INVERSE PROBLEMS IN MARKOV MODELS.

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Abstract.

This paper considers decision of inverse problem of Markov model corresponding to a special medical insurance scheme.

Two mutually opposite problems arise when using Markov processes for modeling. Direct problem is to calculate probabilities of corresponding states and other characteristics of process. It is assumed that parameters of model are available. Inverse problem is to evaluate parameters of model by using experimental output data.

In Markov model input parameters are forces of transition from state to state. When dealing with queuing problems or insurance models these values are unknown in advance. Meanwhile, statistical information about output results for some models can be found in special literature.

This paper considers the parameter estimation for Markov models in medical insurance.

Keywords

Inverse problem, Markov process, actuarial analysis, probability, Chapman-Kolmogorov equations.

A number of widely known applications use the approach allowing to apply the theory of Markov processes for modeling of a situation as a multi-state system. Examples can be the problems of queuing systems, advertising problem, reliability problem, and also different applications in biology, chemistry, physics etc. This approach has also been widely applied in actuarial practice, when the multi-state model is used for modeling of states of the insured.

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Let's consider some aspects of the actuarial analysis.

Actuarial techniques are the systems of numerical methods based on statistics and economics. They are used to calculate insurance premiums, evaluate reserves of insurance companies, price the individual's share in the company fund, for re-insurance arrangement, and developing the strategy for investing company's assets. The main actuarial characteristics are net-premium, insurance funds and cash flows.

One of the main difficulties in calculation of actuarial characteristics is estimation of the insured event occurrence probability.

The premiums and reserves for insurance contracts (especially long-term) and annuities are based on the present value of cash flows associated with policy. Usually occurrence, timing and (or) amount of each payment are not determined at outset. They depend on some random events. It is crucial to estimate the expectation of this present value, which depends on probability of certain event occurrence.

As stated above, many traditional problems of actuarial analysis can be considered in terms of multi-state processes. It is assumed that at any time the individual is in one of a number of states. The current state of the individual or movement (transition) from one state to another may have some financial impact. The task is to quantify this impact. That is, we need to estimate the probabilities of being in a particular state.

Markov process is a very convenient tool for calculation the occurrence probabilities of any events.

In actuarial calculations the simplest example involves only two states of the insured individual: "alive" and "dead". Corresponding scheme of transition is shown in Figure 1.

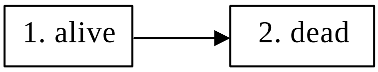

Figure 1.

Model of two states

The transition is possible only in one direction. For a simple life annuity benefits are payable, while the individual is in state 1, and cease upon transition in state 2. In case of a whole life insurance policy premiums are payable while the insured is to state 1, and the death benefit is paid at the time of the transition to state 2. There are simple and well-known approaches to calculation of actuarial values in these cases.

More complicated situation arises for processes with additional states. Figure 2 represents the scheme including three possible states: "active", "disabled" and "dead" commonly used in modeling disability insurance. In this case premiums are payable while the insured is in state 1, and benefits are payable while he is in state 2.

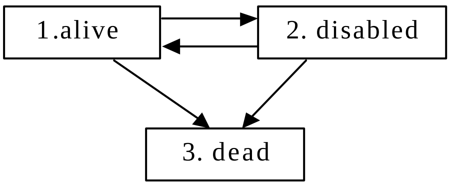

Figure 2.

Model with three states.

The actuarial calculations for this example are more complicated, because an individual can make movements from state 1 to state 2 and back more than ones.

Therefore it is often assumed that transition from state 2 to state 1 is impossible. If Markov condition holds, this problem is also easily solved.

However, Markov models are not always applicable for modeling in insurance. One of the difficulties one faces, is a large number of possible states in the system. For a life insurance the states are individual health conditions, which depend directly on insured's age. Forces of transition from state "active" to state "disable" at age 20-30 are obviously different from ages 50-60.

The errors in calculations can also arise because of large policy duration. For example, if the insurance policy is issued for 20 years, characteristics at the end of the period will be very different from those at the start.

One of the possible approaches in such situation is usage of the semi-Markov process scheme. Stochastic model, in which future of the process depends on a time of transition to current state, is known as semi-Markov model.

