primes
$\{1\}$	$P_1^{\{1\}} = E(x_1)$
$\{2\}$	$P_2^{\{2\}} = \frac{1}{a} \log M_2(a)$
$\{3\}$	$P_3^{\{3\}} = \frac{1}{b} \log M_3(b)$

Coalition	Primes	
$\{1\ 2\ 3\}$	$P_1^{\{123\}} = E(x_1) + E(x_2) + E(x_3)$	$P_2^{\{123\}} = P_3^{\{123\}} = 0$
$\{1\ 2\}$	$P_1^{\{12\}} = E_1(x_1) + E_2(x_2)$	$P_2^{\{12\}} = 0$
$\{1\ 3\}$	$P_1^{\{13\}} = E_1(x_1) + E_3(x_3)$	$P_3^{\{13\}} = 0$
$\{2\ 3\}$	Dans cette coalition, J_2 et J_3 se passent des services de l'assureur. J_2 prend à sa charge une fraction $(1/a)/(1/a+1/b)$ des deux risques, et J_3 la fraction complémentaire $(1/b)/(1/a+1/b)$. Donc $P_2^{\{23\}} = \frac{1}{a}\left[\log M_2\left(\frac{1}{1/a+1/b}\right) + \log M_3\left(\frac{1}{1/a+1/b}\right)\right]$ $P_3^{\{23\}} = \frac{1}{b}\left[\log M_3\left(\frac{1}{1/a+1/b}\right) + \log M_3\left(\frac{1}{1/a+1/b}\right)\right].$	

Nous sommes maintenant en mesure d'écrire les 6 conditions d'existence du coeur.

1. $S = \{1\}$. La condition $y_1(0) \leq P_1^{\{1\}} - P_1^N$ s'écrit

$$P(x_2) + P(x_3) \geq E_2(x_2) + E_3(x_3)$$

ou

$$R_2 + R_3 - \frac{1}{a}\log k_2 - \frac{1}{b}\log k_3 \geq E_2(x_2) + E_3(x_3).$$

Il s'agit de la condition de rationalité individuelle pour l'assureur, qui n'acceptera le marché que si la prime totale perçue est au moins égale à la prime pure.

2. $S = \{2\}$.

$$y_2(0) \leq P_2^{\{2\}} - P_2^N$$

$$\Leftrightarrow P(x_2) \leq \frac{1}{a}\log M_2(a) - 0$$

$$\Leftrightarrow R_2 - \frac{1}{a}\log k_2 \leq \frac{1}{a}\log M_2(a).$$

C'est la condition de rationalité individuelle pour J_2, qui n'acceptera pas une prime trop élevée.

3. $S = \{3\}$.

$$R_3 - \frac{1}{b}\log k_3 \leq \frac{1}{b}\log M_3(b)$$

(condition de rationalité individuelle pour J_3).

4. $S = \{1,2\}$.

$$y_1(0) + y_2(0) \leq [P_1^{\{12\}} - P_1^N] + [P_2^{\{12\}} - P_2^N]$$

$$\Leftrightarrow R_3 - \frac{1}{b}\log k_3 \geq E(x_3).$$

EXEMPLE. Supposons que les deux assurés soient soumis au même risque, distribué selon un loi Γ, de moyenne 1,2 et de variance 1,25 (Baton–Lemaire (1981)). Supposons que $R_2 = 10$, $R_3 = 5$, $a = 0{,}4$, $b = 0{,}8$. On vérifie que le coeur du marché est détermıné par les conditions

$$1{,}2 \leqslant P(x_2) \leqslant 1{,}6$$

$$1{,}2 \leqslant P(x_3) \leqslant 2{,}6$$

$$2{,}4 \leqslant P(x_2) + P(x_3) \leqslant 2{,}8.$$

J_2 est prêt à payer une prime allant jusque 1,6. J_3, ayant plus peur du risque, est même disposé à aller jusque 2,6. En se coalisant, cependant, J_2 et J_3 peuvent limiter la somme de leurs primes à 2,8.

BIBLIOGRAPHIE

BARDOLA, J (1981) Optımaler Rısıkoaustausch als Anwendung fur den Versıcherungsvertrag *MVSV*, 41-66

BATON, B. et LEMAIRE, J (1981) The Core of a Reinsurance Market *ASTIN Bulletin* **12**, 57-71

BORCH, K. (1962) Equilibrium in a Reinsurance Market *Econometrica* **30**, 424–444

BUHLMANN, H et JEWELL, W. (1979) Optimal Risk Exchanges *ASTIN Bulletin* **10**, 243-262

GERBER, H. (1974*a*) On Additive Premium Calculation Principles. *ASTIN Bulletin* **7**, 215-222.

GERBER, H (1974*b*) On Iterative Premium Calculation Principles *MVSV*, 163-172

GERBER, H (1978) Pareto-Optimal Risk Exchange and Related Decision Problems *ASTIN Bulletin* **10**, 25-33

KALAI, E et SMORODINSKY, M (1975) Other Solutions to the Nash Bargaining Problem *Econometrica* **43**, 513-518.

LEEPIN, P. (1975) Uber die Wahl von Nutzenfunktionen fur die Bestimmung von Versicherungs-pramien *MVSV*, 27–45

LEMAIRE, J (1973) A New Concept of Value for Games Without Transferable Utilities. *Int. J of Game Theory* **2**, 205-214

MOFFET, D (1979) The Risk Sharing Problem *Geneva Papers* **11**, 5-13

NASH, J. (1950) The Bargaining Problem *Econometrica* **18**, 155-162

ROTH, A. (1980) *Axiomatic Models of Bargaining*. Springer

A REMARK ON THE PRINCIPLE OF ZERO UTILITY

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Let $u(x)$ be a utility function, i.e., a function with $u'(x)>0$, $u''(x)<0$ for all x. If S is a risk to be insured (a random variable), the premium $P=P(x)$ is obtained as the solution of the equation

$$u(x)=E[u(x+P-S)] \tag{1}$$

which is the condition that the premium is fair in terms of utility. It is clear that an affine transformation of u generates the same principle of premium calculation. To avoid this ambiguity, one can standardize the utility function in the sense that

$$u(y)=0, \quad u'(y)=1 \tag{2}$$

for an arbitrarily chosen point y. Alternatively, one can consider the risk aversion

$$r(x)=-u''(x)/u'(x) \tag{3}$$

which is the same for all affine transformations of a utility function.

Given the risk aversion $r(x)$, the standardized utility function can be retrieved from the formula

$$u(x)=\int_y^x \exp\left(-\int_y^z r(u)\,du\right)dz. \tag{4}$$

It is easily verified that this expression satisfies (2) and (3).

The following lemma states that the greater the risk aversion the greater the premium, a result that does not surprise.

