Long Dated Life Insurance and Pension Contracts

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Summary

- We discuss the "life cycle model" by first introducing a credit market with only biometric risk, and then market risk is introduced via risky securities.

- This framework enables us to find optimal pension plans and life insurance contracts where the benefits are state dependent.

- We compare these solutions both to the ones of standard actuarial theory, and to policies offered in practice.

- The standard representation of preferences gives rise to certain puzzles in insurance:

- The intertemporal substitution of consumption elasticity (ϵ) needs to be larger than one in order for insurance customers to be interested in life and pension insurance.

- The standard model then gives a too low relative risk aversion $\gamma = 1/\epsilon$, which implies that the typical customer will self-insure: He will buy the stock index, and bypass the insurance market.

- In this talk I will introduce an alternative type of preferences that deals with this, and other problems with the standard model.

Outline

- Consumption and saving: No risk.

- Mortality included: Optimal consumption/pension and optimal life insurance contracts, when there is only a credit market.

- Financial model including risky securities: How are the optimal contracts affected by financial market risk?

- The separation of consumption substitution from risk aversion.

- Discussion and conclusions

Introduction

- Four or five decennials back life and pension insurance seemed less problematic than today, at least from the insurance companies' point of view.

- Pricing was done by actuaries using life tables, and a "fixed" calculation interest rate. This rate was not linked to the equilibrium interest (spot) rate of the market.

- The premium reserves of the individual and collective policies were invested in various assets.

- When different contracts were settled, the evolution of the premium reserve connected to the particular contracts gave rise to "bonus".

- In several countries the nominal interest rate was high during some of this period, and considerably higher than the fixed calculation rate used in premium calculations.

- In Norway, for example, more or less by an oversight, this calculation rate (4%) appeared as a guaranteed rate in the contracts.

- During the last two or three decennials this interest rate guarantee has become a major issue for many life insurance companies.

- What initially appeared to be a benefit with almost no value, has become rather valuable for the policy holders, and correspondingly problematic for the insurers.

- In this talk I investigate, using optimal demand theory, if such contracts have any place in the life cycle hypothesis.

- I do not find any evidence about that using the standard model. With recursive utility the picture changes somewhat.

- Taking into account optimal supply, i.e., Pareto optimal contracts, is likely to change this picture entirely.

- Every downturn in the financial market has typically been accompanied by nontrivial problems for the life insurance industry. Life insurance companies seem to prefer to offer defined contribution type policies.

- During the current financial crisis, casual observations seem to suggest that many individuals would rather prefer defined benefit policies.

- Collective pension plans organized by firms on behalf of their workers, are almost exclusively defined contribution plans these days, which appear to be the least costly of the two for the firms, and also the preferred choice to offer by the insurance companies.

- The talk is organized as follows: First we introduce consumption and saving with only a credit market available. Here we also explain some actuarial concepts related to mortality. In particular we study the effects from pooling.

- Next we include mortality risk, i.e., an uncertain planning horizon, in the model. In this setting we derive both optimal life insurance, not commonly studied, and optimal pension insurance, and investigate their properties when there is only a credit market present.

- Then we introduce a market for risky securities in addition to the credit market. Here we solve both the optimal investment and the pension/life insurance problem.

- We show that with pension insurance available, the actual consumption rate at each time is larger than without pension.

- The conflict in in life and pension insurance caused by the close connection between intertemporal consumption substitution and risk aversion is pointed out.

- In Section 5 of the paper we suggest replacing the standard additive and separable expected utility function by recursive, differentiable utility. Here we derive the most original results in the paper.

- Finally we discuss our results, and reflect on longevity risk and cohort risk in relation to the framework presented.

1. Consumption and Saving

- Consider a person having income $e(t)$ and consumption $c(t)$ at time t.
- Given income, possible consumption plans must depend on the possibilities of saving and borrowing/lending.
- We want to investigate the possibilities of using income during one period to generate consumption in another period.
- To start, assume the consumer can borrow and lend to the same interest rate r.