Presenting each state as a collection of two or more sub-states can do approximation of Markov model by semi-Markov. It means, that the age of an individual can be split into periods, for which it is possible to take transition intensities to be constant. Hence, the parameters of Markov model (forces of transition from state to state), will be piecewise constant functions.

Let us first consider a general kind of Markov process and its properties.

Let ? (t) denote the state of individual at age t (t>0). Let us denote stochastic process by {? (t), t>0}. Let us assume that there is a finite number of states: 1,2,...,n, i.e. the process has the state space {1,2, …, n}. Then {? (t), t>0} is a Markov process, if for anyone s, t>0 and i, j, x (u)∈ {1,2, … n},

$$\Pr\{X(s+t)=j \mid X(s)=i, X(u)=x(u), 0 \le u < s\} = \Pr\{X(s+t)=j \mid X(s)=i\}$$

Thus, the future of the process (after time s) depends only on the state at time s and does not depend on the history of the process up to time s. Applicability of the Markov assumption partially depends on the level of details in the states description. For example, consider the three-state process shown on Figure 2. In this case Markov assumption may be inappropriate. The future health of recently disabled individual obviously differs from health of someone of the same age, who has been disabled for a long time. The solution of this problem was considered above.

We define the transition probability function

$$p_{ij}(s,s+t) \equiv \Pr\{X(s+t)=j \mid X(s)=i\},\ i,j \in \{1,2,...n\},$$

and assume, that

$$\sum_{j=1}^{n} p_{ij}(s,s+t) = 1$$

for all t>0.

We also assume the existence of the limits

$$m_{ij}(t) = \lim_{h\to 0} \frac{p_{ij}(t,t+h)}{h},$$

i, j∈{1,2 … n}, i≠j.

At i≠j μ_{ij} is an intensity of transition from state i in state j. It can be easily seen that at s, t, u ? ≥0

$$p_{ij}(s,s+t+u) = \sum_{l=1}^{n} p_{il}(s,s+t)p_{lj}(s+t,s+t+u), \qquad (1)$$

i, j∈{1,2, … n}.

(1) is known as Chapman-Kolmogorov equations.

Transition probability functions are used in calculations of actuarial values. The forces of transition and the transition probability functions are related by the Chapman-Kolmogorov forward and backward equations

$$\frac{\partial}{\partial t} p_{ij}(s,s+t) = \sum_{l=1}^{n} p_{il}(s,s+t)m_{lj}(s+t) \qquad (2)$$

$$\frac{\partial}{\partial t} p_{ij}(s,s+t) = -\sum_{l=1}^{n} m_{li}(s)p_{lj}(s,s+t) \qquad (3)$$

respectively, with boundary conditions

$$p_{ij}(s,s) = d_{ij},$$

where

$$d_{ij} = \begin{cases} 1, & ????\ i=j, \\ 0 & ?????. \end{cases}$$

Generally, these systems of the differential equations can be solved numerically for deriving functions of the transition probability.

The explicit form of probability transition functions can be obtained when $\mu_{ij}(t)=\mu_{ij}$ for all t. Such Markovian process is uniform (homogeneous) in time, or stationary. An assumption that transition forces are constant implies that the time spent in each state has exponential distribution and the functions $p_{ij}(s,s+t)$ are identical for all s, so that they can be written simply as p_{ij}.

In case of step functions for transition forces, it is possible to divide the time interval of our research into intervals on which parameters are constant. In order to solutions (the probabilities functions) to be continuous, it is necessary to join the solutions from different intervals.

When the transition forces are known, the problem reduces to the straightforward task: solving Chapman-Kolmogorov equations. For models with constant intensities it is not so complicated irrespective of number of possible system states. However, if the transition forces are unknown, there is an inverse task, that is to estimate the transition forces from some statistical data.

The estimation of transition forces is a composite problem, especially if the task has a major dimension. So, if we are given n of states, it is possible to derive C_n^2 various couples (i, j), i, j=1..n, i≠j from them. For each couple (i, j) there are two corresponding transition intensities μ_{ij} and μ_{ji}. Thus, in total we need to get $2C_n^2=n(n-1)$ statistical estimates of transition forces being not parameters but rather functions of age x.