LEMMA. *Let $u_1(x)$ and $u_2(x)$ be two utility functions with corresponding risk aversions $r_1(x)$, $r_2(x)$. Let P_i denote the premium that is generated by $u_i (i=1,2)$. If $r_1(x)\geq r_2(x)$ for all x, it follows that $P_1(w)\geq P_2(w)$ for any risk S and all w.*

PROOF. $P_i=P_i(w)$ is obtained as the solution of the equation

$$u_i(w)=E[u_i(w+P_i-S)], \qquad i=1,2. \tag{5}$$

We standardize u_1 and u_2 such that

$$u_i(w)=0,\ u_i'(w)=1. \tag{6}$$

Using (4), with $y=w$, we can express u_i in terms of r_i. Since $r_1(x)\geq r_2(x)$ for all x, it follows that

$$u_1(x)\leq u_2(x) \quad \text{for all } x. \tag{7}$$

Using (5), (6), (7) we see that

$$E[u_2(w+P_2-S)]=E[u_1(w+P_1-S)]\leqslant E[u_2(w+P_1-S)] \tag{8}$$

Since u_2 is an increasing function, the inequality between the first term and the last term means that $P_1 \geqslant P_2$. Q.E.D.

The lemma has some immediate consequences:

APPLICATION 1. *The exponential premium, $P=(1/a)\log E[e^{aS}]$, is an increasing function of the parameter a.*

PROOF. Let $a_1>a_2$. Use the lemma in the special case $r_i(x)=a_i$ (constant) to see that the exponential premium (parameter a_1) exceeds the exponential premium (parameter a_2). Q.E.D.

APPLICATION 2. *Suppose that $r(x)$ is a nonincreasing function. Then $P=P(x)$ as determined from* (1) *is a nonincreasing function of x for any risk S.*

PROOF. Let $h>0$. Use the lemma with $r_1(x)=r(x)$, $r_2(x)=r(x+h)$ to see that $P(x)\geqslant P(x+h)$. Q.E.D.

REMARKS. (1) The last two proofs are simpler than the original proofs given by Gerber (1974, p. 216) for the first application and by Leepin (1975, pp. 31–35) for the second application.

(2) For a small risk S (i.e., a random variable S with a narrow range) $P(x)$ is approximately $E[S]+r(x)\operatorname{var}[S]/2$. Thus the converse of the Lemma ($P_1(w)\geqslant P_2(w)$ for all S implies that $r_1(x)\geqslant r_2(x)$) is trivial.

(3) In Pratt's terminology (1964) the premium P is a (negative) *bid price.* However, Pratt's discussion focusses essentially on (what he calls) the *insurance premium Q*, which is defined as the solution of the equation

$$u(x-Q)=E[u(x-S)] \tag{9}$$

and which should be interpreted as the largest premium someone with fortune x and liability S is willing to pay for full coverage. The counterpart of the Lemma (with $P_i(w)$ replaced by $Q_i(w)$) has been discussed by Pratt (1964, p. 128). A short proof of this counterpart is obtained if one standardizes u_1 and u_2 such that

$$u_i(w-Q_1)=0, \qquad u_i'(w-Q_1)=1. \tag{10}$$

Details are left to the reader.

REFERENCES

GERBER, H U (1974) On Additive Premium Calculation Principles *Astin Bulletin* **7**, 215–222

LEEPIN, P. (1975) Ueber die Wahl von Nutzenfunktionen fur die Bestimmung von Versicherungspramien *Mitteilungen der Vereinigung schweizerischer Versicherunsmathematiker* **75**, 27–45

PRATT, J W. (1964). Risk Aversion in the Small and in the Large *Econometrica* **32**, 122–136

A PRICING MODEL IN A SENSITIVE INSURANCE MARKET

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1. INTRODUCTION

A great attention has been devoted, in the actuarial literature, to premium calculation principles and it has been often emphasized that these principles should not only be defined in strictly actuarial terms, but should also take into account the market conditions (Bühlmann (1980), de Jong (1981)).

In this paper we propose a decision model to define the pricing policy of an insurance company that operates in a market which is stratified in k risk classes $\mathscr{C}_j$.

It is assumed that any class constitutes a homogeneous collective containing $\mathcal{N}_j$ independent risks $S_j(t)$ of compound Poisson type, with the same intensity λ_j. The number n_j of risks of $\mathscr{C}_j$ that are held in the insurance portfolio depends on the premium charged to the class by means of a *demand function* which captures the concept of risk aversion and represents the fraction of individuals of $\mathscr{C}_j$ that insure themselves at the annual premium x_j.

With these assumptions, the return Y on the portfolio is a function of the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$ of the prices charged to the single classes (and of the time) and x is therefore the decision policy instrument adopted by the company for the selection of the portfolio, whose optimal composition is evaluated according to a risk-return type performance criterion.

As a measure of risk we adopt the ultimate ruin probability $q(w)$ that, in the assumptions of our model, can be related to a *safety index* τ, by means of Lundberg-de Finetti inequality. Even though it has been widely debated in the actuarial field, the use of $q(w)$ offers undeniable operational advantages. In our case the safety index τ can be expressed as a function of $\boldsymbol{x}$ and therefore, in the phase of selecting an efficient portfolio, it becomes the function to be maximized, for a given level M of the expected return.

For τ, a quadratic approximation can be given that seems to be acceptable as long as the aggregate loading is not "too high". An assumption that does not exclude, among other things, the possibility of heavy loadings in a number not too large of individual cases.

Once the form of the efficient frontier has been determined, the final step of the decision problem of the company is to select the portfolio that maximizes a utility function of the form $u(M, V)$, that is the portfolio represented as the tangency point between the efficient frontier and the "highest possible" indifference curve. It could be pointed out that, in the model, the validity of variance as a risk measure of the portfolio does not depend on the possibility of achieving an acceptable quadratic approximation of the utility function, but on the goodness of the approximation obtained for the ruin probability, that we have chosen as a stability criterion for the company.

It is to be noted that, in our assumptions, we can achieve a stratification more refined than the one obtained solely based on the characteristics of the risk process $S(t)$, since we can consider classes that differ only for the risk aversion. This can be related to the introduction of a multivariate measure of risk aversion inside the collective, as suggested, e.g., by Sengupta (1981).

On the other hand the form and the rationale of the results continue to be valid in all the cases in which we can obtain an expression that relates $q(w)$ to the decision variables (i.e., the prices) and the endogenous quantities of the risk process. This is the case, for example, in the martingale assumptions on $Y(t)$, as discussed by de Finetti (1939) and developed by Gerber (1981) in the study of an autoregressive model.