- Given any e and c, the consumer's net saving $W(t)$ at time t is

$$W(t) = \int_0^t e^{r(t-s)}(e_s - c_s)ds$$

- Assuming the person wants to consume as much as possible for any e, not any consumption plan is possible.

- A constraint of the type $W(t) \geq a(t)$ may seem reasonable: If $a(t) < 0$ for some t, the consumer is allowed a net debt at time t.

- Another constraint could be $W(T) \geq B \geq 0$, where T is the planner's horizon. He/she is then required to be solvent at time T.

- The objective is to optimize the utility $U(c)$ of lifetime consumption, subject to a budget constraint. There could also be a bequest motive, but this is not the only explanation underlying life insurance.

2. Uncertain planning horizon

- For this we need to discuss: *Mortality*

- Yaari (1965), Hakansson (1969) and Fisher (1973) were of the first to introduce an uncertain lifetime into the theory of the consumer.

- The remaining lifetime of an x year old consumer/investor at time zero, T_x, is a random variable. Its support is $(0, \tau)$.

- It has cumulative probability distribution function $F^x(t) = P(T_x \leq t)$, $t \geq 0$, and the survival function we denote by $\bar{F}^x(t) = P(T_x > t)$.

- Ignoring selection effects, it can be shown that

$$\bar{F}^{x}(t) = \frac{l(x+t)}{l(x)} \tag{1}$$

for some function $l(\cdot)$ of one variable only.

- The decrement function $l(x)$ can be interpreted as the expected number alive in age x from a population of $l(0)$ newborne.

- The force of mortality or death intensity is defined as

$$\mu_x(t) = \frac{f_x(t)}{1 - F^x(t)} = -\frac{d}{dt} \ln \bar{F}^x(t), \qquad F^x(t) < 1 \tag{2}$$

where $f_x(t)$ is the probability density function of T_x.

- Integrating yields the survival function in terms of the force of mortality

$$\bar{F}^x(t) = \frac{l(x+t)}{l(x)} = \exp\Big\{ - \int_0^t \mu_x(u)\,du \Big\}. \tag{3}$$

- Suppose $y \geq 0$ a.s. is a process in L. Then the formula

$$E\Big(\int_0^{T_x} y_t dt \Big) = \int_0^{\tau} E(y_t) \frac{l(x+t)}{l(x)} dt = \int_0^{\tau} E(y_t) e^{-\int_0^t \mu_x(u) du} dt \tag{4}$$

follows essentially from integration by parts, our independence assumption regarding mortality and the Fubini Theorem.

- The single premium of an annuity paying one unit per unit of time is given by the actuarial formula

$$\bar{a}_x^{(r)} = \int_0^{\mathcal{T}} e^{-rt}\frac{l_{x+t}}{l_x}dt, \tag{5}$$

- The single premium of a "temporary annuity" which terminates after time n is

$$\bar{a}_{x:\bar{n}|}^{(r)} = \int_0^{n} e^{-rt}\frac{l_{x+t}}{l_x}dt. \tag{6}$$

- Under a typical pension plan the insured will pay a constant, or "level" premium p up to some time of retirement n, and from then on he will receive an annuity b as long as he lives.

- *The principle of equivalence* gives the following relationship between premium and benefit:

$$p\int_0^{n} e^{-rt}\frac{l_{x+t}}{l_x}dt = b\int_n^{\mathcal{T}} e^{-rt}\frac{l_{x+t}}{l_x}dt.$$

- In standard actuarial notation this is written

$$p\bar{a}^{(r)}_{x:\bar{n}|} = b(\bar{a}^{(r)}_x - \bar{a}^{(r)}_{x:\bar{n}|}). \tag{7}$$

- The following formulas are standard in life insurance computations

$$\mu_x(t) = -\frac{l'(x+t)}{l(x+t)}, \qquad \text{and} \qquad f_x(t) = -\frac{l'(x+t)}{l(x)} = \frac{l(x+t)}{l(x)}\mu_{x+t}, \tag{8}$$

- Here $l'(x+t)$ is the derivative of $l(x+t)$ with respect to t.