Therefore, this problem should be solved at the particular assumptions, namely:

1. Transition intensities μ_{ij} are approximated by step functions. For this purpose we shall divide the whole time period on the intervals such that transition forces are constant in each of them. Let (x_1, x_2) - one of such intervals, and $\mu_{ij}(x)=\mu_{ij}$ for all $x\in(x_1, x_2)$.
2. We are observing individuals with the actual age x over each corresponding interval.
3. Each individual corresponds to an independent realization of stochastic process.

4. The individual's behaviour does not depend on how the observation started and for what reason it stopped.

One more assumption, which would make a solution much simpler, is that we have the complete information about each observable individual's history. In [1] the model of many states is applied to modeling the process of disability insurance. The estimates of transition forces are carried out by the maximum likelihood method with the complete data – all individual's transition states and time spent in them are assumed known.

However, it is very rare when we have the complete data. In the problems of life insurance experimental information is likely to be represented in so-called mortality tables only. From them one can obtain the information on values of probability of death in a particular age for individuals of a particular group.

Mortality table is created on the basis of the following data:

Large enough group of people, for example 100,000 of identical age is observed over some time period. l_x denotes the number of individuals of age x, remaining alive at age x out of the original group of 100,000. Then, $d_x=l_x-l_{x+1}$ is the number of deaths aged x over one year. Then the probability that any observable individual dies within one year after attained age x is: $q_x=d_x/l_x$. Correspondingly, $p_x=1-q_x$ is the survival probability of the individual aged x during one year.

One of the most common methods of statistical estimation is the least squares method. The idea being is that the unknown parameters are estimated by minimizing the standard error vector. Let us consider an example of applying this approach to estimation of mortality of smokers and non-smokers. The statistical data for calculations was taken from the British Actuarial Journal [2]. The BAJ provides with mortality tables for four groups: males/females and smokers/non-smokers. Here is an example:

Mortality table for non-smokers (males).

Age x	l_x	d_x	q_x
25	96570	65	0.00067
26	96505	63	0.00066
27	96442	62	0.00065
28	96380	62	0.00065
29	96318	64	0.00066
30	96254	65	0.00067
31	96189	67	0.00070
32	96122	70	0.00072
33	96052	73	0.00076

34	95979	77	0.00080
35	95902	82	0.00086

The data for male smokers and non-smokers, age 25-35 was used for this research. The tables allow us to estimate mortality probabilities for any age using the above formulas. The model is the three-states process shown in Figure 3:

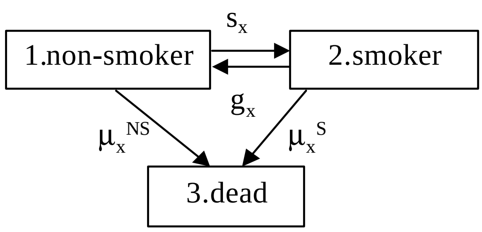

Figure 3.

Model for Smokers / Non-Smokers

The following assumptions correspond to this model:

1. There are three possible states. Each individual can be in any of them. Two "alive" states - 1 and 2, and the state "dead", which is an absorbing one.
2. It is possible for a non-smoker to start smoking, and then to quit, and people who smoked at the beginning of the investigation can give up as well.
3. Death can be caused by smoking or any other reason.
4. All forces of transition are assumed constant in the age interval: (25,35).

This model includes four unknown parameters: s_x , g_x, μ_x^{NS} , μ_x^{S} – transition intensities. Mortality tables give us the values of death probabilities for age x. In Figure 3 these are the probabilities of transition from state 1 into state 3, and from state 2 into state 3.

