SOME NUMERICAL ASPECTS IN TRANSIENT RISK THEORY*

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ABSTRACT

We give some actual possibilities for computing numerical values in the classical risk models both in transient and asymptotical cases by introducing the concept of normed model. Some recent approximations are tested on numerical examples.

We also emphasize the interest of these methods to compute waiting time distributions (transient and stationary cases) in queueing theory.

1. MODELS CONSIDERED

1.1. *Risk Model*

We will limit our attention to the classical Cramér–Lundberg model for which we have the following characteristics:

(i) The claim number process is a Poisson one with parameter λ. Let $(A_n)_{n\geq 1}$ be the sequence of interarrival times between claims so that

$$\mathbb{E}(A_n)=\lambda^{-1}. \tag{1.1}$$

Following the current notation, $N(t)$ $(t\geq 0)$ represents the total number of claim occurrences on $(0, t]$.

(ii) The process of successive claim amounts is a sequence of non negative i.i.d. random variables $(B_n)_{n\geq 1}$ with d.f. $B(\cdot)$ such that

$$\mathbb{E}(B_n)=\beta \tag{1.2}$$

and this process is independent of $(A_n)_{n\geq 1}$.

(iii) The premium income process has a constant rate per unit of time: c. To avoid certain ruin on $[0, \infty)$, we must have:

$$\frac{\lambda\beta}{c}<1. \tag{1.3}$$

So, we can define η, the security loading by

$$c=\lambda\beta(1+\eta). \tag{1.4}$$

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Every risk model is thus characterized by a triple $(\lambda, B(x), \eta)$. Now define,

(1.5) $$S(t)=\sum_{n=1}^{N(t)} B_n$$

with the usual convention that a summation over a void indice set is 0, and

(1.6) $$R(t)=u+c\cdot t-S(t)$$

where u, supposed to be positive, is the initial reserve. Of course, if $F(x, t)$ is the d.f. of $S(t)$, we have

(1.7) $$F(x,t)=\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}B^{n*}(x)$$

where B^{n*} represents the n-fold convolution of B.

If T is the random variable, possibly defective, defined by

(1.8) $$T=\inf\{t: R(t)<0\}$$

we have for the probabilities of non-ruin the following definitions:

(a) on a finite horizon time $[0, t]$

(1.9) $$\phi(u,t)=\mathbb{P}[T>t]$$

(b) on a finite horizon time $[0, \infty)$

(1.10) $$\phi(u)=\lim_{t\to\infty}\phi(u,t).$$

For the ruin probabilities, we have, of course

(1.11) $$\psi(u,t)=1-\phi(u,t)$$

(1.12) $$\psi(u)=1-\phi(u).$$

1.2. *Normed Risk Models*

1.2.1. *First Semi-Normed Relation*

Let R_0 and R_1 be two risk models characterized respectively by $(1, B(\cdot), \eta)$ and $(\lambda, B(\cdot), \eta)$.

If $\phi_0(u, t)$ and $\phi_1(u, t)$ are corresponding non-ruin probabilities, we want to find a relation between ϕ_0 and ϕ_1. To do so; let us remark that from (1.7)

(1.13) $$F_0(x,t)=\sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}B(x)$$

(1.14) $$F_1(x,t)=\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}B(x)$$

so that

(1.15) $$F_1(x,t)=F_0(x,\lambda t)$$

or $S_1(t)$ has the same distribution as $S_0(\lambda t)$.

Now, from (1.9)

$$\phi_1(u,t)=\mathbb{P}[S_1(t')\leq u+c_1\cdot t',\ t'\in[0,t]] \tag{1.16}$$

with $c_1=\lambda\cdot\beta\cdot(1+\eta)$ by (1.4). For R_0, we have $c_0=(1+\eta)\cdot 1\cdot\beta$. Using (1.15), we get

$$\begin{aligned}\phi_1(u,t)&=\mathbb{P}[S_0(\lambda t')\leq u+\lambda\cdot c_0\cdot t',\ t'\in[0,t]]\\&=\mathbb{P}[S_0(t'')\leq u+c_0t'',\ t''\in[0,\lambda t]]\end{aligned}$$

and finally

$$\phi_1(u,t)=\phi_0(u,\lambda t). \tag{1.17}$$

1.2.2. *Second Semi-Normed Relation*

Following Pfenninger (1974), we can also normalize the claim size distribution. Let us consider the risk model R_1 and R_2, where R_2 is characterized by $(\lambda, B'(\cdot), \eta)$ with

$$B'(x)=B(\beta x) \tag{1.18}$$

i.e., $B'(x)$ is the d.f. of the random variables B_n/β.