2. THE MODEL

2.1. *Preliminaries*

Let us consider the risk process $\{S(t); t \geq 0\}$, that represents the sum of claim amounts incurred in $[0, t)$ in a given insurance portfolio. The accumulated claims up to time t can be represented as a random sum

$$S(t) = \sum_{r=1}^{N(t)} X_r,$$

with d.f. $F_S(x, t) = P\{S(t) \leq x\}$. The process $\{N(t); t \geq 0\}$, with distribution $p_n(t)$, $(n = 0, 1, \ldots)$, counts the number of claims in $[0, t)$ and the set of r.v. $\{X_r; r = 1, 2, \ldots\}$ represents the amount of the rth claim incurred in $[0, t)$. We can suppose that the m.g.f. $\chi_r(u) = E\{e^{uX_r}\}$ is finite for some $u \neq 0$.

We shall assume that the collective premium function of the risk (sum of premiums earned in the time interval $[0, t)$) is non-random and we shall denote it by $\pi(t) = E\{S(t)\} + l(t)$, that is as a sum of the (aggregate) net premium $E\{S(t)\}$ and the (aggregate) risk loading $l(t)$. As generally accepted in the actuarial literature, we shall assume $l(t) \geq 0$, since we shall disregard investment income in premium calculation. In fact, as shown by Kahane (1979), negative loadings could be justified by considerations on the cost of the capital and on the rates of investment. Meaningful loading formulas are obtained for instance by choosing $l(t)$ to be proportional to the expected value (supposed as positive) or to the variance of $S(t)$, that is

$$l(t) = \eta E\{S(t)\}, \qquad \eta \geq 0,$$

or

$$l(t) = \beta \operatorname{Var}\{S(t)\}, \qquad \beta \geq 0.$$

Besides the investment income, we shall neglect also the administrative costs and we shall indicate by $Y(t) = \pi(t) - S(t)$ the return on the insurance portfolio up to time t. Then the liquidity of the company can be represented by the risk reserve $R(t)=w+Y(t)$, being $w=R(0)$ the initial free capital; namely we have

$$R(t)=w+E\{S(t)\}+l(t)-\sum_{r=1}^{N(t)} X_r.$$

One of the most natural assumptions on the process $S(t)$ is that the r.v. X_r are independent with common d.f. $F_X(x)$ independent on time and such that $F_X(0)=0$ (positive risk sums). If we suppose, following F. Lundberg, that $N(t)$ is a Poisson process with intensity λ, the risk process $S(t)$ becomes a compound Poisson process, with m.g.f.

$$\varphi_S(u,t)=\exp\{\lambda t[\chi(u)-1]\},$$

where $\chi(u)$ is the common m.g.f. of the X_r, and with expected value $E\{S(t)\}=\lambda E\{X\}t$. Furthermore the risk loading becomes a linear function of time, i.e., $l(t)=l\cdot t$, whether one uses the expected value principle or the variance principle.

2.2. *The Risk Classes*

The foregoing classical model can be used to describe the riskiness of the portfolio of a given insurance line. Let us now suppose that the insurance market relative to this line is stratified in k risk classes $\mathscr{C}_j$, $(j=1,2,\ldots,k)$ according to the following hypotheses

(a) $\mathscr{N}$ stochastically independent individual risks are in the market.

(b) The class $\mathscr{C}_j$ is a homogeneous collective consisting of $\mathscr{N}_j$ (being $\sum_{j=1}^{k}\mathscr{N}_j=\mathscr{N}$) risks $S_j(t)$ which are compound Poisson processes with the same intensity λ_j. The classes are assumed to be ordered in such a way that $\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_k$.

(c) The m.g.f. $\chi(u)$ is the same for all the classes.

(d) For any individual risk in the class $\mathscr{C}_j$ the premium $x_jt=(\lambda_jE\{X\}+l_j)t$ is charged. Therefore x_j and l_j represent the annual premium and the annual loading relative to these risks, respectively.

Denoting then by n_j the number of risks of the class $\mathscr{C}_j$ that are held in the portfolio, for the property of infinite divisibility one has $\lambda=\sum_{j=1}^{k}\lambda_jn_j$ and the return on the whole portfolio

$$Y(t)=\left(\sum_{j=1}^{k}x_jn_j\right)t-\sum_{j=1}^{k}\sum_{\mathscr{C}_j}S_j(t)$$

has m.g.f.

$$\varphi_Y(u,t)=\exp\left\{\left(\sum_{j=1}^{k}x_jn_j\right)ut+[\chi(-u)-1]\left(\sum_{j=1}^{k}\lambda_jn_j\right)t\right\}. \tag{2.1}$$

2.3. *Anti-Selection*

In this situation, if a company A decided to collect an aggregate annual premium $\pi(1)=\pi$ to protect itself against unfavourable outcome of the risk process, it could be led to charge to each contractholder the "average" premium π/n, with $n = \sum_{i=1}^{k} n_i$. In fact such a choice would offer the advantage of minimizing the administrative costs. But if it were possible to know the risk class to which the contracts (not necessarily all of them) belong, it would be easy for a competing firm B to collect policies concerning low-risk individuals, by charging them a premium less then π/n. On the other hand, the individuals that are more exposed to risk would be spurred to insure themselves with the company A, considering as advantageous the average premium π/n. The effect of such an anti-selection mechanism would then be an alteration of the company A's portfolio composition, such that it would increase even considerably the probability of a negative evolution of the process $Y(t)$. So the choice of the value π would turn out to be inadequate.

Therefore if we make the necessarily schematic and simplifying assumption that the company and the policyholders are in a state of *perfect information* on the parameters of the risk process and in particular on the value of the intensity λ_j, the choice of the premiums will have a significant influence on the composition of the portfolio.

REMARK. The assumption of perfect information finds a different formulation within the subjectivistic theory of probability. In fact in this context it means that the parts are *in agreement* on the values of the probabilities. The problem was discussed, e.g., by Pressacco (1979), who questioned whether a subjective fair price can be given an objective meaning.

The possibility of different probability evaluations has been considered, e.g., by de Ferra (1968) and Volpe di Prignano (1974). In these cases the "advantageousness" of an insurance contract can be studied by defining an *indifference premium* that differs from the net premium both in consideration of the risk aversion and because of the diverse evaluations of the probabilities. The importance of these problems has also been emphasized by Rothschild and Stiglitz (1976), who studied the equilibrium in a competitive insurance market in a state of imperfect information.

In any case, the dependence of premium determination upon the market conditions is the basic assumption in the economic models of insurance market proposed, e.g., by Bühlmann (1980) and de Jong (1981).

2.4. *Demand Function and Risk Aversion*

We are thus led to introduce in the model a dependence of n_j on the premium charged to the class $\mathscr{C}_j$, i.e., $n_j = n_j(x_j)$, $(j = 1, 2, \ldots, k)$. Following Cacciafesta (1970), we shall make the rather natural assumption

$$n_j(x) = \mathcal{N}_j d_j(x), \qquad (j = 1, 2, \ldots, k),$$

where the *demand function* $d_j(x)$ (that we, for sake of simplicity, shall treat as a real-valued function) represents the fraction of individuals of the class $\mathscr{C}_j$ that insure themselves at the annual premium x and therefore it expresses the sensitivity of $\mathscr{C}_j$ to the price that is charged. If one were to represent this function (for given j) as a set of random variables $\{d_j(x); x \geq 0\}$, with probability distributions that are chosen based on statistical observations, besides procedural and estimative complications, the highly significant relation between demand function and risk aversion would be mistreated.

