INDEX-LINKED ANNUITIES AGAIN

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1 INTRODUCTION

I presented a paper in 1988 to the IACA Conference in Munich and to the International Congress of Actuaries in Helsinki (Wilkie, 1988) in which I described the use of option pricing methodology to value certain types of annuity, now common in the United Kingdom, for which the amount of each payment depended in some way on the value of a Retail Prices Index (RPI), but with upper and lower limits, which I described as having a "cap and collar" guarantee.

Since then I have found that it is easier to explain the method of valuation, and to use it, by using expected value methodology, rather than option pricing methodology. On suitable assumptions, the two methods give the same answers.

The purpose of this paper is to describe the expected value approach, show that it gives the same results as the previous method, and demonstrate some further applications of the method. In order to do this a small amount of repetition of the previous paper is necessary.

I used the term "cap and collar" guarantee to describe any payment whose actual amount, Z, say, is dependent on some fluctuating market rate, X, say, but between limits A and B so that

$$Z = \text{Max}(A, \text{Min}(X,B)),$$

ie
$$Z = \begin{cases} A & \text{if } X < A \\ X & \text{if } A \leq X \leq B \\ B & \text{if } B < X. \end{cases}$$

Some authors use the term "floor" to describe the lower limit, the whole forming a "collar".

Legislation about pensions in the United Kingdom had introduced two forms of "cap and collar" benefit which pension schemes are obliged to pay to beneficiaries in prescribed circumstances. The first example related to deferred pensions, provided where a member of a pension scheme had left the service of his employer, and it was required that his deferred pension should be partially indexed to the RPI, over the period from leaving service to the retirement date.

The second example related to certain pensions provided by pension schemes, in consequence of the Social Security Act 1986, as a way of "contracting-out" of part of the State Scheme, which are required to be increased each year in line with the change in the RPI, but by no more than 3% in each year, and, since the word "increased" is used, by no less than 0%. Each year is taken on its own and there is no provision for catching up or pulling back any differences between the RPI and the current level of pension.

This type of pension has been extended by the Social Security Act 1990, which provides that in certain circumstances pensions in payment provided by pension schemes will have to be increased each year in line with the RPI, but by no more than 5% nor less than 0% in any one year. This is known as Limited Price Indexation (LPI). Because of various uncertainties about pensions in the United Kingdom, the requirement for LPI has not yet come into force.

The proposed introduction of LPI has meant that the managers of pension schemes have explored alternative forms of giving pension increases, linked to the RPI, but with different levels of upper limit. Examples of these are discussed below.

2 LIMITED PRICE INDEXATION

I shall first discuss pensions with a straightforward LPI formula. The following discussion is derived directly from the Munich/Helsinki paper.

Let us first assume that pensions are payable yearly. Let the money amount of pension in year t be X(t), and the value of the RPI at that time be Q(t). Let the change in the RPI in the year (t, t+1) be $J(t+1) = Q(t+1)/Q(t)$. Successive values of X are related by the formula

$$X(t+1) = j(t+1)\,X(t)$$

$$\text{where } j(t+1) = \begin{cases} 1.0 & \text{if } J(t+1) < 1.0 \\ J(t+1) & \text{if } 1.0 \leq J(t+1) \leq 1+c \\ 1+c & \text{if } 1+c < J(t+1), \end{cases}$$

where c represents the upper limit of pension increase, e.g. $c = 0.05$ if pension increases are limited to 5% (in the Munich/Helsinki paper I used $c = 0.03$ throughout). In the Munich/Helsinki paper I used the option pricing method to value such a benefit. I assumed that at time t, $X(t) = 1$ and $Q(t) = 1$. The payment at time t+1 will be j(t+1). I shall show below how, on suitable assumptions, the expected value of j(t+1) can be calculated. Let us denote this expected value by E(t+1).