We have

$$\begin{aligned}\phi_1(u,t)&=\mathbb{P}[S_1(t')\leq u+\lambda\cdot\beta\cdot(1+\eta)t',\ t'\in[0,t]]\\&=\mathbb{P}\left[\sum_{n=0}^{N(t')}B_n\leq u+\lambda\cdot\beta\cdot(1+\eta)t',\ t'\in[0,t]\right]\\&=\mathbb{P}\left[\sum_{n=0}^{N(t')}\frac{B_n}{\beta}\leq\frac{u}{\beta}+\lambda(1+\eta)t',\ t'\in[0,t]\right]\\&=\mathbb{P}\left[S_2(t')\leq\frac{u}{\beta}+\lambda(1+\eta)t',\ t'\in[0,t]\right]\end{aligned}$$

and finally

$$\phi_1(u,t)=\phi_2\left(\frac{u}{\beta},t\right). \tag{1.19}$$

1.2.3. *Normed Relation*

Combining the two preceding steps, we get the so-called normed relation for the risk models R_1 and R_3 respectively characterized by $(\lambda, B(\cdot), \eta)$ and $(1, B'(\cdot), \eta)$:

$$\phi_1(u,t)=\phi_3\left(\frac{u}{\beta},\lambda\cdot t\right) \tag{1.20}$$

R_3 is called the *normed model.*

This relation gives some simplification for numerical computation, especially for tabulation purposes. For example, in the M/M/1 model for which

$$B(x)=\begin{cases}0 & x<0\\ 1-e^{-\mu x} & x\geq 0\end{cases}$$

characterized shortly by the triple (λ, μ, n), the normed model R_3 is given by the triple $(1, 1, \eta)$ so that we have only one parameter, the security loading.

From the numerical point of view, it suffices to treat this model to obtain results for any model with triple (λ, μ, η).

2. THE QUEUEING MODEL

We will only consider the classical M/G/1 model for which $\bar{\lambda}$ is the rate of arrivals and $\bar{B}(\cdot)$ the d.f. of the service time, with mean $\bar{\beta}$. If $\bar{N}(t)$ $(t \geq 0)$ represents the total number of arrivals on $[0, t]$ and W_n the waiting time of customer number n (we suppose that $W_0 = 0$, i.e., a time 0, a service is just beginning) it can be shown (Janssen (1977)) that

$$\mathbb{P}[W_{\bar{N}(t)} \leq u] = \bar{\phi}(u, t) \tag{2.1}$$

$$\lim_{t\to\infty} [W_{\bar{N}(t)} \leq u] = \bar{\phi}(u) \tag{2.2}$$

where $\bar{\phi}(u, t)$ and $\bar{\phi}(u)$ are the non-ruin probabilities of a risk model characterized by $\bar{\lambda}$ as claim number process parameter, by $\bar{B}(x)$ as claim size distribution and by $c = 1$ as premium rate. The security loading of this corresponding risk process is, of course, given by

$$c = (1+\eta)\bar{\lambda}\cdot\bar{\beta} \quad \text{or} \quad \eta = \frac{1}{\bar{\lambda}\cdot\bar{\beta}} - 1. \tag{2.3}$$

Consequently, to every M/G/1 queueing model, characterized by $\bar{\lambda}$ and $\bar{\beta}(x)$, corresponds a risk process with parameters $(\bar{\lambda}, \bar{\beta}(x), (1/\bar{\lambda}\bar{\beta})-1)$. Inversely every result for the Cramér–Lundberg model $(\lambda, B(x), \eta)$ can be transposed for a M/G/1 queueing model with parameters

$$\bar{\lambda} = \frac{1}{(1+\eta)\beta}$$

$$\bar{B}(x) = B(x).$$

For a fixed η and a given $B(x)$, we can see the relation between the normed-model non-ruin probability $\phi_3(u, t)$ and the waiting time distribution. We have:

$$\mathbb{P}[W_{\bar{N}(t)} \leq u] = \phi_3\left(\frac{u}{\beta}, \frac{t}{(1+\eta)\beta}\right). \tag{2.4}$$

3. NON-RUIN PROBABILITY IN THE TRANSIENT CASE FOR THE M/G/1 MODEL

Theoretically, two principal methods are used to solve this problem: the first method is based upon the double Laplace transform of $\phi(u, t)$ and the second one upon the previous determination of $\phi(0, t)$.