Instead of strictly describing the function $d_j(x)$, it therefore appears more significant to refer to a deterministic model characterized by

$$d_j(x) = \begin{cases} 1, & \text{for } 0 \leq x < \lambda_j E\{X\}, \\ \dfrac{x_j^* - x}{l_j^*}; & \text{for } \lambda_j E\{X\} \leq x \leq x_j^*, \\ 0, & \text{for } x > x_j^*, \end{cases} \tag{2.2}$$

with $x_j^* = \lambda_j E\{X\} + l_j^*$, $l_j^* \geq 0$, $(j = 1, 2, \ldots, k)$. Evidently x_j^* can be seen as a measure of the risk aversion of the class $\mathscr{C}_j$ as a whole. It is interesting to note that the function $d_j(x)$ can be interpreted as the probability that an individual of $\mathscr{C}_j$ chosen at random insures himself, provided that the decisions of the individuals are stochastically independent; in this case $n_j(x)$ is to be understood as the expected number.

If we accept the assumption that the risk to which an individual is exposed is small relative to his wealth c, i.e., $E\{S(1)\} \ll c$, if his utility function $u(z)$ can be expanded in a Taylor series around c and if we limit ourselves to a second-order approximation, then we obtain a quadratic utility function

$$u(c+z) = z - \tfrac{1}{2} r(c) z^2,$$

where $r(c) = -u''(c)/u'(c)$ is the Arrow–Pratt (*local*) *risk aversion*, or, in other terms, the (*local*) *propension to insurance* (in the actuarial applications r is generally supposed as a decreasing function of c). If all the individuals of the class $\mathscr{C}_j$ have the same value r_j of risk aversion, then x_j^* and l_j^* represent respectively the *maximum acceptable premium* and the *maximum acceptable loading* by each one of them.

Because of the Poisson assumptions on the risk process, one can prove that

$$l_j^* \simeq \tfrac{1}{2} r_j \lambda_j E\{X^2\}; \tag{2.3}$$

by expressing l_j^* according to the variance principle, i.e., $l_j^* = \beta_j^* \operatorname{Var}\{S_j(1)\}$, relation (2.3) gives: $\beta_j^* \simeq \frac{1}{2} r_j$.

The foregoing considerations suggest, among other things, that it includes in the model the possibility of a stratification more refined than the one obtained solely based on the characteristics of the risk process $S(t)$, since one can take in consideration classes that differ only for the risk aversion (without contradicting the hypotheses made in (2.2)).

The introduction of the functions $n_j(x)$ brings about that all the variables endogenous to the risk come to depend upon the choice of the vector $\boldsymbol{x}$ of the prices charged to the classes. From relation (2.1) one can, for example, derive the expression of the *expected return on the portfolio*

$$M(\boldsymbol{x}, t) = E\{Y(\boldsymbol{x}, t)\} = \left[\sum_{j=1}^{k} N_j l_j d_j(x_j)\right] t \tag{2.4}$$

and that of the *variance of the portfolio return*

$$V(\boldsymbol{x}, t) = \operatorname{Var}\{Y(\boldsymbol{x}, t)\} = \left[\sum_{j=1}^{k} \mathcal{N}_j \lambda_j d_j(x_j)\right] E\{X^2\} t. \tag{2.5}$$

2.5. *The Probability of Ruin*

Because of the form of the demand curves that we have assumed, the charging of a premium $x_j > x_j^*$ is entirely equivalent to a refusal by the company of the risks belonging to the class $\mathscr{C}_j$; the choice of the price vector thus seems to be a significant and reliable means for the portfolio selection.

The process $Y(\boldsymbol{x}, t)$ can be evaluated in terms of risk-return, that is by defining a performance criterion explicitly in terms of expected return and of portfolio risk and by choosing the best composition according to this criterion.

Many and plausible measures of risk can be proposed and adopted, but in our case it is natural to consider the probability of ruin before time t, $q(w, t)$, which moreover is the most investigated stability criterion in the actuarial literature and is also widely adopted in the administrative policy of the insurance companies. As can be seen, for example, in Seal (1979), it is generally rather complicated to evaluate $q(w, t)$ and this is also the case in models based on Poisson assumptions. It is instead rather easy to obtain useful results in the asymptotic case, i.e., for $q(w) = \lim_{t\to\infty} q(w, t)$.

In fact, with the assumptions of our model, the following classical result holds

$$q(w) \leq e^{-\tau w}, \tag{2.6}$$

$-\tau$ being the negative root of

$$E\{e^{uY(t)}\} = 1. \tag{2.7}$$

The inequality (2.6) was derived by F. Lundberg (1909) and by de Finetti (1939) using different methods. τ, known as *safety index*, is also called *adjustment coefficient*, e.g., by Gerber (1981), who proposed a martingale theoretic approach to the ruin problem.

Generally, the right-hand side of (2.6) does not represent the ruin probability but provides an upper bound for it. However we are dealing with an "efficient" bound, because relation (2.6) becomes an equality when the graph of the realizations of the process $Y(t)$ can not jump the barrier $-w$, that is if at the time of ruin there remains no margin of insolvency (Dubourdieu (1952)).

From equations (2.1) and (2.7), $-\tau$ is the negative root of

$$\chi(-u)-1=-u\frac{\sum_{j=1}^{k} x_j n_j(x_j)}{\sum_{j=1}^{k} \lambda_j n_j(x_j)}. \tag{2.8}$$

Because of the independence of such an expression from the time variable it is therefore sufficient to refer to a single-period model, as it was reasonable to expect, due to the fact that $Y(t)$ is a process with independent increments. All the endogenous quantities characteristic to the model will then be single-period (annual) quantities.

It is interesting to derive a quadratic approximation by using the property $\chi(-u)=1-E\{X\}+u^2E\{X^2\}/2+o(u^2)$. From (2.8) we then obtain

$$uE\{X\}-\frac{u^2}{2}E\{X^2\}\simeq u\frac{\sum_{j=1}^{k} x_j n_j(x_j)}{\sum_{j=1}^{k} \lambda_j n_j(x_j)},$$

which provides

$$\tau\simeq 2\frac{1}{E\{X^2\}}\frac{\sum_{j=1}^{k} \mathcal{N}_j l_j d_j(x_j)}{\sum_{j=1}^{k} \mathcal{N}_j \lambda_j d_j(x_j)}=2\frac{M(\boldsymbol{x})}{V(\boldsymbol{x})}, \tag{2.9}$$

where we denote $M(\boldsymbol{x})=M(\boldsymbol{x},1)$ and $V(\boldsymbol{x})=V(\boldsymbol{x},1)$. It should be pointed out that since the approximation is valid near the origin, then the less the quantity $\sum_{j=1}^{k} x_j n_j/\sum_{j=1}^{k} \lambda_j n_j$ exceeds the value of the derivative of $\chi(u)$ at the point $u=0$, the better the approximation is. This means that the results which we shall obtain will be much better, the closer we get to the fairness condition in the whole portfolio.