For this type of pension, the increases each year are independent of the progression of the RPI in previous years. (This is not necessarily the case for all types of index-linked annuity with guarantees and limits, see Wilkie, 1984.) At the end of one year the amount of pension is X(t+1), and the increase in the following year, from t+1 to t+2, denoted j(t+2), is related to the increase in the RPI over that year, J(t+2), in the same way as j(t+1) was related to J(t+1). The expected value of the increase can be calculated in the same way, and will be denoted E(t+2). Provided that our assumptions from year to year are the same (and they need not be, as will be shown below), then the expected rates of pension increase are the same from year to year, being all equal to say $E = E(t+1) = E(t+2) \dots$.

The expected value of such an annuity can therefore be calculated as if the annuity consisted of a series of payments of 1, E, E^2, E^3, ... If these are valued using a force of interest for fixed money payments of δ, so that the one year discounting factor $v = e^{-\delta}$, then the product Ev plays the role of a discounting factor in the usual way, equivalent to what is denoted W in the Munich/Helsinki paper. How do we calculate the value of E? This depends on our assumptions about the way in which the RPI changes.

In the Munich/Helsinki paper I described how options on the RPI were analogous to currency options, as described by Garman and Kohlhagen (1982). It was assumed that the exchange rate, equivalent to Q(t), followed the diffusion process

$$d \ln Q(t) = \mu(t,Q(t))\, dt + \sigma\, dz$$

where the mean rate of change μ, could be a function of t and Q(t), but the standard deviation of the diffusion process, σ, was constant. I also explained in the Munich/Helsinki paper that I had shown elsewhere (Wilkie, 1986) that the movement in the RPI in the UK could be defined by a first order autoregressive process

$$I(t) = \mu + \alpha(I(t-1) - \mu) + \epsilon(t)$$

where I(t), the force of inflation in the year (t−1,t), is defined as

$$I(t) = \ln Q(t) - \ln Q(t-1),$$

μ and α are constants, with values of about 0.05 and 0.6 respectively, and $\epsilon(t)$ is normally distributed with zero mean and standard deviation $\sigma = 0.05$. Although this process applies to the RPI at annual intervals, it is analogous with the continuous diffusion process stated above, and the value of σ can be taken as the same. The autoregressive part of the discrete process falls into the $\mu(t,Q(t))$ term.

Actuaries responsible for valuing pension schemes in the United Kingdom do not, in general, use this type of process to describe the future progress of inflation. Instead they assume a uniform rate of inflation, of, e.g. 5%, 6%, 7% per year, and use that assumption as if it were a guaranteed fixed rate. An assumption is also made about the valuation rate of interest, which is a rate of interest suitable for valuing fixed money payments (rather than index-linked payments), and no specific assumption is made about the yield on index-linked securities (although they exist in the UK).

In the Munich/Helsinki paper I defined the option values in terms, inter alia, of the fixed force of interest, δ, on money payments, and the fixed force of interest, η, earned on index-linked stock. I now wish to drop the η parameter, and replace it by an assumed mean rate of inflation.

For the time being, we shall ignore the autoregressive part of the difference equation for inflation given above, or, equivalently, assume that the mean of the diffusion process is a constant, equal to μ per year, and consider the distribution of $J(t+1) = Q(t+1)/Q(t)$. Since $\ln J(t+1)$ is distributed normally, with mean μ and variance σ^2, $J(t+1)$ is distributed lognormally with parameters μ and σ.

It is convenient now to introduce a notation to represent limited expected values. Let the limited expected value of a general function g(.) of a general random variable X, from a to b, be denoted

$$E(g(X);a,b) = \int_a^b g(x).f_X(x)dx,$$

where $f_X(x)$ is the density function of X.

We can now derive the expected value of $j(t+1)$, defined above in relation to $J(t+1)$. It can be expressed as

$$\begin{aligned} E(t+1) &= E[j(t+1)] \\ &= E(1;0,1.0) + E(J(t+1);1.0,1+c) + E(1+c;1+c,\infty) \end{aligned}$$

It is shown in the Appendix that if X is distributed lognormally, with parameters μ and σ, then

$$E(X^r;a,b) = m_r\{N(g(b) - r\sigma) - N(g(a) - r\sigma)\}$$

where $m_r = \exp(r\mu + \tfrac{1}{2}r^2\sigma^2)$, $g(x) = (\ln x - \mu)/\sigma$, and N(.) is the distribution function of the normal (Gaussian) distribution.

