The Use of Option Pricing Theory for Valuing Benefits with "cap and collar" Guarantees

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(A slightly altered version of a paper with the same title to be presented to the 23rd Congress of the International Actuarial Association, Helsinki 1988)

The term "cap and collar" guarantee derives from practice in the mortage loan market in the United Kingdom. Typically such loans carry rates of interest that fluctuate with short-term market conditions. Recently lenders have offered loans whose rates of interest fluctuate as usual, but within the first few years are guaranteed not to rise above a certain rate (the cap), nor to fall below a certain rate (the collar). I therefore wish to use the term to describe any payment whose actual amount, Z, say, is dependent on some fluctuating market rate, X, say, but between limits A and B so that

$$Z = \text{Max}\ (A, \text{Min}\ (X,B)),$$

ie.

$$Z = \begin{cases} A & \text{if } X<A, \\ X & \text{if } A\leq X\leq B, \\ B & \text{if } B<X. \end{cases}$$

Legislation about pensions in the United Kingdom has introduced two forms of "cap and collar" benefit which pension schemes are obliged to pay to beneficiaries in prescribed circumstances. I shall describe these, and show how they can be valued using certain principles from option pricing theory.

Deferred Pensions

The first of these benefits was introduced by the Social Security Act 1985. Earlier legislation had provided that, where a member of a pension scheme left the service of his employer, and where certain other conditions were fulfilled, he must be provided with a preserved deferred pension, payable from a specified retirement age. The 1985 Act required that, where a member left service after 1 January 1986, the deferred pension pertaining to service after 1 January 1985 must be increased over the period from leaving service to the retirement date by the proportionate increase of the Retail Prices Index over the same period but with a limit of 5% per annum compound.

For the purpose of discussion I ignore the practical details of the timing of the increases, and I assume that the value of the Retail Prices Index (RPI), denoted by $Q(t)$, is available for all t. I assume that the member leaves service at time 0, and is due to retire n years later, so the pension is deferred n years. I assume that $Q(0) = 1$, and I put

$$R(n) = 1.05^n$$

Then a pension of 1 at leaving must be revalued to the lesser of $Q(n)$ and $R(n)$. The amount would not be reduced if the value of the RPI fell, so the amount of 1 deferred n years is given by

$$C(n) = \mathrm{Max}(1, \mathrm{Min}(Q(n), R(n))).$$

Note that the revaluation takes place over the whole period, not year by year. The maximum value of $C(n)$ is $R(n)$ and the minimum is 1.

How should one value C(n)? An easy assumption, particularly when inflation has recently been well above 5%, is to put C(n) = R(n), ie. to assume that the 5% limit always applies. If one assumes that inflation will always be at or above 5% then this would be valid. But in reality future inflation is uncertain, and it has been below 5% in the U.K. in the last few years, and also in the more distant past.

There is more than one way of allowing for the fluctuation in the inflation of retail prices. The option pricing method I shall describe relies on a feature peculiar to the U.K. bond market. The government issues two kinds of stock: conventional fixed money stock, where the interest payments and redemption amounts are fixed in pound sterling terms; and, since 1981, index-linked stock where the interest payments and redemption amounts are linked to the value of the RPI at the time of payment (strictly the value of the RPI about eight months before the payment date, but we can ignore this detail). There is neither a cap nor a collar on the index-linked payments. Both kinds of stock are freely traded, so it is possible to establish rates of interest applicable to each kind for an appropriate term. The two kinds of stock define two currencies; pounds sterling which I shall assume earn a fixed force of interest, δ; and RPI units, which I shall assume earn a fixed force of interest, η.

Let us look at the benefit from the point of view of the deferred pensioner. He left the scheme at time 0, say, with a deferred benefit of 1 unit, say (let us take this as a lump sum of 1; we can multiply by an appropriate factor to turn it into pension later). But this is one unit of the RPI currency. If there were no

cap and no collar he would be due to receive from the scheme one unit of RPI, whose amount in pounds at time n would be Q(n), and whose amount at some intermediate date, t, is Q(t). Since a unit of RPI earns interest at force η, the present value of one unit due in T years time is $e^{-\eta T}$.

But the pensioner will not actually receive one unit of RPI. At the bottom end, if Q(n) is less than a lower limit, say A(=1), then the pensioner has effectively a "put option" provided for him by the scheme. He can require the scheme to take back the RPI unit and give him A for it. This is a put option on the RPI currency at an exercise price of A, exercisable only at time n.

At the other end, if Q(n) exceeds some upper limit B(= R(n)), then the pensioner has effectively written a call option for the scheme. The scheme has the right to take back the RPI unit and pay the pensioner B for it.