REMARK. The evaluation of the stability of an insurance company with an infinite planning horizon can raise doubts of a conceptual nature and in fact, in the past, the suitability of using the ultimate ruin probability has been widely debated (for a review, see Ammeter, Depoid and de Finetti (1957, p. 59)). The question has not remained limited strictly to the actuarial setting; for example, Massé (1964) has made use of the index $q(w)$ to compare the two notions of complete strategy and incomplete strategy. More recently, Ammeter (1970) has applied the ultimate ruin probability criterion in the study of the solvency problem of the european life insurance companies.

The parameter τ has been used even lately by Amsler (1978), who introduced it in his "general equilibrium equation of a collective risk", obtaining from it the definition of a solvency index.

3. SELECTING THE OPTIMAL INSURANCE PORTFOLIO

3.1. *The Programming Problem*

In the foregoing model the main problem faced by the company is to choose the price vector $\boldsymbol{x}$ so as to constitute an *efficient portfolio*, which has the maximum safety index for a given level M of the expected return. It has then to solve the following programming problem

$$(3.1)\qquad \begin{cases} \max \tau(\boldsymbol{x}) \\ M(\boldsymbol{x}) = M \geq 0 \\ x_j \geq \lambda_j, \qquad (j = 1, 2, \dots, k), \end{cases}$$

where the lower bounds on the x_j are due to the fact that we have excluded negative loadings.

Since the factor $1/E\{X^2\}$ seems to be irrelevant in the optimization problem, one can put this quantity equal to 1, as for example would be the case if $S(t)$ were an ordinary Poisson process (unit jumps, $\chi(u) = e^u$). For sake of simplicity we shall indeed confine ourselves to this case, from now on, by putting moreover $E\{X\} = 1$. Obviously, with these limitations the expected value principle and the variance principle turn out to coincide and we shall write $l_j = \eta_j \lambda_j$. It can be noted that in this case η_j directly represents the Arrow–Pratt risk aversion in so far as, within the limits of the quadratic approximation of the utility function, one has $\eta_j \simeq \frac{1}{2} r_j$.

Recalling equation (2.9), problem (3.1) is equivalent to

$$(3.2)\qquad \begin{cases} \min V(\boldsymbol{x}) \\ M(\boldsymbol{x}) = M \geq 0 \\ x_j \geq \lambda_j, \qquad (j = 1, 2, \dots, k); \end{cases}$$

we are thus led to a mean-variance model.

Obviously, it is sufficient to study the problem (3.2) within the interval D of the Euclidean k-space $\mathbb{R}^k$:

$$D \equiv \{\boldsymbol{x} \mid \lambda_j \leq x_j \leq x_j^*;\ j = 1, 2, \dots, k\}.$$

In fact all the intervals for which $x_j > x_j^*$ for one or more values of j, that correspond to the exclusion of some risk classes, are equivalent to the cases $x_j = x_j^*$ and therefore are represented by intervals on the boundary of D.

Hence the problem (3.2) takes on the following explicit formulation

$$(3.3)\qquad \begin{cases} \min \sum_{j=1}^{k} \dfrac{\mathcal{N}_j}{\eta_j^*}[-x_j + (1+\eta_j^*)\lambda_j] \\ \sum_{j=1}^{k} \dfrac{\mathcal{N}_j}{\eta_j^*}\left[-\dfrac{1}{\lambda_j}x_j^2 + (2+\eta_j^*)x_j - (1+\eta_j^*)\lambda_j\right] = M \\ \lambda_j \leq x_j \leq \lambda_j(1+\eta_j^*), \qquad (j = 1, 2, \dots, k). \end{cases}$$

This programming problem differs from those typical to the mean-variance models that are used in the portfolio analysis in that the objective function is linear, whereas the constraint is a quadratic function which contains the linear terms but in which the mixed terms are missing. The latter characteristic depends on the hypothesis of independence among the risks.

3.2. *Deriving the Efficient Prices*

The constraint equation represents, when M varies in $\mathbb{R}^+$, a portion of elliptic paraboloid in $k+1$ dimensions, whose vertex has the following coordinates

$$\begin{cases} M^* = \frac{1}{4} \sum_{j=1}^{k} \mathcal{N}_j \lambda_j \eta_j^*, & (3.4a) \\ x_j = \left(1 + \frac{\eta_j^*}{2}\right) \lambda_j, \qquad (j = 1, 2, \ldots, k). & (3.4b) \end{cases}$$

Therefore the maximum expected return M^* is obtained by choosing $\boldsymbol{x}$ coinciding with the center C of D, whose coordinates are just given in (3.4b) (see fig. 1). Furthermore, in C we have

$$V^* = \frac{1}{2} \sum_{j=1}^{k} \mathcal{N}_j \lambda_j. \tag{3.5}$$

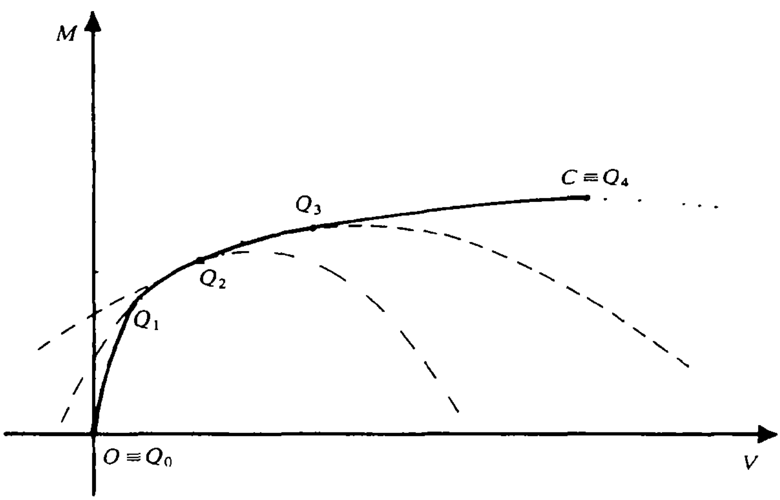

FIGURE 1 Efficient frontier with four risk classes.