Hence

$$\begin{aligned} E(t+1) = N(g(1.0)) &+ m_1\{N(g(1+c) - \sigma) - N(g(1.0) - \sigma)\} \\ &+ (1+c)\{1 - N(g(1+c))\} \end{aligned}$$

where $m_1 = \exp(\mu + \tfrac{1}{2}\sigma^2)$.

Now if $\delta - \eta = \mu + \tfrac{1}{2}\sigma^2$ so that $e^\delta = m_1e^\eta$, or the expected return on fixed interest stock is equal to the expected return on index linked stock times the expected "return" on the RPI in the year, then a little manipulation shows that

$$E(t+1).e^{-\delta} = W = F + G,$$

where W, F and G are as defined in the Munich/Helsinki paper, namely

$$F = e^{-\delta} N(f_1) + (1+c)\, e^{-\delta} N(d_2)$$

$$G = e^{-\eta} (1 - N(d_1) - N(f_2))$$

$$d_1 = \ln(e^{-\eta}/(1+c)e^{-\delta})/\sigma + \sigma/2$$

$$d_2 = \ln(e^{-\eta}/(1+c)e^{-\delta})/\sigma - \sigma/2$$

$$f_1 = \ln(e^{-\delta}/e^{-\eta})/\sigma + \sigma/2$$

$$f_2 = \ln(e^{-\delta}/e^{-\eta})/\sigma - \sigma/2.$$

An advantage of the original formulation is that it shows the way in which W should be partitioned into a fixed money part, F, and an index-linked part, G, which, on the assumption of continuous hedging, creates a risk-free hedge.

Apart from the theoretical and practical difficulties of a system of continuous hedging, it is unlikely that the actuary carrying out the valuation of a pension fund will have any control over the actual investment policy of the pension fund. At the best, he may be asked to advise on what the appropriately matching investments might be.

It is, therefore, useful to be able to use this new formulation to estimate the uncertainty in the rate of pension increase, $j(t+1)$, to be granted. Higher moments can readily be calculated, for example

$$\begin{aligned} E[j(t+1)^2] &= E(1;0,1.0) + E(J(t+1)^2;1.0,1+c) + E((1+c)^2;1+c,\infty) \\ &= N(g(1.0)) + m_2\{N(g(1+c) - 2\sigma) - N(g(1.0) - 2\sigma)\} \\ &\quad + (1+c)^2\{1 - N(g(1+c)\} \end{aligned}$$

where $m_2 = \exp(2\mu + 2\sigma^2)$.

From this the variance and standard deviation can readily be derived. Higher moments, if desired, can be calculated on the same lines.

A further advantage of the new formulation is that the values of E depend only on μ and σ, whereas in the original formulation the value of W depended on δ, η and σ. A complete grid of results can therefore be presented in a single table, as in Table 1. Instead of using the median force of inflation, μ, as one of the parameters, it is more convenient to use the mean rate of inflation, expressed as a percentage,

$$m = 100(\exp(\mu + \tfrac{1}{2}\sigma^2) - 1)\%$$

whence

$$\mu = \ln(1 + m/100) - \tfrac{1}{2}\sigma^2.$$

Instead of showing the value of E, it is convenient to show the mean rate of pension increase, also expressed as a percentage;

$$e = 100(E - 1)\%.$$

The values of p, the proportion to be invested in index-linked, and the standard deviation of the rate of increase, s, are also shown in the table.

For Table 1, the value of c is taken as 0.05, i.e. pension increases are limited to 5% per annum. It is not difficult to calculate a similar table for other values of the cap.

A suitable value to take for σ is somewhere between 0.04 and 0.05. This results from the analysis of inflation rates in my own stochastic investment model (see Wilkie, 1986). But it can also be thought of in the following way: one may be reasonably confident about what the rate of inflation might be over the coming year, but what about inflation in, say, five or ten years time? If one chooses a range within which one is 95% certain that the rate of inflation will lie in some year ten years ahead from the date of calculation, and treat this as a range from -2σ to $+2\sigma$, then one can calculate σ as one quarter of the size of the range.