Chapman-Kolmogorov equations for this scheme are:

$$\begin{cases} \frac{d}{dt} p_{11} = p_{12} m_{21} + p_{11} m_{11} \\ \frac{d}{dt} p_{12} = p_{12} m_{22} + p_{11} m_{12} \\ \frac{d}{dt} p_{13} = p_{11} m_{13} + p_{12} m_{23} \\ \frac{d}{dt} p_{21} = p_{21} m_{11} + p_{22} m_{21} \\ \frac{d}{dt} p_{22} = p_{21} m_{12} + p_{22} m_{22} \\ \frac{d}{dt} p_{23} = p_{21} m_{13} + p_{22} m_{23} \end{cases}.$$

Norm condition:

$$\sum_{i,j} p_{ij} = 1.$$

The initial conditions can be derived as follows:

In the initial moment (t=0) the individual does not smoke, i.e. is in a state 1:

$$p_{11}(0) = 1,\ p_{12}(0) = 0,\ p_{13}(0) = 0,$$
$$p_{21}(0) = 1,\ p_{22}(0) = 0,\ p_{23}(0) = 0.$$

The least squares method gives the following numerical estimates of the transition intensities:

μ_{12} = 0,0164, μ_{21}= 0,0592, μ_{13}=0,00107, μ_{23}= 0,00921.

In the case considered parameters of the model will depend on age x. In the more general case they will depend on the duration of stay in the particular state (i.e. they will differ for individuals of the same age but with the different past information). This case is obviously more complicated and requires more data about observed individuals.

So, let us assume that the intensities of transition between the states differ for different ages, i.e. $\mu_{ij} \cong \mu_{ij}(?)$.

Individual's age x is a discrete number (1, 2, … years). It is assumed that during one life year transition intensities stay constant, i.e. $\mu_{ij}(?)$ functions are ???????-??????????.

If we consider an individual's age as a continuos parameter and not discrete, we can try to build "smooth" transition intensities. It is possible to calculate the values of the transition functions in the discrete points from a given data and then interpolate these points by a smooth enough curve.

We can consider the [0, 1] interval as a subclass on which we are looking for an approximate solution (this is justified by the prior considerations). The following numerical estimates of the transition intensities [2] were obtained by the trial and error method. They were approximated, and the following results were obtained:

$$\mu_{12} \in [0{,}000171;0{,}00036],$$

$$\mu_{21} \in [0{,}000121;0{,}00051],$$

$$\mu_{13} \in [0{,}000671;0{,}00086],$$

$$\mu_{23} \in [0{,}001180;0{,}00193].$$

Age *x*	P_{13_stat}	P_{13_calc}	P_{23_stat}	P_{23_calc}
25	0.00067	0.00060	0.00118	0.00111
26	0.00066	0.00067	0.00117	0.00120
27	0.00065	0.00071	0.00118	0.00123
28	0.00065	0.00072	0.00121	0.00131
29	0.00066	0.00078	0.00128	0.00132
30	0.00067	0.00081	0.00135	0.00141
31	0.00070	0.00083	0.00145	0.00150
32	0.00072	0.00090	0.00154	0.00159
33	0.00076	0.00092	0.00165	0.00171
34	0.00080	0.00094	0.00178	0.00192
35	0.00086	0.00095	0.00193	0.00201

As the above solution is not unique, we have to estimate the interval of values of μ_{ij}, in which we obtain a good approximation of the statistical data. The analysis of the corresponding Markov scheme and the set of equations shows that the substantial changes in μ_{13}, μ_{23} will cause the large changes in the values of p_{13}, p_{23}. The changes in μ_{12}, μ_{21} will influence these variables in the much smaller extend.

The numerical experiment gave the following results:

Large (of the 2 orders) increase and decrease of the parameter values for μ_{12} and μ_{21} causes insignificant changes in the calculated probabilities (of the order 1-3%), probabilities p_{12} and p_{21} change a lot, proportionally to changes of μ_{12} and μ_{21}. Large changes in μ_{13} and μ_{23} cause proportional changes of p_{13}, and p_{23}. Such dependence is explained by the characteristics of the model.

All the above allows to conclude that solving the inverse problem for the given model requires additional information apart from statistical data about mortality probabilities. Such information could be the data on duration of stay in the states 1 and 2 (fig.2). More precise solution could be found by estimating the probabilities p_{12} and p_{21} for various ages. The source of such information could be a study of smoker and non-smoker population.

There will be more than one solution of the inverse problem corresponding to this Markov scheme. This is typical for most of inverse tasks.