The liability of the scheme is equal to the pensioner's asset and is:

one RPI unit,
plus one put option at exercise price A,
minus one call option at exercise price B.

Since the uniform force of interest on fixed money investments is taken to be δ, the present values of the exercise prices at a point T years before the exercise date are $Ae^{-\delta T}$ and $Be^{-\delta T}$.

The options are analogous to currency options. Garman and Kohlhagen (1982) have shown that these can be valued by a formula similar to the Black-Scholes

formula for valuing options on shares. The exchange rate, equivalent to $Q(t)$, is assumed to follow the diffusion process:

$$d \log Q(t) = \mu(t, Q(t))\, dt + \sigma\, dz$$

where the mean rate of change, μ, can be a function of t and $Q(t)$, but the standard deviation of the diffusion process, σ, is constant.

The value of the call option with exercise price B, at time $T = n-t$ before the exercise date, when the RPI has value $Q = Q(t)$, and δ and η are the forces of interest already defined is:

$$Q\, e^{-\eta T}\, N(d_1) - B\, e^{-\delta T}\, N(d_2),$$

$$\text{where } d_1 = \frac{\log\,(Qe^{-\eta T}/Be^{-\delta T})}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = \frac{\log\,(Qe^{-\eta T}/Be^{-\delta T})}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

and $N(\cdot)$ is the normal probability integral.

The value of the put option with exercise price A is:

$$A\, e^{-\delta T}\, N(f_1) - Q\, e^{-\eta T}\, N(f_2)$$

$$\text{where } f_1 = \frac{\log\,(Ae^{-\delta T}/Qe^{-\eta T})}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$f_2 = \frac{\log\,(Ae^{-\delta T}/Qe^{-\eta T})}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

Hence the value of the whole liability is:

$$W = Q\, e^{-\eta T}\, (1 - N(d_1) - N(f_2))$$

$$+ A\, e^{-\delta T}\, N(f_1) + B\, e^{-\delta T}\, N(d_2).$$

Note that the value of $\mu(t,Q(t))$ does not enter into the formula. This could be explained by assuming that the values of δ and η are consistent in some way with the expected force of inflation, but such an explanation is not in fact necessary for the derivation of the formula, which is based on the assumption that the liability can be exactly "matched" or "hedged" by the portfolio consisting of

$$G = Q\, e^{-\eta T}\, (1 - N(d_1) - N(f_2))$$

invested in index-linked securities and

$$F = A\, e^{-\delta T}\, N(f_1) + B\, e^{-\delta T}\, N(d_2)$$

invested in fixed money securities. Provided that this hedge portfolio can be maintained at all times then the mean rate of inflation does not matter. In fact such a hedge portfolio cannot be changed continuously and costlessly, but an approximate hedge presumably could be maintained.

I have shown elsewhere (Wilkie, 1986) that the movement in the retail prices index in the U.K. can be defined by a first order autoregressive process:

$$D \log Q(t) = \mu + \alpha\, (D \log Q(t-1) - \mu) + \varepsilon(t)$$

where $D \log Q(t) = \log Q(t) - \log Q(t-1)$,

μ and α are constants, with values of about 0.05 and 0.6 respectively, and $\varepsilon(t)$ is normally distributed with zero mean and standard deviation, $\sigma = 0.05$. This process applies to the RPI at annual intervals, but it is analogous with the continuous diffusion process described above, and the value of σ can be taken as the same. The autoregressive part of the discrete process falls into the $\mu(t,Q(t))$ term.

In order to calculate the value of the liability, W, at the date of leaving service, we need to put $A = 1$, $B = R(n) = 1.05^n$, $T = n$, $Q = Q(0) = 1$, and choose values for δ, η and σ. It is convenient to calculate the value of W, the values of the amounts in the hedge portfolio, G and F, and to show the percentage of investments required in index-linked, say, which is $p = 100\ G/W$. It is also convenient to express W as a discount factor at an equivalent rate of interest, j, where $j = 100\ (1/W^{1/n} - 1)$ per cent.

Table 1 shows these values, with $\delta = \log(1.085)$, $\eta = \log(1.03)$, $\sigma = 0.05$. The simple assumption that inflation will always be higher than 5%, so that the upper limit would always apply, would produce an equivalent rate of interest of 3.333%, (1.085 = 1.05 x 1.0333). It can be seen that this would always overvalue the benefit, since the value of j always exceeds this. So also would the assumption that there was neither cap nor collar, so that the benefit was a pure index-linked benefit, when the appropriate interest rate would be 3%.