My own, subjective, views on this are that the range in the UK might be from -4% to $+20\%$, giving a range of 24%, or a value of σ of 0.06.

3 ALTERNATIVE FORMS OF LPI

It is not difficult to use the same method for alternative forms of limited price indexation. Consider, for example, the following question, posed in examination style.

A pension fund proposes to grant pension increases on the following basis: fully indexed to the RPI up to a 6% increase, 75% of any increase over 6% and up to 10%, and 50% of any increase over 10%, with no decreases. Derive formulae for the mean and second moment of the annual rate of increase of pensions, and calculate the mean and standard deviation of that rate assuming that the increase in the RPI each year is lognormally distributed with mean rate of increase 6%, standard deviation (i.e. 100σ) 5% and independence from year to year.

Answer: We first specify the actual pension increase each year, $j(t+1)$, in terms of the increase in the RPI each year, $J(t+1)$, viz:

$$j(t+1) = \begin{cases} 1.0 & \text{if } J(t+1) < 1.0 \\ J(t+1) & \text{if } 1.0 \le J(t+1) \le 1.06 \\ 0.265 + 0.75J(t+1) & \text{if } 1.06 < J(t+1) \le 1.10 \\ 0.54 + 0.5J(t+1) & \text{if } 1.10 < J(t+1). \end{cases}$$

TABLE 1

Rate of increase of pension subject to Limited Price Indexation

Lower limit of increase each year is 0% and upper limit is 5%

m is mean annual rate of inflation
σ is standard deviation of annual force of inflation
e is mean annual rate of increase of pension
s is standard deviation of rate of increase of pension
p is proportion to be invested in index-linked assets for matching

σ	m:	3.0%	4.0%	5.0%	6.0%	7.0%	8.0%
0.03	e %	2.77	3.32	3.81	4.21	4.50	4.71
	s %	1.91	1.81	1.62	1.37	1.10	0.84
	p %	57.71	52.99	44.87	35.11	25.41	17.03
0.04	e %	2.69	3.14	3.55	3.90	4.20	4.44
	s %	2.07	2.00	1.87	1.69	1.48	1.26
	p %	45.49	43.18	38.99	33.54	27.52	21.56
0.05	e %	2.64	3.01	3.35	3.67	3.95	4.19
	s %	2.16	2.11	2.02	1.89	1.74	1.56
	p %	37.28	36.04	33.69	30.48	26.73	22.73
0.06	e %	2.60	2.91	3.21	3.49	3.74	3.97
	s %	2.22	2.19	2.12	2.03	1.91	1.77
	p %	31.49	30.76	29.35	27.36	24.93	22.24
0.07	e %	2.57	2.84	3.10	3.35	3.58	3.79
	s %	2.26	2.24	2.19	2.12	2.03	1.92
	p %	27.21	26.77	25.86	24.56	22.94	21.08

The mean value of j(t+1), E, is given by the formula

$$E = E(1.0;0,1.0) + E(J;1.0,1.06) + E(0.265+0.75J;1.06,1.10) \\ + E(0.54+0.5J;1.10,\infty),$$

in which J denotes J(t+1).

The mean rate of increase of pensions, e, is given by $e = 100(E - 1)\%$.

After substitution and simplification, we obtain

$$E = 0.54 - 0.275N(g(1.10)) - 0.265N(g(1.06)) + N(g(1.0)) \\ + m_1\{0.5 + 0.25N(g(1.10) - \sigma) + 0.25N(g(1.06) - \sigma) \\ - N(g(1.0) - \sigma)\},$$

where $m_1 = 1.06$, $\sigma = 0.05$, $\mu = \ln m_1 - ½\sigma^2 = 0.057019$ and $g(x) = (\ln x - \mu)/\sigma$.