There are different methods of finding the intervals for estimates, that is selection of a subspace in the space of admissible solutions. Each point of such subspace satisfies a set of equations, initial conditions and system of limitations.

Thus, it is possible to assign different values from solutions subspace to the parameters and obtain equally good description of the statistical data. Such "variation" of parameters can be done arbitrarily or using some functional connections between parameters.

For example, for the three states process in Figure 4,

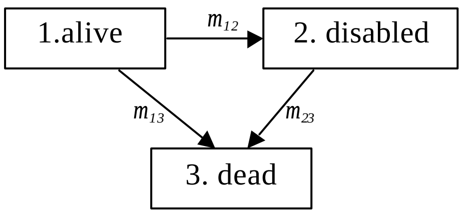

Figure 4.

the force of mortality for people of the given age group will depend on transition forces as follows:

$$m(t)=\frac{(m_{23}-m_{13})(m_{12}+m_{13})e^{-(m_{12}+m_{13})t}-m_{12}m_{23}e^{-m_{23}t}}{(m_{23}-m_{13})e^{-(m_{12}+m_{13})t}-m_{12}e^{-m_{23}t}}.$$

It is seen from here, that for any chosen set of values for three parameters, there is another choice, which gives exactly the same value for *m(t)*. If $m_{12} = \tilde{m}_{12}, m_{13} = \tilde{m}_{13}, m_{23} = \tilde{m}_{23}$, then the same *m(t)* can be obtained, assuming $\hat{m}_{12} = \tilde{m}_{23} - \tilde{m}_{13}, \hat{m}_{13} = \tilde{m}_{13}, ?\, \hat{m}_{23} = \tilde{m}_{12} + \tilde{m}_{13}$. In lack of the prior information about parameter values, we can choose any values. Our purpose is simply to find the best value of *m(t)*, based on the three-state model. Usually, there is data available not for all three transitions from Figure 4, but for transitions from states 1 and 2 into state 3 only.

Figure 5 is illustrate the intensity of change from state "smoking" to state "nonsmoking" – μ_{12}.

Figure 6 is illustrate the intensity of change from state "nonsmoking" to state "smoking" – μ_{21}.

Figure 7 is illustrate the intensity of change from state "nonsmoking" to state "death" – μ_{13}.

Figure 8 is illustrate the intensity of change from state "smoking" to state "death" – μ_{23}.