3.1. *Cramér–Arfwedson–Thorin*

The equation of Thorin (1968), valid in the general case GI/G/1 is:

$$\phi(u, t) = \int_0^t dK(v) \int_{-\infty}^{u+cv} \phi(u+cv-x, t-v)\, dB(x) + 1 - K(t) \tag{3.1}$$

where $K(t) = 1 - e^{-\lambda t}$. It gives the double Laplace–Stieltjes transform of $\phi(u, t) =$

$$\bar{\bar{\phi}}(s, z) = -z(1-s/s_1(z))/(1-cs-z-\bar{B}(s)) \tag{3.2}$$

where $s_1(z)$ is the only root with a negative real part in the Lundberg equation:

$$1 - z + c \cdot s - \bar{B}(s) = 0, \tag{3.3}$$

$\bar{B}(s)$ being the Laplace–Stieltjes transform of $B(x)$.

For the M/G/1 model, Cramér (1955) and Arfwedson (1950) obtained this result by using the integro-differential equation

$$c\frac{\partial}{\partial u}\phi(u, t) = \frac{\partial}{\partial t}\phi(u, t) + \phi(u, t) - \int_0^u \phi(u-y, t)\, dB(y). \tag{3.4}$$

This was also found by Beekman (1966) using results of Donsker and Baxter (1957) about processes with stationary independent increments.

Theoretically, thus, the problem is worked out, but we have to use twice the Laplace inversion. However, we dispose of fiable algorithms for this inversion (Piessens (1969), Stroud and Secrest (1966)), but this needs some care: the Laplace inversion of a good approximation of a given function is not surely a good approximation of the Laplace inversion of this function. Some precautions are thus required if we want to compute $\phi(u, t)$ by means of a double inversion of $\bar{\bar{\phi}}(s, z)$; probably for this reason, there are few results needing such double transformation in the risk theory literature.

However, if $B(x)$ is an exponential polynomial, i.e., if

$$B(x) = 1 - \sum_{v=1}^{m} b_v e^{-\beta_v x}, \qquad b_v > 0, \beta_v > 0 \quad v = 1, 2, \ldots m, \quad \sum b_v = 1 \tag{3.5}$$

then the problem can be solved with only one inversion. In this case, $\bar{\phi}(u, z)$, the Laplace transform of $\phi(u, t)$, is given by

$$\bar{\phi}(u, z) = 1 - \sum_{v=1}^{m} g_v(z)\, e^{-u\, s_2(z)} \tag{3.6}$$

where $s_{2v}(z)$ are the m roots of the Lundberg equation with a positive real part.

Furthermore, in this case, this equation is a polynomial one and the roots are easily obtained by well-known algorithms (Bairstow, Newton–Raphson) (see e.g., Wikstad (1977), Stroeymeyt (1977)).

It is also possible to approximate a general claim size distribution by an exponential polynomial; this was tested by Thorin and Wikstad (1977) for a lognormal distribution.

3.2. *Prabhu–Beňes–Seal*

The well-known relations of Prabhu (1961) can be used here:

$$\phi(u,t)=F(u+ct,t)-c\int_0^t \phi(0,t-\theta)f(u+c\theta,\theta)\,d\theta \tag{3.7}$$

$$\phi(0,t)=\frac{1}{ct}\int_0^t F(y,t)\,dy, \tag{3.8}$$

where $f(x,t)=\partial/\partial x\,F(x,t)$.

Although the function $F(x,t)$ is very difficult to handle directly, the use of the Laplace transform and an integration give the non-ruin probability.

3.3. *Direct Results for* M/M/1 *and* M/D/1 *Models*

M/M/1 model, i.e., the model with the following claim size distribution:

$$B(x)=\begin{cases}0 & x<0,\\ 1-e^{-x} & x\geq 0\end{cases}$$

is the really well-known model in risk theory, it has a direct solution in terms of a modified Bessel function of first class; some subroutines give very accurate values of this function (see e.g. Stroeymeyt (1977)).

The M/D/1 model with a deterministic claim amount can also be directly solved (see e.g., Seal (1974)).