The second moment of j(t+1), E_2, is given by the formula

$$E_2 = E(1.0^2;0,1.0) + E(J^2;1.0,1.06) + E((0.265+0.75J)^2;1.06,1,10) \\ + E((0.54+0.5J)^2;1.10,\infty)$$

which after substitution and simplification can be expressed as

$$E_2 = 0.2916 - 0.221375N(g(1.10)) - 0.070225N(g(1.06)) + N(g(1.0)) \\ + m_1\{0.54 - 0.1425N(g(1.10) - \sigma) - 0.3975N(g(1.06) - \sigma)\} \\ + m_2\{0.25 + 0.3125N(g(1.10) - 2\sigma) + 0.4375N(g(1.06) - 2\sigma) \\ - N(g(1.0) - 2\sigma)\},$$

where $m_2 = \exp(2\mu + 2\sigma^2) = 1.126413$.

After further substitution and calculation we find that the mean rate of increase, e, is 5.60%, and the standard deviation

$$s = \sqrt{(E_2 - E^2)} = 0.0377 \text{ or } 3.77\%.$$

For manual calculation a practical difficulty is that it is hard to obtain tables of the normal distribution function which are sufficiently accurate. This is usually not a problem for computer calculation, within which accurate values are usually readily available. For example, the FORTRAN language has a built-in function ERF(x), which is related to the normal distribution function by

$$N(x) = ½(1 + ERF(x/\sqrt{2}))$$

for $x > 0$.

The calculations shown above look a little cumbersome, but they are easy to program. Rather than attempt any simplification, it is easier to deal with each tranche separately.

The standard deviation of the rate of increase, calculated for the example above, or for the standard cases in Table 1, can be compared with the standard deviation of the annual rate of increase of the unrestricted index-linked annuity. For a mean rate of increase of the RPI of 6% and a value of σ of 0.05, the unrestricted annuity has a standard deviation of 5.30%, as compared with 1.89% for the corresponding entry in Table 1 and 3.77% calculated for the example above. This shows the extent to which the limitation on the annual rate of indexation reduces the uncertainty about its value.

This calculated standard deviation, however, does not immediately allow the standard deviation of the present value of a pension increasing according to one of these formulae to be calculated. This calculation is not considered here.

4 VARYING THE MEAN INFLATION RATE

The discussion so far has been in terms of a constant mean rate of increase of the RPI. This seems to be the only reasonable assumption in the long term, but it is possible that the actuary valuing pensions of this type has views about what the rate of inflation will be over the next year or two. A conventional valuation assumption might be on the following lines: if inflation has been rather high recently, it will be, say, 12% next year, 9% the year after and 6% for each year after that. It is easy to incorporate these specific assumptions into the calculations for pensions with LPI, using the formulae above, and incorporating the mean rate of increase each year into the valuation.

If the autoregressive model for inflation, described in Wilkie (1986), and referred to above, is used, then the mean rate of inflation in each successive year depends on the rates of inflation actually experienced in the preceding year, moving exponentially towards the mean rate. Thus, if we take the time "now" to be 0, and the mean force of inflation in the preceding year to be I_0, then the mean force of inflation in each successive year t = 1, 2, 3, ... is given by

$$E[I_t] = \mu + \alpha^t(I_0 - \mu).$$

Although this tends exponentially back to μ, with a value for α of about 0.6 the convergence is quite fast, and it might only be necessary to use a specific value for the first few years, with a constant rate thereafter.

The existence of the autoregressive term complicates the calculation of the standard deviation of the present value of an LPI pension. This, too, is not considered here.

5 DEFERRED PENSIONS

I now turn to the first example of the Munich/Helsinki paper. This related to deferred pensions which pension schemes must provide for those who have left the service of the employer under certain conditions. The deferred pension must be increased over the period from leaving service to the retirement date by the proportionate increase of the RPI over the same period, but with a limit of 5% per annum compound, and with no decreases.

It can be seen that in principle this is the same problem as discussed above, over a longer timescale. We are concerned here with valuing only the amount of pension due at retirement in relation to the amount due on leaving service, taking into account changes in the RPI over the deferred period. The valuation of the pension in payment then depends on how it is to be increased from year to year.