Table 1

Value of deferred benefit of 1 at leaving service

Deferment period n	Value per unit W	Equivalent rate of interest j%	Proportion of hedge in index-linked p%
1	0.9536	4.87	32.7
5	0.8174	4.11	43.5
10	0.6852	3.85	42.1
15	0.5768	3.74	40.5
20	0.4866	3.67	39.2
25	0.4111	3.62	38.0
30	0.3475	3.59	37.0
35	0.2940	3.56	36.0
40	0.2488	3.54	35.2

After the member has left service for some time, say t years, the value of Q will have altered to Q(t). We can use the same formula for W, keeping A = 1, B = R(n) = 1.05^n, where n is the original term, Q(t) is the current value of the RPI, T = n-t, and δ, η and σ are chosen as before. Table 2 shows the same values as before, for t = 5 years, for an original benefit of 1, with Q(5) = 1.02^5, 1.05^5 and 1.08^5 respectively, ie assuming inflation has averaged 2%, 5% and 8% respectively over the five years since leaving service. In this case the equivalent rate of interest, j, is taken as:

$$j = 100\ ((Q(t)/W)^{1/T} - 1).$$

ie it is the equivalent rate to use for valuing the benefit revalued in line with Q(t).

When we consider first the cases with one year to go, it is easy to see that when inflation has been low, as in section A of the table, the chance of reaching the 5% limit is small and the benefit is valued almost as fully index-linked, with a discount rate just above 3%, and almost the entire hedge portfolio in index-linked. When inflation has been very high, as in section C of the table, it is almost certain that the 5% limit will apply, the benefit is valued almost as $1.05^6 / 1.085 = 1.2351$, and almost all the hedge portfolio is in fixed interest stock. When inflation has been intermediate, at 5%, the position is also intermediate. As the term to go lengthens the three examples come rather closer together, though the proportions in the hedge portfolio still remain fairly far apart.

Pensions in payment

My second example also derives from U.K. legislation. The Social Security Act 1986 requires that certain pensions provided by pension schemes as a way of "contracting-out" of part of the state scheme shall be increased each year in line with the change in the RPI, but by no more than 3% in each year, and (since the word "increased" is used) by no less than 0%. Each year is taken on its own and there is no provision for catching up or pulling back any differences between the RPI and the current level of pension. Such pensions are similar to my Type 4 index-linked annuity (with a cap as well as a collar) as described in Wilkie (1984).

Let us first assume that pensions are payable yearly. Let the money amount of pension in year t be X(t), and the value of the RPI at that time be Q(t). Let the

change in the RPI in the year (t,t+1) be R(t+1) = Q(t+1)/Q(t). Successive values of X are related by the formula:

$$X(t+1) = r(t+1)\ X(t)$$

where

$$r(t+1) = \begin{cases} 1.0 & \text{if } R(t+1) < 1.0 \\ R(t+1) & \text{if } 1.0 \le R(t+1) \le 1.03 \\ 1.03 & \text{if } 1.03 < R(t+1). \end{cases}$$

We can again use the option pricing method to value such a benefit. Assume that at time t, X(t) = 1 and Q(t) = 1. The payment at time t+1 will be r(t+1). The pensioner, if he survives, has the right to receive an amount of R(t+1), but he has given the pension scheme (or insurance company) the option to repurchase the R(t+1) for 1.03, if it wishes, and he has himself the right to demand 1.0 instead of R(t+1) if he wishes.

The pensioner's assets equal the scheme's liability and are:

one unit of RPI, with a current price of 1.0,
plus one put option on the RPI at an exercise price of 1.0,
minus one call option on the RPI at an exercise price of 1.03.

The same formula as before can be used to value this, with the simplification that T = 1. It splits into an index-linked part

$$G = e^{-\eta}\ (1 - N(d_1) - N(f_2))$$

and a fixed money part

$$F = e^{-\delta}\ N(f_1) + 1.03\ e^{-\delta}\ N(d_2)$$

which total

$$W = F + G$$

where

$$d_1 = \log(e^{-\eta}/1.03e^{-\delta})/\sigma + \sigma/2.$$

$$d_2 = \log(e^{-\eta}/1.03e^{-\delta})/\sigma - \sigma/2$$

$$f_1 = \log(e^{-\delta}/e^{-\eta})/\sigma + \sigma/2$$

$$f_2 = \log(e^{-\delta}/e^{-\eta})/\sigma - \sigma/2.$$

This allows us to value one future payment, payable certainly. But we have to deal with a life annuity. Let us consider the position at time t, just after the payment of X(t) has been made. Let the present value of future annuity payments be a(t) per unit of payment now, or X(t)a(t) in total. This can be taken as comprising the payment due in one year's time, X(t+1), plus future annuity payments beyond that date, whose value is X(t+1)a(t+1). Let the probability that the annuity terminates within one year be q(t) and put p(t) = 1 - q(t).