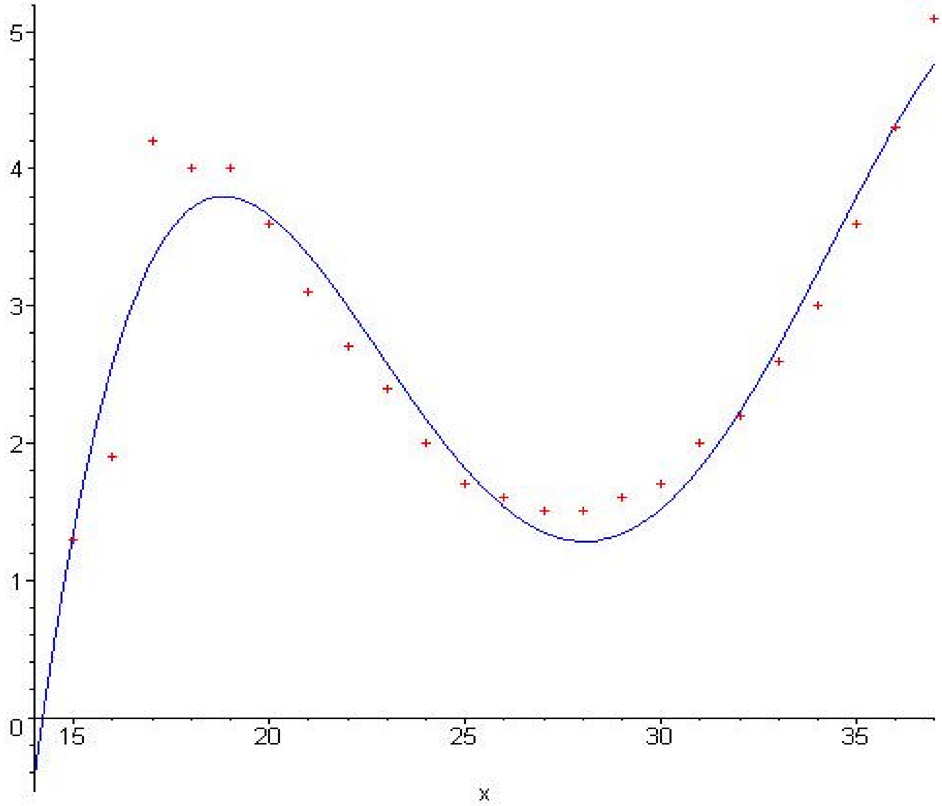

Figure 5

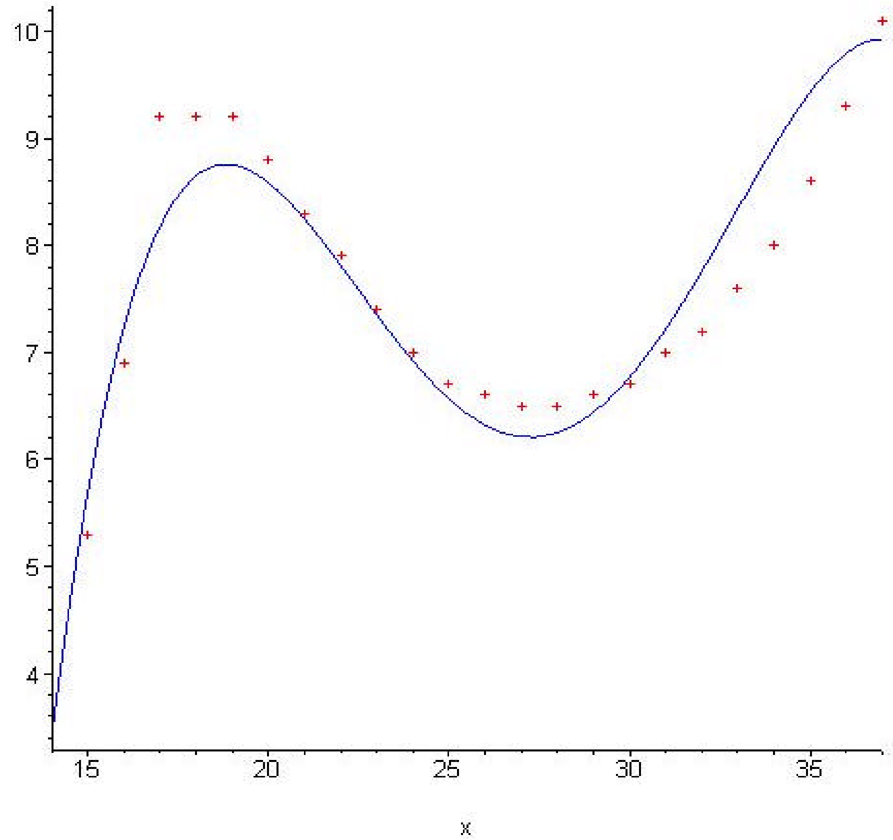

Figure 6.