- The present value of one unit payable at time of death is denoted $\bar{A}_x$. Using (8) and integration by parts, it can be written

$$\bar{A}_x = \int_0^\tau e^{-rt} f_x(t)dt = 1 - r\bar{a}^{(r)}_x. \tag{9}$$

- This insurance contract is called *Whole life insurance.*

- If the premium rate p is paid until the retirement age n for a combined life insurance with z units payable upon death, and an annuity of rate b per time unit as long as the insured lives, we have the following relationship between p, b and z:

$$p\bar{a}^{(r)}_{x:\bar{n}|} = b(\bar{a}^{(r)}_{x} - \bar{a}^{(r)}_{x:\bar{n}|}) + z(1 - r\bar{a}^{(r)}_{x}). \tag{10}$$

- Pension insurance and life insurance can now be integrated in the life cycle model in a natural way, as we shall see.

The effects from Pooling

- Continuing our discussion of consumption and saving, the following quantity plays an important role:

$$E\big(W(T_x)e^{-rT_x}\big) = \text{expected, discounted net savings.}$$

- In the absence of a life insurance/pension insurance market, one would, as before, consider consumption plans c such that

$$W(T_x)e^{-rT_x} \geq b \geq 0 \quad \text{almost surely}$$

e.g., debt must be resolved before the time of death.

- If pension insurance is possible, on the other hand, then one can allow consumption plans where

$$E\big(W(T_x)e^{-rT_x}\big) = 0 \quad \text{(no life insurance.)}$$

- The individual's savings possibilities are exhausted, by allowing gambling on own life length.

- Those individuals who live longer than average are guaranteed a pension as long as they live via the pension insurance market.

- The financing of this benefit comes from those who live shorter that average, which is what pooling is all about.

- Integration by parts gives the following expression for the expected discounted net savings

$$E\big(W(T_x)e^{-rT_x}\big) = \int_0^{\mathcal{T}} \big(e(t) - c(t)\big)e^{-\int_0^t (r+\mu_{x+u})du}dt.$$

- This we may interpret as the present value of the consumer's net savings: It takes place at a "spot" interest rate

$$r + \mu > r$$

since the mortality rate $\mu > 0$. This is a result of the the pooling effect of (life and) pension insurance.

- The existence of a life and pension insurance market allows the individuals to: Save at a higher interest rate than the spot rate r;

- With a pure pension insurance contract, the policyholder can consume *more* while alive, since terminal debt is resolved by pooling. This is illustrated later.

- Example 1: A Pension Contract, or an Annuity:

  Suppose $e(t) = 0$ for $t > n$. The condition $E\big(W(T_x)e^{-rT_x}\big) = 0$ can be interpreted as the *Principle of Equivalence*:

$$\int_0^n \big(e(t) - c(t)\big)P[T_x > t]e^{-rt}dt = \int_n^\tau c(t)e^{-rt}P[T_x > t]dt.$$

  Here $(e_t - c_t) = p_t$ is interpreted as a premium (intensity) while working, paying for the pension c after time n.

- This way the pension is paid out to the beneficiary as long as necessary, and only then, i.e., only as long as the policy holder is alive.

3. Optimal life and pension insurance: No risky securities

- In order to analyze the problem of optimal consumption, we need some assumptions about the preferences of the consumer:

$$U(c) = E\Big\{ \int_0^{T_x} e^{-\rho t} u(c_t)dt + e^{-\kappa T_x} v(W_{T_x}) \Big\}.$$

  Here ρ and κ are subjective impatience rates, u is a strictly increasing and concave utility function, and v is connected to life insurance.

- The variable $z = W(T_x)$ is the amount of life insurance. It is often assumed to be a given constant (e.g., 1), but we shall here allow it to be a decision variable.

- First we focus on pensions and annuities and set $v \equiv 0$.

The pension problem may be formulated as:

$$\max_c E\Big\{\int_0^{T_x} e^{-\rho t} u(c_t)dt\Big\}$$

subject to (i) $E\big(W(T_x)e^{-rT_x}\big) = 0$, and (ii) $c_t \geq 0$ for all t.