4. THE ASYMPTOTIC NON-RUIN PROBABILITY

For a general M/G/1 model, we have:

$$\phi(u)=\lim_{t\to\infty}\phi(u,t)=\bar{\phi}(u,0) \tag{4.1}$$

where

$$\bar{\phi}(u,z)=\int_{0^-}^{\infty} e^{-zt}\,d_t\phi(u,t).$$

Thus, only one inversion of a Laplace transform is needed and we avoid some problems raised by the double inversion. Furthermore, in some special cases, the value is explicitly given. If $B(x)$ is an exponential polynomial (3.5), Cramér

(1955) gives an explicit formula:

$$\phi(u) = 1 - \sum_{k=1}^{m} C_k e^{-R_k u} \tag{4.2}$$

where $R_k, k = 1, 2, \ldots m$, denote the m roots of the Lundberg equation, a polynomial one in this case; $C_k, k = 1, 2, \ldots m$, are simple functions of those roots. Especially, if $B(x)$ is an exponential, we have the following expression:

$$\phi(u) = 1 - \frac{1}{1+\eta} e^{-\eta/(1+\eta)\, u}. \tag{4.3}$$

For the M/D/1 model, a recursive formula exists to compute $\phi(u)$.

As pointed out by Bohmam (1971) the computation of asymptotic non-ruin probabilities is now easy to do even with a common desk computer.

5. SOME NUMERICAL RESULTS IN THE TRANSIENT CASE

We will restrict ourselves to three models already treated in the literature:

Model A or M/M/1 model (see e.g., Seal (1974), Stroeymeyt (1977))

$$\lambda = 1$$

$$B(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

$$\eta = 0.1.$$

Model B or M/D/1 model (see e.g., Seal (1974))

$$\lambda = 1$$

$$B(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}$$

$$\eta = 0.$$

Model C (see e.g., Stroeymeyt (1977))

$$\lambda = 2$$

$$B(x) = \begin{cases} 0 & x < 0 \\ 1 - 0.8\, e^{-0.7x} - 0.2\, e^{-x} & x \geq 0 \end{cases}$$

$$\eta = 0.037234.$$

These models do not give rise to special computational difficulties, they are useful to test some approximations and bounds, and to test different methods.

5.1. *The Accuracy of the Laplace Inversion Methods*

To test the precision of the Laplace inversion methods, we give in Table 1 the real values of the non-ruin probability computed by means of a Bessel modified function for the model A (Column 1.i). The same values are computed by the

TABLE 1

MODEL A VALUES OF $\phi(0, T)$

T	(1 1)	(1 2)	(1 3)
0 1	0.90965	0 90321	0.89887
0 2	0 83561	0 82978	0 82586
0 3	0 77429	0 76905	0 76547
0 4	0 72295	0 71817	0 71497
0.5	0 67952	0.67527	0.67230
0.6	0.64242	0.63589	0 63589
0 7	0 61043	0 60688	0 60453
0 8	0.58260	0.57942	0 57726
0 9	0.55819	0 55530	0 55335
1 0	0.53660	0 53400	0 53223
2 0	0 40714	0 40621	0.40554
3 0	0 34479	0 34442	0.34421
4 0	0 30669	0.30656	0.30649
5 0	0 28040	0 28035	0 28034
6 0	0 26088	0.26086	0.26086
7 0	0 24566	0 24566	0 24566
8.0	0 23337	0 23337	0 23337
9 0	0 22319	0 22319	0.22319
10 0	0 21457	0 21457	0 21457
100	0.11001	0 11001	0.11002
200	0.09902	0.09902	0 09897

(1.1) Direct computation
(1 2) Stroud and Secrest method
(1 3) Piessens method

Stroud and Secrest method (Column 1.2) and by the Piessens method (Column 1.3) for the model A, for different values of t and for $u = 0$.

To obtain those values, the Prabhu–Beňes–Seal relations (3.7) and (3.8), were used. It can be pointed out that the non-ruin probabilities obtained by Laplace inversion are quite similar to the non-ruin probabilities "directly" computed, except for small values of t.

In Table 2, we give the non-ruin probabilities for the model C obtained by the Stroud and Secrest method (2.1) and by the Piessens method (2.2), for $u = 0$. Here also, it can be remarked that those methods give nearly the same values except for small values of t.