Then:

$$X(t).a(t) = p(t).X(t+1)(1+a(t+1)) + q(t).0$$
$$= p(t).X(t).W(1+a(t+1))$$

Hence $\quad a(t) = p(t).W(1+a(t+1)).$

But this is the usual recurrence relation connecting successive annuity values, with W taking the place of the usual "v". Thus we can use W as if it were a one year discount factor, calculate an equivalent rate of interest, j = 100 (1/W - 1)%, and use that interest rate in the usual way.

The values of w depend only on δ, η and σ, and it is convenient to quote only the equivalent interest rate,

j, and the fraction of the hedge portfolio that should be invested in index-linked securities. It is also convenient to express δ and η in terms of equivalent rates of interest, i and k, where $\delta = \log(1+i/100)$ and $\eta = \log(1+k/100)$. Table 3 shows values for $\sigma = 0.04$, 0.05 and 0.06, for i = 6, 7, 8, 9 and 10% and for k = 2, 2.5, 3, 3.5 and 4%.

It can be seen that j is always between i and k, and is also always higher than 100((1+i/100)/1.03 - 1), ie. the rate that could be used to value an annuity increasing at a fixed 3% per annum. The equivalent rate j is higher, the higher the value of σ; but the proportion, p, to be invested in index-linked in the hedge portfolio is more stable with higher σ than with a lower σ.

Practical adjustments may be necessary to deal with annuities paid monthly, but whose amount is revised annually, or with annuities where the valuation date does not coincide with the revision date. I do not deal with these here.

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Table 2

Value of deferred benefit of 1, after 5 years

Outstanding term T = n-5	Value per unit W	Equivalent rate of interest j%	Proportion of hedge in index-linked p%
A: when inflation has been at 2%, Q(5) = 1.1041			
1	1.0719	3.00	99.6
5	0.9454	3.15	86.9
10	0.8023	3.24	75.4
20	0.5766	3.30	63.4
30	0.4145	3.32	56.5
B: when inflation has been at 5%, Q(5) = 1.2763			
1	1.2124	5.27	47.5
5	1.0429	4.12	44.4
10	0.8745	3.85	42.2
20	0.6211	3.67	39.2
30	0.4435	3.59	38.0
C: when inflation has been at 8%, Q(5) = 1.4693			
1	1.2351	18.97	0.2
5	1.0785	6.38	8.5
10	0.9085	4.92	14.5
20	0.6466	4.19	19.0
30	0.4617	3.93	20.6

Table 3

Interest rates for valuing annuity, j%
(and proportion to be invested in index-linked, p%)

k%	2	2.5	3	3.5	4
i%			A : σ = 0.04		
6	3.78(24.5)	3.91(26.1)	4.04(27.3)	4.19(28.2)	4.33(28.7)
7	4.54(21.0)	4.65(22.9)	4.77(24.6)	4.90(26.1)	5.04(27.3)
8	5.33(17.1)	5.42(19.1)	5.53(21.2)	5.64(23.1)	5.76(24.8)
9	6.16(13.2)	6.23(15.2)	6.32(17.3)	6.41(19.3)	6.51(21.3)
10	7.02(9.8)	7.08(11.6)	7.14(13.5)	7.21(15.5)	7.30(17.5)
			B : σ = 0.05		
6	3.91(21.0)	4.02(21.8)	4.13(22.5)	4.25(22.9)	4.36(23.2)
7	4.69(19.0)	4.79(20.1)	4.90(21.1)	5.09(21.9)	5.12(22.5)
8	5.50(16.6)	5.59(17.9)	5.68(19.1)	5.78(20.2)	5.89(21.1)
9	6.32(14.1)	6.40(15.4)	6.48(16.8)	6.57(18.0)	6.67(19.2)
10	7.17(11.6)	7.24(12.9)	7.31(14.3)	7.39(15.6)	7.47(16.9)
			C : σ = 0.06		
6	4.00(18.2)	4.10(18.7)	4.19(19.0)	4.29(19.3)	4.38(19.4)
7	4.81(16.9)	4.90(17.6)	4.99(18.2)	5.08(18.7)	5.18(19.0)
8	5.63(15.4)	5.71(16.3)	5.80(17.0)	5.89(17.7)	5.98(18.2)
9	6.46(13.8)	6.54(14.7)	6.62(15.5)	6.70(16.3)	6.78(17.1)
10	7.31(12.0)	7.38(13.0)	7.45(13.9)	7.52(14.8)	7.60(15.6)