To solve the conditional extremum problem, let us set up the Lagrangian function

$$L(\boldsymbol{x}, \mu) = V(\boldsymbol{x}) + \mu[M - M(\boldsymbol{x})].$$

From the equations $\partial/\partial x_j L(\boldsymbol{x}, \mu) = 0$ we obtain

$$x_j = \frac{\lambda_j}{2}\left(\frac{1}{\mu} + \eta_j^* + 2\right), \qquad (j = 1, 2, \ldots, k). \tag{3.6}$$

These are the parametric equations of a straight line $\mathcal{O}$ which passes through the center C of the interval D and coincides with its "upward" diagonal (i.e., the straight line passing through points $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $(x_1^*, x_2^*, \ldots, x_k^*)$ only if the values of η_j^* are all equal.

By substituting (3.6) in the constraint equation, one has

$$\frac{1}{\mu} = \pm 2\sqrt{\frac{M^* - M}{\sum_{j=1}^{k} \frac{\mathcal{N}_j \lambda_j}{\eta_j^*}}}. \tag{3.7}$$

The negative roots of (3.7) are to be discarded because, as it is easy to verify, they correspond to points with maximum variance. Therefore all the points of the straight line $\mathcal{O}$ that are lower than C are to be discarded as inefficient. Then equation (3.7), modified in this manner, leads to the parametric equations of $\mathcal{O}$, $x_j = x_j(M)$. To obtain the *efficient frontier* it is sufficient to substitute these expressions of x_j into the objective function, thus attaining $V = V(M)$ and therefore, passing to the inverse function, the equation

$$M = -\frac{V^2}{\sum_{j=1}^{k} \frac{\mathcal{N}_j \lambda_j}{\eta_j^*}} + \frac{\sum_{j=1}^{k} \mathcal{N}_j \lambda_j}{\sum_{j=1}^{k} \frac{\mathcal{N}_j \lambda_j}{\eta_j^*}} V + M^* - \frac{1}{4} \frac{\left(\sum_{j=1}^{k} \mathcal{N}_j \lambda_j\right)^2}{\sum_{j=1}^{k} \frac{\mathcal{N}_j \lambda_j}{\eta_j^*}}. \tag{3.8}$$

It is to be noted that the constant term in (3.8) is nonnegative and vanishes if and only if the η_j^* values are all equal. In fact, by indicating by $A(\{\eta_j^*\})$ the weighted arithmetic mean of the η_j^*, with weighting factors $\mathcal{N}_j \lambda_j / \sum_{j=1}^{k} \mathcal{N}_j \lambda_j$, it can be written, keeping in mind (3.4a)

$$\frac{1}{4} \sum_{j=1}^{k} \mathcal{N}_j \lambda_j \left[A(\{\eta_j^*\}) - \frac{1}{A(\{1/\eta_j^*\})} \right]$$

and the conclusion is drawn by observing that the quantity between square brackets is the difference between the arithmetic and the harmonic mean.

In order that equation (3.8) represents an efficient frontier it is necessary to bound it to suitable values of V. Above all we shall disregard values greater than V^*, in so far as they provide levels of expected return less than M^* (and in fact they are the points lower than C, which we have discarded because of the inversion of $V = V(M)$). Values of the variance that are decreasing from V^* corresponds to points of $\mathcal{O}$ which move upwards away from C, until they reach the boundary of D. We shall denote by Q_{k-1} the intersection point between $\mathcal{O}$ and this boundary. If all the η_j^* were to be equal, the point Q_{k-1} would coincide with the vertex $(x_1^*, x_2^*, \ldots, x_k^*)$ of D, that we shall indicate by Q_0 and that corresponds to values of M and V equal to zero (empty portfolio). Instead, in the general case, the first class to be excluded will be the one corresponding to the least η_j^*.

Let us then consider a permutation q of the subscripts $\{j\}$ such that

$$\eta_{q1}^* \leq \eta_{q2}^* \leq \cdots \leq \eta_{qk}^*.$$

The coordinates of the point Q_{k-1} will then be expressed by

$$x_j(Q_{k-1}) = \left(1 + \frac{\eta_{q1}^* + \eta_j^*}{2}\right)\lambda_j, \qquad (j = 1, 2, \ldots, k),$$

which provide

$$V(Q_{k-1}) = V^* - \frac{\eta_{q1}^*}{2} \sum_{j=1}^{k} \frac{\mathcal{N}_j \lambda_j}{\eta_j^*},$$

$$M(Q_{k-1}) = M^* - \left(\frac{\eta_{q1}^*}{2}\right)^2 \sum_{j=1}^{k} \frac{\mathcal{N}_j \lambda_j}{\eta_j^*}.$$

The value $V(Q_{k-1})$ is the minimum possible variance relative to an efficient portfolio made up of k risk classes. For lower values of V we are led in practice to a problem in $k-1$ dimensions, until the increase of the prices charged will not bring to the exclusion of the class $\mathscr{C}_{q2}$. This will happen in a point Q_{k-2} with coordinates

$$x_j(Q_{k-2}) = \begin{cases} \left(1 + \dfrac{\eta_{q2}^* + \eta_j^*}{2}\right)\lambda_j, & \text{for } j \neq q_1 \\ x_j^*, & \text{for } j = q_1. \end{cases}$$

The efficient portfolios composed of $k-1$ risk classes are represented by the points of the line-segment $Q_{k-1}Q_{k-2}$, laying on the boundary of D and the efficient frontier has the same expression as in (3.8), provided that now we bound it to the values of V contained between $V(Q_{k-2})$ and $V(Q_{k-1})$ and the sums range over the remaining $k-1$ classes. By continuing to increase the prices, the progressive elimination of all the risk classes will be brought about, until one reaches, in Q_0, the emptying out of the portfolio.

The complete efficient frontier can be expressed by

$$(3.9)\qquad \begin{cases} M = \left(\displaystyle\sum_{j=1}^{k} \frac{\mathcal{N}_{qj}\lambda_{qj}}{\eta_{qj}^*}\right)^{-1} \left\{ -V^2 + \left(\displaystyle\sum_{j=1}^{k} \mathcal{N}_{qj}\lambda_{qj}\right) V \right. \\ \qquad \left. + \dfrac{1}{4}\left[\displaystyle\sum_{j=1}^{k} \frac{\mathcal{N}_{qj}\lambda_{qj}}{\eta_{qj}^*} \sum_{j=1}^{k} \mathcal{N}_{qj}\lambda_{qj}\eta_{qj}^* - \left(\displaystyle\sum_{j=1}^{k} \mathcal{N}_{qj}\lambda_{qj}\right)^2\right]\right\}, \\ \text{for } V(Q_{k-s}) < V \leq V(Q_{k-s+1}), \qquad (s = 1, 2, \ldots, k), \end{cases}$$

where we denote $C = Q_k$, and the points Q_{k-s} have coordinates

$$x_j(Q_{k-s}) = \begin{cases} \left(1 + \dfrac{\eta_{qs}^* + \eta_j^*}{2}\right)\lambda_j, & \text{for } j \neq q_1, q_2, \ldots, q_{s-1} \\ x_j^*, & \text{for } j = q_1, q_2, \ldots, q_{s-1}. \end{cases}$$

In the space $\mathbb{R}^k$ the efficient portfolios are represented by the points of the broken line $CQ_{k-1} \ldots Q_1Q_0$.