- Ignoring (ii) for the moment, we may use Kuhn-Tucker: The Lagrangian is:

$$\mathcal{L}(c;\lambda) = \int_0^{\tau} u(c_t)e^{-\int_0^t(\rho+\mu_{x+s})ds}dt$$
$$+\lambda\Big(\int_0^{\tau}\big(e(t)-c(t)\big)e^{-\int_0^t(r+\mu_{x+s})ds}dt\Big).$$

- If $c^*(t)$ is optimal, there exists a Lagrange multiplyer λ such that $\mathcal{L}(c;\lambda)$ is maximized at $c^*(t)$ and complementary slackness holds.

- Denoting the directional derivative of $\mathcal{L}(c;\lambda)$ in the direction c by $\bigtriangledown\mathcal{L}(c;\lambda;c)$, the first order condition of this unconstrained problem is

$$\bigtriangledown\mathcal{L}(c^*;\lambda;c) = 0 \quad \text{in all 'directions' } c,$$

which is equivalent to

$$\int_0^T \left(u'(c_t^*)e^{-\int_0^t(\rho+\mu_{x+s})ds} - \lambda e^{-\int_0^t(r+\mu_{x+s}ds}\right)c(t)dt = 0, \ \forall c.$$

- This gives the first order conditions

$$u'(c_t^*) = \lambda e^{-(r-\rho)t}, \quad t \geq 0.$$

which does not depend on mortality.

- Differentiating this function in t along the optimal path c^*, we may deduce the following differential equation for c^*

$$\frac{dc_t^*}{dt} = (r - \rho)T(c_t^*),$$

where $T(x) = -\frac{u'(x)}{u''(x)}$ is the risk tolerance function of the consumer, the reciprocal of the absolute risk aversion function of the customer.

- The differential equation for the pension c_t^* tells us that the value of the interest rate r is a crucial border value for the subjective impatience rate ρ.

- When $\rho > r$ the optimal consumption c_t^* is always a decreasing function of time t, but when $\rho < r$ the optimal consumption increases with time.

- In the first case, the 'impatient' one has already consumed so much, that he can only look forward to a decreasing consumption path.

- The 'patient' one can, on the other hand, look forward to a steadily increasing future consumption path.

- Exampel 2: CEIS Consumer; Pension Contract.

- Assume that the income process e_t is

$$e_t = \begin{cases} y, & \text{if } t \leq n; \\ 0, & \text{if } t > n \end{cases} \tag{11}$$

and $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$. The constant y is salary while working.

- The optimal consumption and pension is

$$c_t^* = k e^{\frac{1}{\gamma}(r-\rho)t}$$

where k in an integration constant.

- Equality in constraint (i) determines this constant: The optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau)$) is

$$c_t^* = y \, \frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_0)}_{x}} \, e^{\frac{1}{\gamma}(r-\rho)t} \quad \text{for all } t \geq 0.$$

- Here $r_0 = r - \frac{r-\rho}{\gamma}$ and $\bar{a}^{(r)}_{x:\bar{n}|}$ and $\bar{a}^{(r_0)}_{x}$ are actuarial formulas:
$\bar{a}^{(r)}_{x:\bar{n}|} = \int_0^n e^{-rt} \frac{l_{x+t}}{l_x} dt, \quad \text{and} \quad \bar{a}^{(r_0)}_{x} = \int_0^\tau e^{-r_0 t} \frac{l_{x+t}}{l_x} dt$

- Although the FOC does not depend on mortality, the optimal pension does, via the budget constraint.

- In Example 2 we see that the impatient one has an optimal consumption path that is a decreasing exponential, while the patient insurance customer has an increasing exponential consumption path.

- This seems to suggest that it may be difficult to compare consumption paths between different consumers across time.

- That this is not so clear-cut as this example might suggest, follows from a securities market, (to be introduced later)

- In Example 2 we notice that the above effects are enforced as the EIS-coefficient $\epsilon := \frac{1}{\gamma}$ increases.

- EIS = Elasticity of Intertemporal Substitution in Consumption. This quantity deals with the willingness to postpone consumption by saving in order to increase total consumption by increasing total capital gains.