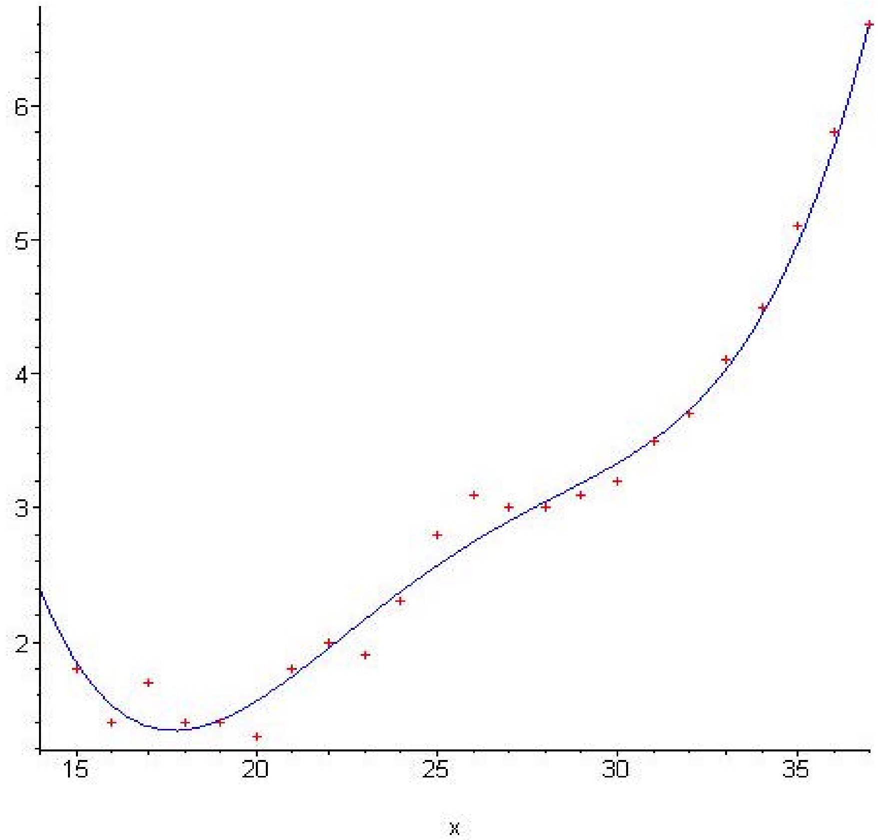

Figure 7.

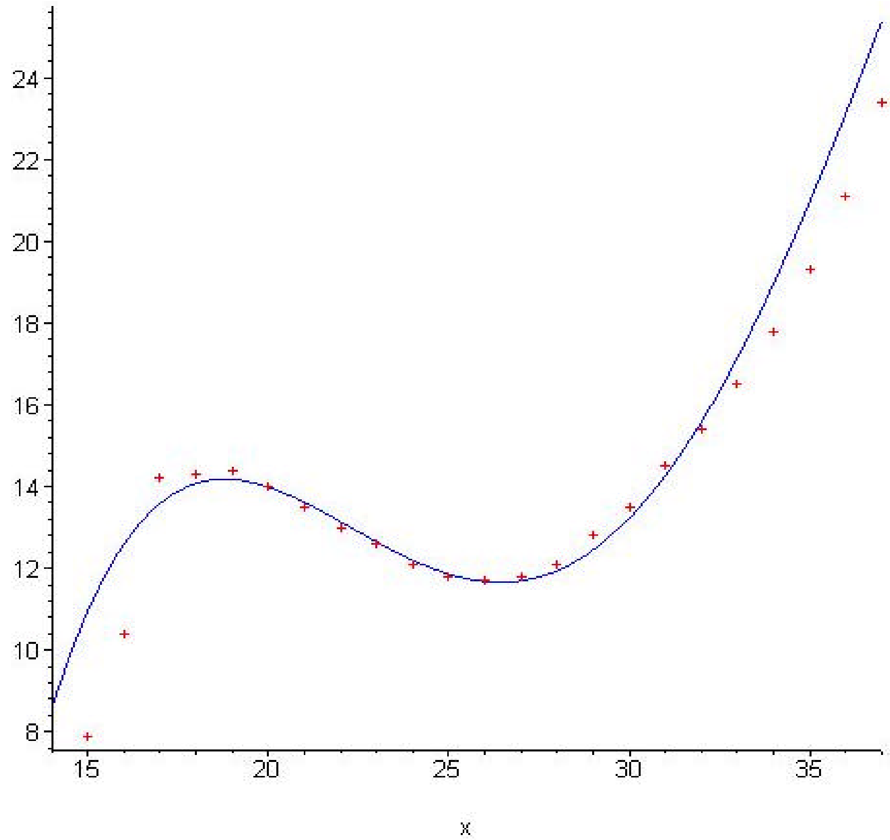

Figure 8.

The presence of ambiguity in a solution of the inverse task reduces leads to the following question: Is it possible to realize such a choice from the subspace of solutions, which whenever possible optimizes the properties of the given process?
This question will be the topic of our further researches.

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