In the plane (V, M) the efficient frontier has the shape of a "chain" of arcs of parabola that are joined together in the points Q_{k-s} and which are ever more convex from C to Q_0.

3.3. *Maximizing the Expected Utility of the Company*

Once the efficient frontier has been determined, the analysis of the decision problem of the company is concluded by choosing the portfolio that represents the best trade-off between mean and variance, that is by maximizing a utility function of the form $u(M, V)$. By introducing a set of indifference curves in the (V, M) space, the optimal portfolio is represented by the tangency point between the efficient frontier and the indifference curve corresponding to the highest possible level of the utility.

If we suppose, for example, that the initial free capital w is large relative to the expected return on the portfolio, i.e., if $w \gg M^*$, then it is possible, analogously as was done in sect. (2.4), to approximate the utility function of the company by the quadratic utility function

$$u(w+z) = z - \tfrac{1}{2} r_c(w) z^2,$$

with the related indifference curves

$$V = -M^2 + \frac{2}{r_c(w)}(M - U),$$

where U is the level of expected utility corresponding to the curve and $r_c(w)$ represents the Arrow–Pratt measure of risk aversion of the insurance company.

However it is to be noted that, with our assumptions, the suitability in using the variance as a measure of the riskiness of the portfolio does not rely on the accuracy of the quadratic approximation of the company's utility function, but on the goodness of the approximation made for the probability of ruin.

4. ILLUSTRATION OF RESULTS IN THE TWO-CLASSES CASE

Let us now discuss and illustrate the results obtained in section (3) in the case in which the risk market is made up of only two risk classes $\mathscr{C}_1$ and $\mathscr{C}_2$, with $\lambda_1 < \lambda_2$. Furthermore, let suppose that $\mathscr{C}_1$ is characterized by a level of risk aversion greater t: an $\mathscr{C}_2$, i.e., $\eta_1^* > \eta_2^*$.

The problem finds a simple geometric representation in the plane (x_1, x_2) (see fig. 2). We see that the level lines $M(\boldsymbol{x}) = M$ of the expected return constitute a set of ellipses with center C (the center of rectangle D), axes parallel to the coordinate axes and size decreasing as M increases. The maximum expected return will then be attained by choosing the premiums $x_i = x_i(C)$. The level lines $V(\boldsymbol{x}) = V$ of the variance instead form a set of parallel straight lines with slope $-\mathcal{N}_1\eta_2^*/\mathcal{N}_2\eta_1^*$ that come closer to the origin as V increases.

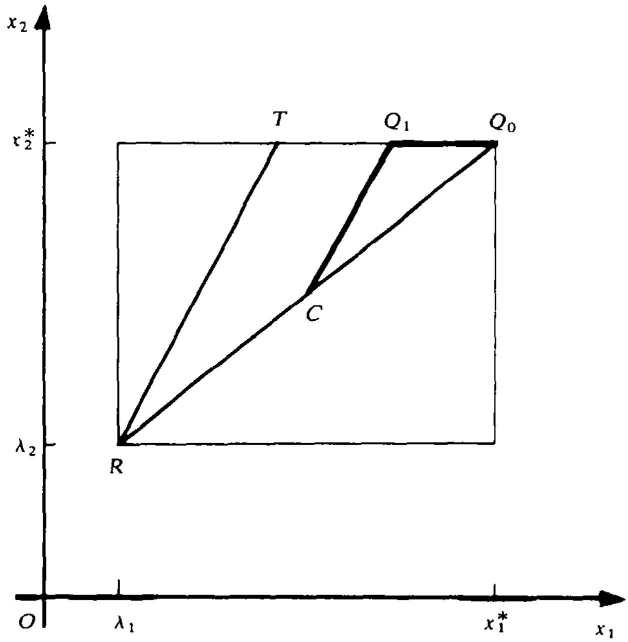

FIGURE 2. CQ_1Q_0. Efficient prices, RQ_0 Market portfolios, RT Proportional loadings

The efficient portfolio relative to the choice of a return level M is therefore represented by the tangency point between the ellipse $M(\boldsymbol{x}) = M$ and the "highest" possible variance level line. In this manner we obtain the straight line $\mathcal{C}$

$$x_2 = \frac{\lambda_2}{\lambda_1} x_1 - \frac{1}{2}\lambda_2(\eta_1^* - \eta_2^*),$$

which passes through C and intersects the boundary $x_2 = x_2^*$ of D in the point Q_1, having abscissa $x_1(Q_1) = [1 + (\eta_2^* - \eta_1^*)/2]\lambda_1$. Of course, from $\mathcal{C}$ are to be discarded, besides the points above Q_1, even those below C, that correspond to inefficient portfolios (maximum V for given M). The prices indicated by Q_1 generate the mixed portfolio with minimum variance; in order to achieve lesser values of V it is necessary to operate with only one class, choosing the prices on the line-segment Q_1Q_0.

If the two classes were to have an equal degree of risk aversion, i.e., if $\eta_1^* = \eta_2^* = \eta^*$, the locus of the efficient solutions would be given by

$$x_j = \lambda_j\left(1 + \frac{\eta^*}{2} + \frac{1}{\mu}\right),$$

with

$$0 \leqslant \frac{1}{\mu} \leqslant \frac{\eta^*}{2}, \qquad (j = 1, 2)$$

and the points Q_1 and Q_0 would coincide. In this manner a premium-making policy of rather intuitive significance would be confirmed, that is the charging to both the classes of a loading equal to the same percentage η of the net premium. In our case, instead, being $\eta_1^* > \eta_2^*$, it turns out that the efficient choices consist in overloading the more risk averse class $\mathscr{C}_1$ by increasing the percent loading $\eta\lambda_1$ by the quantity $(\eta_1^* - \eta_2^*)/2$.

Another interesting result consists in the fact that a diversification of the portfolio is not always efficient, because if small values of the variance (line-segment Q_1Q_0) are desired, then the expected return is maximized by insuring only individuals that are of the more risk averse class.