Assume that the amount of pension calculated as at the date of leaving service, and as if there were no further inflation, is 1, and that time is measured from the date of leaving service. Assume that the value of the RPI at this date is also 1.

Denote the amount of pension due on retirement in n years time by R(n). Denote the value of the RPI at time t by Q(t). Denote the limiting factor by c, so that a 5% upper limit, as used in the UK, corresponds to c = 0.05.

Then

$$R(n) = \text{Max}\,(1, \text{Min}\,(Q(n), C(n)))$$

where $C(n) = (1+c)^n$.

Note that I have altered the notation from that used in the Munich/Helsinki paper. The value of R(n) is therefore given by

$$R(n) = \begin{cases} 1.0 & \text{if } Q(n) < 1.0 \\ Q(n) & \text{if } 1.0 \le Q(n) \le C(n) \\ C(n) & \text{if } C(n) < Q(n) \end{cases}$$

Assuming now that Q(n) is lognormally distributed with parameters M = M(n) and $\Sigma = \Sigma(n)$ (corresponding to μ and σ over one year) then the expected value of C(n), denoted E, is given by

$$\begin{aligned} E &= E[R(n)] \quad = E(1;0,1.0) + E(Q(n);1.0,C(n)) + E(C(n);\, C(n),\infty) \\ &= N(g(1.0)) + m_1(N(g(c(N)) - \Sigma) - N(g(1.0) - \Sigma)) \\ &\quad + C(n)(1 - N(g(C(n)))) \end{aligned}$$

where $g(x) = (\ln x - M)/\Sigma$ and $m_1 = \exp(M + \tfrac{1}{2}\Sigma^2)$

What should be used for M(n) and $\Sigma(n)$? If we assume that the changes in the RPI from year to year are independent, and are all lognormally distributed with parameters μ and σ, then over n years we can put $M(n) = n\mu$ and $\Sigma(n) = \sigma\sqrt{n}$. This is consistent with the diffusion process on which the option pricing method is based, with the assumption of a constant mean, and it gives the same answers as in the Munich/Helsinki paper if appropriate parameter values are chosen.

However, if we assume that the autoregressive model described above applies, we get a different result.

We note that I(t) represents the force of inflation in year (t-1,t), and that it follows the autoregressive model

$$I(t) = \mu + \alpha(I(t-1) - \mu) + \epsilon(t),$$

where $\epsilon(t)$ is distributed normally with zero mean and standard deviation σ.

Then

$$\ln Q(n) = \ln Q(0) + I(1) + I(2) + \ldots + I(n)$$

Substitution and manipulation gives us

$$M(n) = E[\ln Q(n) - \ln Q(0)] = n\mu + \alpha(1 - \alpha^n)(I_0 - \mu)/(1 - \alpha)$$

and

$$\begin{aligned}\Sigma(n)^2 &= \mathrm{Var}[\ln Q(n) - \ln Q(0)] \\ &= \sigma^2[n - 2\alpha(1 - \alpha^n)/(1 - \alpha) + \alpha^2(1 - \alpha^{2n})/(1 - \alpha^2)]/(1 - \alpha)^2\end{aligned}$$

If $I_0 = \mu$, so that inflation at the starting point is already at its mean value, then $M(n) = n\mu$, as under our previous assumption. Similarly, if $\alpha = 0$, then $\Sigma(n)^2 = n\sigma^2$, as before.

However, if α is non-zero, then we can express the standard deviation over n years as

$$\Sigma(n) = \sigma\sqrt{n}.k(n;\alpha),$$

where the value of $k(n;\alpha)$ depends only on n and α. A value of 0.6 for α is roughly appropriate to the United Kingdom over the past 70 years (see Wilkie, 1986). Values of $k(n;\alpha)$ for $\alpha = 0.5$, 0.6 and 0.7 for selected years are as shown in Table 2.

As n increases, so $k(n;\alpha)$ tends towards a limiting value of $1/(1-\alpha)$. This limiting value is also shown in the table.