5.2. *Approximations of $F(x, t)$ by Means of Normal Power Approximation and Γ-function*

The form of the Prabhu–Beňes–Seal relations suggests that an approximation of $F(x, t)$ can provide a good approximation of the non-ruin probability. But those approximations of $F(x, t)$ are only valid for large t, and thus they bring a lot of imprecision in the integral

$$\int_0^t f(c\theta + u, \theta)\, \phi(0, t-\theta)\, d\theta \quad \text{in (3.7).}$$

TABLE 2

MODEL C VALUES OF $\phi(0, t)$

t	(2 1)	(2 2)
0 1	0.82524	0.82914
0 2	0 71305	0 71638
0.3	0 63429	0 63511
0 4	0 57240	0 57458
0.5	0.52601	0.52779
0.6	0 48913	0 49057
0.7	0 45904	0 46020
0 8	0 43396	0.43489
0 9	0.41267	0.41347
1	0 39432	0 39496
2	0 29016	0.29023
3	0 24180	0 24180
4	0 21252	0 21251
5	0 19239	0 19239
6	0 17748	0 17748
7	0.16586	0 16586
8	0 15648	0.15648
9	0.14871	0 14871
10	0 14213	0 14213

(2 1) Stroud and Secrest method
(2 2) Piessens method

However, some of those methods will provide an acceptable approximation of $\phi(0, t)$, when t is not too small.

Bohman and Esscher (1963) and Cramér (1955) give approximations of $F(x, t)$ in terms of $\Phi(x)$. the reduced normal distribution function. Normal Power approximations are proposed by Pesonen (1975) and by Taylor (1978). A Γ-function was also proposed by Seal (1978).

In our examples, the best method to calculate $\phi(0, t)$ seems to be the Normal Power approximation from Taylor (1978).

Table 3 contains some values of $\phi(0, t)$ and of this approximation for the M/M/1 model. The method of Taylor consisting of an approach of $\phi(u, t)$ by means of $\phi(0, t)+(1-\phi(0, t))\cdot G(w, t)$ involves some numerical complications: for one certain value of the security loading, η, negative numbers are obtained for a variance. Furthermore, this method occasionally involves some surprising results: an approximation for $\phi(1, 10)$ is smaller than the approximation for $\phi(1, 100)$. Taylor thinks that the consideration of higher order moments could give more accuracy but, of course, this will lead to complications from the numerical point of view.

5.3. *The De Vylder Approximation*

De Vylder (1978) proposed to approach the asymptotic non-ruin probability of a M/G/1 model by non-ruin probability of a M/M/1 model with such parameters

TABLE 3

	Model A					Model C			
T	(1)	(2)	(3)	(4)	T	(1)	(2)	(3)	(4)
0.1	0.46003	1.55294	0.90136	0.90965	0.1	0.44025	1.20985	0.85731	0.82914
0.2	0.44370	1.19918	0.86913	0.83561	0.2	0.41618	0.93522	0.75184	0.71638
0.3	0.43134	1.03439	0.81417	0.77429	0.3	0.39816	0.80241	0.65527	0.63511
0.4	0.42104	0.93164	0.75267	0.72295	0.4	0.38333	0.71731	0.58645	0.57458
0.5	0.41208	0.85861	0.70029	0.67952	0.5	0.37057	0.65557	0.53537	0.52779
0.6	0.40408	0.80266	0.65686	0.64242	0.6	0.35930	0.60755	0.49576	0.49057
0.7	0.39681	0.75765	0.62056	0.61043	0.7	0.34918	0.56851	0.46396	0.46020
0.8	0.39013	0.72021	0.58981	0.58260	0.8	0.33998	0.53578	0.43772	0.43489
0.9	0.38394	0.68828	0.56340	0.55819	0.9	0.33154	0.50774	0.41561	0.41347
1.0	0.37814	0.66052	0.54043	0.53660	1.0	0.32373	0.48330	0.39665	0.39496
2.0	0.33420	0.49466	0.40750	0.40714	2.0	0.26745	0.33895	0.29046	0.29023
3.0	0.30422	0.41030	0.34484	0.34479	3.0	0.23245	0.27019	0.24185	0.24180
4.0	0.28162	0.35665	0.30668	0.30669	4.0	0.20804	0.22980	0.21253	0.21251
5.0	0.26372	0.31910	0.28038	0.28040	5.0	0.18993	0.20341	0.19239	0.19239
6.0	0.24910	0.29131	0.26086	0.26088	6.0	0.17590	0.18488	0.17748	0.17748
7.0	0.23690	0.26995	0.24564	0.24566	7.0	0.16470	0.17112	0.16586	0.16586
8.0	0.22655	0.25306	0.23335	0.23337	8.0	0.15553	0.16045	0.15648	0.15648
9.0	0.21764	0.23941	0.22317	0.22319	9.0	0.14785	0.15187	0.14871	0.14871
10.0	0.20989	0.22815	0.21455	0.21457	10.0	0.14133	0.14477	0.14213	0.14213
20.0	0.16577	0.17309	0.16815	0.16816	20.0	0.10583	0.10785	0.10648	0.10649
40.0	0.13453	0.13913	0.13621	0.13621	40.0	0.08092	0.08233	0.08143	0.08143