Finally, let us compare the policy of the efficient prices with that of the prices that determine a *natural,* or *market, portfolio,* that is a portfolio that contains both the risk classes in the same proportion with which they are present on the market. By solving the equations

$$\frac{n_j(x_j)}{\sum_{j=1}^{k} n_j(x_j)} = \frac{\mathcal{N}_j}{\sum_{j=1}^{k} \mathcal{N}_j}, \qquad (j = 1, 2),$$

one easily obtains the parametric equations

$$\begin{cases} x_j = x_j^* - d\eta_j^* \lambda_j, \\ 0 \leq d \leq 1, \\ (j = 1, 2), \end{cases}$$

that represent the diagonal of D passing through Q_0. As can be seen, it is a matter of charging to the two classes a loading which is equal to a same fraction $(1-d)$ of the respective maximum percent loading η_j^* and this policy will turn out to be efficient if the classes are characterized by a different degree of risk aversion.

REFERENCES

AMMETER, H (1970) Das Solvibilitatsproblem in der Risiko-Lebenversicherung, *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker* **70**, 41–56.

AMMETER, H., DEPOID, P and DE FINETTI, B (1959) L'Étude Mathématique des Assurances Non Viagères dans l'Europe Continentale Occidentale *Astin Bulletin* **1**, 46–70

AMSLER, M -H (1978) L'equation générale d'équilibre d'un risque collectif *Mitteilungen der Vereinigung schweizerischer Versicherugsmathematiker* **78**, 221–246

BUHLMANN, H (1980) An Economic Premium Principle *Astin Bulletin* **11**, 52–60

CACCIAFESTA, R. (1970) Sulla Determinazione delle Classi di Rischio *Giornale dell'Istituto Italiano degli Attuari* **70**, 38–48.

DUBOURDIEU, J. (1952) *Théorie Mathématiques des Assurances I, Théorie Mathématique du Risque dans les Assurances de Répartition* Gauthier-Villars Paris.

FERRA, C DE (1968) Sulla Vantaggiosità delle Operazioni Assicurative, *Giornale dell'Istituto Italiano degli Attuari* **68**, 1–16.

FINETTI, B DE (1939) La Teoria del Rischio e il Problema della "Rovina dei Giocatori" *Giornale dell'Istituto Italiano degli Attuari* **39**, 41–51

GERBER, H U (1981) On the Probabılıty of Ruın ın an Autoregressıve Model. *Mıtteılungen der Vereınıgung schweızerıscher Versıcherungsmathematıker* **81**, 213–219

JONG, P DE (1981) Insurance Premıums under Demand Constraınts *Scandınavıan Actuarıal Journal* **81**, 123–125

KAHANE, Y (1979) The Theory of Insurance Rısk Premıums—A Re-examınatıon ın the Lıght of Recent Developments ın Capıtal Market Theory *Astın Bulletın* **10**, 223–239

LUNDBERG, F (1909) Uber dıe Theorıe der Ruckversıcherung. *Trans VI Intern Congr. Act.* **1**, 877–948.

MASSÉ, P (1964) *Le Choıx des Investıssements (Crıtéres et Méthodes)*. Dunod Parıs

PRESSACCO, F (1979) Value and Prıces ın a Reınsurance Market *Astın Bulletın* **10**, 263–273

ROTHSCHILD, M and STIGLITZ, J (1976) Equılıbrıum ın Competıtıve Insurance Markets an Essay on the Economıcs of Perfect Informatıon. *Quarterly Journal of Economıcs* **90**, 629–649

SEAL, H L (1979) *Survıval Probabılıtıes (The Goal of Rısk Theory)* Wıley New York

SENGUPTA, J K (1981) *Optımal Decısıons under Uncertaınty* Lecture Notes ın Economıcs and Mathematıcal Systems N 193, Sprınger-Verlag Berlın

VOLPE DI PRIGNANO, E (1974) Sulla Mısurabilıtà del Premio dı Indıfferenza neı Ramı Elementarı *Gıornale dell'Istıtuto Italıano deglı Attuarı* **74**, 25–45

CORRIGENDUM

E. KREMER (1982). Rating of Largest Claims and ECOMOR Reinsurance Treaties for Large Portfolios. *Astin Bulletin* **13**, 47–56.

Ragnar Norberg pointed out to me that there is an incorrectness in the proof of the basic theorem on pages 50–51. Nevertheless by a slight modification the proof becomes complete. Instead of (3.6) one should use the bound

$$\frac{|r_k|}{\nu_k} \leq R_{k1}^{\varepsilon} + R_{k2}^{\varepsilon}.$$

With (for $\varepsilon > 0$)

$$R_{k1}^{\varepsilon} := E\left(\frac{1}{\nu_k} \cdot \sum_{i=1}^{N_k} |X_i| \cdot 1_{U_\varepsilon^c \cap [Y_{k1}, Y_{k2})}(X_i)\right)$$

$$R_{k2}^{\varepsilon} := E\left(\frac{1}{\nu_k} \cdot \sum_{i=1}^{N_k} |X_i| \cdot 1_{U_\varepsilon \cap [Y_{k1}, Y_{k2})}(X_i)\right)$$

$U_\varepsilon := [P_s - \varepsilon, P_s + \varepsilon]$ (U_ε^c denoting the complementary set of U_ε).

By the reasoning following formula (3.6) one can conclude with the theorem of dominated convergence:

$$\lim_{k\to\infty} R_{k1}^{\varepsilon} = 0$$

$$\limsup_{k\to\infty} R_{k2}^{\varepsilon} \leq E(|X_i| \cdot 1_{U_\varepsilon}(X_i)).$$

Since F is by assumption continuous, the last expression can be made arbitrarily small (by suitable choice of ε), implying statement (3.7) of the proof.

ERRATUM

P. TER BERG (1980). Two Pragmatic Approaches to Loglinear Claim Cost Analysis. *Astin Bulletin* **11**, 77–90.

Formula (5.7) contains an annoying printing error The correct formula reads:

$$(5.7) \qquad \frac{\partial^2 \log L}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}'} = -\tfrac{1}{2} \sum \varphi_r \left(\frac{y_r}{\mu_r} + \frac{n_r^2 \mu_r}{y_r} - 2n_r\right) z_r z_r'.$$

This correction is important if one maximizes the loglikelihood function via Newton's method, which needs the inverse of (5.7).

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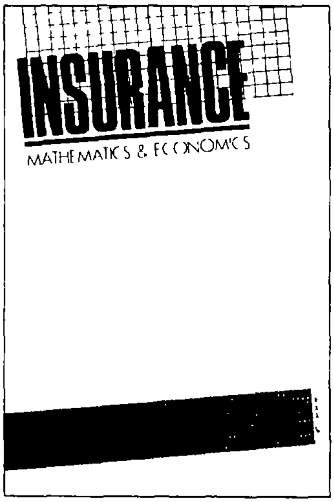

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   FOLKS, J L, and R S CHHIKARA (1978) The Inverse Gaussian Distribution and its Statistical Application — A Review (with Discussion), *Journal of the Royal Statistical Society B* **40**, 263–289

   ROTHENBERG, T J (1973) *Efficient Estimation with A Priori Information* New Haven Yale University Press

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