A higher standard deviation generally results in lower average values of the increase in the pension over the deferred period. However, before relying on this effect in order to place a lower value on deferred pensions, one would wish to be sure that the autoregressive model was likely to remain valid for the future. Alternative models of inflation are possible, and have been suggested (see, for example, Clarkson, 1991). Some alternative models may result in lognormal distributions of the RPI in future years; in these cases the methodology of this paper is applicable, with appropriate parameter values being chosen. Other models for changes in the RPI may not result in a lognormal distribution, in which case alternative methods specific to that model need to be investigated.

After a deferred pension of the sort under consideration has been running for a number of years, say t years, the actual value of the RPI, Q(t), will become known. The outstanding term, originally n, will have reduced to $n-t$. The same methodology can be used, except that we are now interested in the increase in the RPI over the outstanding term, and the limits, originally 1.0 and C(n), have altered to 1/Q(t) and C(n)/Q(t). No new principles are involved.

TABLE 2

Values of $k(n;\alpha)$ for selected values of n and α

Term, n (years)	α=0.5	α=0.6	α=0.7
1	1.00	1.00	1.00
2	1.27	1.33	1.39
3	1.45	1.57	1.70
4	1.57	1.74	1.94
5	1.65	1.87	2.14
10	1.83	2.18	2.67
15	1.89	2.29	2.90
20	1.91	2.34	3.01
25	1.93	2.38	3.08
30	1.94	2.40	3.12
35	1.95	2.41	3.15
40	1.96	2.42	3.18
∞	2.00	2.50	3.33

APPENDIX

X is distributed lognormally, with parameters μ and σ, if Y = ln X is distributed normally (μ, σ^2).

X = exp Y.

The density function of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\ \sigma} \exp\left[\frac{-(y-\mu)^2}{2\sigma^2}\right]$$

The density function of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\ \sigma x} \exp\left[\frac{-(\ln x-\mu)^2}{2\sigma^2}\right]$$

Then

$$\begin{aligned} E(X^r;a,b) &= \int_a^b x^r f_X(x)dx \\ &= \int_a^b x^r \frac{1}{\sqrt{2\pi}\ \sigma x} \exp\left[\frac{-(\ln x-\mu)^2}{2\sigma^2}\right] dx \\ &= \int_{a'}^{b} \frac{1}{\sqrt{2\pi}\ \sigma} \exp(yr) \exp\left[\frac{-(y-\mu)^2}{2\sigma^2}\right] dy \\ &= \int_a^b \frac{1}{\sqrt{2\pi}\ \sigma} \exp\left[\frac{-(y-\mu-r\sigma^2)^2}{2\sigma^2}\right] \exp(r\mu + \tfrac{1}{2}r^2\sigma^2)dy \\ &= \exp(r\mu + \tfrac{1}{2}r^2\sigma^2) \left[N(\frac{(\ln b-\mu)}{\sigma} - r\sigma) - N(\frac{(\ln a-\mu)}{\sigma} - r\sigma)\right] \end{aligned}$$

where N(.) is the distribution function for the normal (Gaussian) distribution.

Putting $m_r = E(X^r;0,\infty) = \exp(r\mu + \tfrac{1}{2}r^2\sigma^2)$,

and $g(x) = (\ln x - \mu)/\sigma$, we get

$$E(X^r;a,b) = m_r\{N(g(b) - r\sigma) - N(g(a) - r\sigma)\}.$$

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SUMMARY

In a paper to the Munich Conference and the Helsinki Congress in 1988 the author described a method of using option pricing methodology for valuing pensions, both in payment and deferred, subject to increases in line with the Retail Prices Index, but subject to various limits. In this paper an alternative approach is explored, in which the expected rate of increase of the pension is calculated on the assumption that changes in the RPI are lognormally distributed. Under suitable assumptions the results are the same as those given by the option pricing method, but they seem to be more direct, more compatible with the usual methods of valuation used by actuaries, and therefore more intelligible to them.

In addition, the new approach makes it easier to calculate higher moments of the rates of pensions increase, and easier to calculate the values of pensions subject to alternative forms of limitation on the increase.