(1) Normal-Power Approximations of $\phi(0, t)$ (two terms)
(2) Normal-Power Approximations of $\phi(0, t)$ (one term)
(3) G. C Taylor Approximation of $\phi(0, t)$.
(4) $\phi(0, t)$.

that the two reserve processes $R(t)$ have the same first moments. De Vylder emphasized the fact that the initial reserve must be large and supposed that this approximation can also be used for transient probabilities. In Table 4, we compare some results of this approximation for the model B and the model C. If this approximation is not good for small values of u, this very simple method gives acceptable values for important values of u ($u = 10$).

5.4. *Some Easily Computable Bounds in Transient Case*

We found it interesting to examine some easily computable bounds, to test approximations or calculations by means of Laplace inversion and to eliminate some aberrant results.

(1) *Gerber Minoration*: Gerber (1973) gives a minoration based upon martingales. It can be improved for the M/M/1 model. This minoration cannot be used with a null initial reserve except for the M/M/1 model. For the M/M/1 normed model, the Gerber minoration takes the following form:

$$\phi(u, t) \leq 1 - \min_{(c-1)/c < r < 1} (1-r) \exp\left(-ru - crt + t\frac{r}{1-r}\right).$$

TABLE 4

Model B

	$u=0$			$u=1$			$u=2$	
t	(1)	(2)	t	(1)	(2)	t	(1)	(2)
1	0.73576	0.26218	1	0.91970	0.75288	1	0.98101	0.93381
2	0.60901	0.18674	2	0.83457	0.61680	2	0.94171	0.84879
3	0.53106	0.15284	3	0.76548	0.53420	3	0.89866	0.77767
4	0.47697	0.13252	4	0.70988	0.47763	4	0.85758	0.72010
5	0.43662	0.11861	5	0.66437	0.43584	5	0.82000	0.67297
6	0.40503	0.10833	6	0.62638	0.40337	6	0.78607	0.63369
7	0.37944	0.10032	7	0.59411	0.37721	7	0.75552	0.60040
8	0.35815	0.09387	8	0.56630	0.35554	8	0.72796	0.57176
9	0.34008	0.08852	9	0.54201	0.33722	9	0.70300	0.54680
10	0.32450	0.08399	10	0.52057	0.32147	10	0.68031	0.52481

	$u=3$			$u=4$			$u=5$			$u=6$	
t	(1)	(2)	t	(1)	(2)	t	(1)	(2)	t	(1)	(2)
1	0.99634	0.98491	1	0.99941	0.99695	1	0.99992	0.99944	1	0.99999	0.99991
2	0.98231	0.94840	2	0.99528	0.98438	2	0.99888	0.99573	2	0.99976	0.99893
3	0.96124	0.90623	3	0.98669	0.96449	3	0.99586	0.98776	3	0.99883	0.99612
4	0.93698	0.86513	4	0.97461	0.94094	4	0.99063	0.97627	4	0.99681	0.99118
5	0.91181	0.82717	5	0.96024	0.91613	5	0.98342	0.96237	5	0.99358	0.98428
6	0.88695	0.79275	6	0.94455	0.89139	6	0.97466	0.94701	6	0.98917	0.97581
7	0.86298	0.76168	7	0.92822	0.86741	7	0.96475	0.93091	7	0.98372	0.96615
8	0.84016	0.73363	8	0.91171	0.84449	8	0.95405	0.91453	8	0.97741	0.95564
9	0.81859	0.70822	9	0.89533	0.82278	9	0.94283	0.89822	9	0.97039	0.94458
10	0.79827	0.68513	10	0.87925	0.80230	10	0.93132	0.88216	10	0.96283	0.93319

	$u=7$			$u=8$			$u=9$			$u=10$	
t	(1)	(2)	t	(1)	(2)	t	(1)	(2)	t	(1)	(2)
1	1	0.99999	1	1	1	1	1	1	1	1	1
2	0.99995	0.99975	2	0.99999	0.99995	2	1	0.99999	2	1	1
3	0.99969	0.99886	3	0.99993	0.99968	3	0.99998	0.99992	3	1	0.99998
4	0.99899	0.99694	4	0.99970	0.99901	4	0.99992	0.99969	4	0.99998	0.99991
5	0.99768	0.99386	5	0.99921	0.99774	5	0.99975	0.99921	5	0.99993	0.99974
6	0.99566	0.98962	6	0.99836	0.99579	6	0.99942	0.99838	6	0.99980	0.99941
7	0.99291	0.98434	7	0.99708	0.99314	7	0.99886	0.99714	7	0.99958	0.99887
8	0.98948	0.97819	8	0.99535	0.98981	8	0.99805	0.99546	8	0.99922	0.99807
9	0.98543	0.97132	9	0.99317	0.98586	9	0.99695	0.99334	9	0.99870	0.99700
10	0.98082	0.96389	10	0.99055	0.98136	10	0.99555	0.99078	10	0.99799	0.99563

TABLE 4 (*continued*)

		Model C			
t	(1)	(2)	*t*	(1)	(2)
1	0 99164	0 99166	21	0 66608	0.66583
2	0.97316	0.97313	22	0.65748	0 65724
3	0.95019	0.95009	23	0.64930	0 64905
4	0.92596	0.92581	24	0 64150	0 64125
5	0 90206	0.90187	25	0.63405	0.63380
6	0.87917	0.87895	26	0.62693	0.62668
7	0 85758	0 85734	27	0.62012	0 61987
8	0.83735	0.83711	28	0 61359	0 61334
9	0 81847	0 81821	29	0 60733	0 60708
10	0 80084	0.80059	30	0 60132	0 60106
11	0 78440	0 78413	31	0 59554	0 59528
12	0 76902	0 76876	32	0 58998	0.58972
13	0 75463	0.75437	33	0 58463	0 58437
14	0 74115	0 74089	34	0 57947	0.57921
15	0.72848	0.72821	35	0 57450	0.57423
16	0 71655	0.71629	36	0 56970	0 56942
17	0 70531	0 70505	37	0.56506	0 56478
18	0 69469	0 69443	38	0.56058	0.56029
19	0.68464	0 68439	39	0 55624	0 55595
20	0 67512	0 67487	40	0.55204	0.55174

(1) $\phi(10, t)$.
(2) De Vylder approximation of $\phi(10, t)$

Taking the derivative, it can be easily proved that the minimum is attained for

$$\rho = 1 - \frac{\sqrt{1+4(u+ct)t}-1}{2(u+ct)}.$$

(2) *Gerber Majoration*: when the initial reserve is null, Gerber (1979) gives a majoration of $\phi(0, t)$

$$\phi(0, t) \leq \left(1 - \frac{\lambda\beta}{c}\right) + \frac{1}{ct}\,\frac{\lambda\sigma^2}{c - \lambda\beta}.$$

(3) *Beekman–Bowers Minoration*: Beekman and Bowers (1972) proposed a very simple minoration of $\phi(u, t)$

$$1 - \frac{\alpha_2 t}{u^2} \leq \phi(u, t).$$

Of course, for large values of t, this minoration becomes negative.

(4) Bounds based upon the asymptotic non-ruin probability: The asymptotic non-ruin probability is generally easy to compute: either explicit formula exist or only one Laplace inversion provides it. With these probabilities, it is possible to construct bounds for small values of t, bearing in mind that, especially in this case, different values were observed for Laplace inversion (see Delfosse 1980).

(4a) Minoration:

$$\frac{\phi(u)}{\phi(u+ct)} \leq \phi(u, t).$$

TABLE 5

BOUNDS AND APPROXIMATIONS DESCRIBED IN 5 4

Model A $u = 0$

t	(1)	(4a)	$\phi(0, t)$	(4b)	(2)
0 1	0 77724	0.90950	0.90965	0 90992	1
0 2	0 62707	0 83472	0 83561	0 83727	↑
0 3	0 52256	0 77188	0 77429	0 77857	
0.4	0.44734	0 71834	0.72295	0 73076	
0 5	0.39140	0 67218	0.67952	0 69139	
0 6	0 34857	0 63197	0.64242	0 65855	
0 7	0.31491	0.59664	0 61043	0.63079	
0 8	0 28788	0.56534	0.58260	0.60699	
0 9	0 26576	0 53744	0 55819	0 58635	
1 0	0 24736	0 51239	0 53660	0 56822	
2.0	0 15722	0.35553	0.40714	0.45858	
3.0	0 12549	0.27841	0 34479	0 40224	
4 0	0 11015	0 23273	0 30669	0 36567	
5 0	0.10166	0.20265	0 28040	0.33926	
6 0	0 09666	0 18143	0.26088	0.31894	
7 0	0.09369	0.16572	0 24566	0 30263	
8 0	0 09199	0.15369	0.23337	0.28915	
9.0	0.09115	0 14421	0 22319	0 27775	↓
10	0 09091	0 13659	0.21457	0.26794	1
100	0.09091	0 09091	0 11001	0.13061	0.18181
200	0 09091	0 09091	0.09902	0 11145	0 13636

Model A $u = 10$

t	(4a)	(3)	(1)	$\phi(10, t)$	(4b)
0.1	0.99423	0 99800	0.99998	0 99999	0 99999
0 2	0 98869	0 99600	0 99994	0 99998	0.99998
0 3	0.98321	0 99400	0.99988	0.99997	0 99997
0 4	0 97784	0 99200	0 99980	0 99995	0 99995
0 5	0 97259	0 99000	0.99969	0 99992	0.99993
0.6	0.96744	0 98800	0 99956	0.99989	0.99990
0 7	0 96420	0 98600	0.99941	0.99985	0.99987
0.8	0 95746	0 98400	0.99923	0 99980	0 99983
0 9	0.95261	0 98200	0 99902	0.99975	0.99979
1	0.94787	0 98000	0 99879	0 99969	0.99974
2	0 90517	0 96000	0.99488	0.99865	0 99895
3	0.86972	0 94000	0 98833	0 99677	0.99757
4	0.83996	0 92000	0.97965	0 99410	0 99566
5	0 81473	0.90000	0 96942	0 99077	0 99332
6	0.79317	0 88000	0.95816	0 98689	0 99061
7	0 77463	0 86000	0 94629	0.98258	0.98761
8	0 75858	0 84000	0 93413	0.97796	0.98440
9	0 74462	0 82000	0 92190	0 97311	0.98103
10	0 73242	0.80000	0 90978	0 96810	0 97754
100	0 63375	0	0 63374	0.73947	0 78760
200	0.63374	0	0 67334	0 68217	0 72116

(4b) Majoration.

$$\phi(u,t) \leq \phi(u) \cdot F(u+ct,t) \cdot \frac{1}{(\phi * F(x,t)|_{x=u+ct})} \quad \text{(when } t \leq u/c\text{)}.$$

In Table 5, we present these bounds for the model A for $u = 0$ and for $u = 10$; in Table 6, the same bounds for the model C, for $u = 10$.

TABLE 6

BOUNDS AND APPROXIMATIONS DESCRIBED IN 5 4

		Model C	$u = 10$		
t	(4a)	(3)	(2)	$\phi(10, t)$	(4b)
1	0 83201	0 89871	0 93398	0 99164	0 99346
2	0 71966	0.79741	0 85465	0 97316	0 98010
3	0.63941	0 69612	0 78205	0 95019	0 96382
4	0 57937	0.59482	0 71969	0 92596	0 94665
5	0 53287	0 49353	0 66688	0 90206	0.92956
6	0.49588	0 39223	0 62208	0 87917	0 91300
7	0 46582	0 29094	0 58387	0 85758	0 89715
8	0.44098	0.18964	0 55101	0 83735	0 88210
9	0 42016	0 08835	0 52254	0 81847	0.86784
10	0 40250	0	0 49767	0 80084	0 85436
20	0.31268	0	0 35744	0 67512	0 75213
30	0 28239	0	0.29932	0 60132	0 68634
40	0 26982	0	0 26935	0 55204	0 63950

COMMENTS

These bounds are rather crude for certain values, but a package of these majorations and minorations does not take much computer time and allows to eliminate some inexact values. Our minoration $\phi(u)/\phi(u+ct)$ shows that $\phi(0, 0.1)$ and $\phi(0, 0.2)$ are too small in Model A and in Model C; for those values, the obtained non-ruin probabilities were the most different.

These minorations and majorations are also interesting to limit the use of precise but time-consuming methods: those bounds can be used to restrain the area of possible computations, if we allow some parameters of the model to vary. For example, the calculation of the bounds (4a), (4b) takes 16 times less calculation time than the computation of an exact value by the Laplace inversion.

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