- In the present case with no risk, it is indeed the EIS interpretation that is relevant. When risk is introduced, this parameter will play more than one role, which we return to later.

- Recall the standard interpretation of the EIS-coefficient ϵ: In a two period setting without risk, if $\epsilon < 1$ will a one per cent increase in the interest rate factor $R = 1 + r$ lead to a higher increase in consumption today than tomorrow.

- If $\epsilon > 1$ the substitution effect will dominate, and consumption tomorrow increases the most.

- If $\epsilon = \gamma = 1$ the income and the substitution effects cancel, and consumption at the two time points increase equally much.

- In our setting we choose to look at an increase in the risk free interest rate r. We then get

$$\frac{\partial}{\partial r} c_t^* = y \, \frac{\bar{a}_{x:\bar{n}|}^{(r)} e^{\frac{1}{\gamma}(r-\rho)\,t}}{(\bar{a}_x^{(r_0)})^2} \, \epsilon \left(t - \left(\frac{1}{\epsilon} \tau_1^{(n)} + (1 - \frac{1}{\epsilon}) \tau_2 \right) \right), \tag{12}$$

  The time points $\tau_1^{(n)}$ and τ_2 are defined by first mean value theorem.

- The break-point-in-time

$$\tilde{t} = \Big(\frac{1}{\epsilon}\tau_1^{(n)} + (1 - \frac{1}{\epsilon})\tau_2\Big)$$

is seen to be a convex combination of the two time points $\tau_1^{(n)}$ and τ_2 when $\epsilon \geq 1$, and when $\epsilon = 1$, $\tau = \tau_1^{(n)}$.

- When ϵ decreases below 1 the individual's propensity to substitute consumption across time diminishes; the income effect dominates.

- This is of course an important observation for the insurance industry, since, according to this interpretation, for $\epsilon < 1$ people should be less willing to engage in life and pension insurance contracts.

- It seems to be part of conventional wisdom that it is primarily *risk aversion* that induces people to demand insurance.

- Presumably the insurance industry target their products to individuals with ϵ larger than one *and* γ larger than two.

- Since $\gamma = \frac{1}{\epsilon}$ this is impossible! The parameter γ parameter is seen to play two conflicting roles, of particular relevance to life insurance and pension products.

Including life insurance

- We can now introduce life insurance. The problem is to solve

$$\max_{c(t),z} E\Big\{ \int_0^{T_x} e^{-\rho t} u(c_t) dt + e^{-\kappa T_x} v(z) \Big\}$$

subject to (i) $E\big(W(T_x)e^{-rT_x}\big) \geq E\big(ze^{-rT_x}\big)$, and (ii) $c_t \geq 0$ for all t and $z \geq 0$.

- The Lagrangian for the problem is

$$\mathcal{L}(c, z; \lambda) = \int_0^{\mathcal{T}} u(c_t) e^{-\int_0^t (\rho + \mu_{x+s}) ds} dt + v(z)(1 - \kappa \bar{a}_x^{(\kappa)})$$
$$- \lambda\Big((1 - r\bar{a}_x^{(r)})z - \int_0^{\mathcal{T}} \big(e(t) - c(t)\big) e^{-\int_0^t (r + \mu_{x+s}) ds} dt\Big).$$

- The first order condition in c is as for pensions. The FOC in the life insurance amount z is:

$$v'(z^*) = \lambda \; \frac{1 - r\bar{a}_x^{(r)}}{1 - \kappa\bar{a}_x^{(\kappa)}}.$$

- We can now determine both the optimal life time consumption, including pension and and the optimal amount of life insurance. An example will illustrate.

<u>Example 3:</u> CEIS consumer

- Assume $e_t = y$ for $t \leq n$, $e_t = 0$ for $t > n$, and $u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$, and the life insurance index is $v(x) = \frac{1}{1-\psi}x^{1-\psi}$.

- The optimal life insurance amount and optimal consumption/pension are given by

$$z^* = \lambda^{-\frac{1}{\psi}} \left(\frac{1 - r\bar{a}_x^{(r)}}{1 - \kappa\bar{a}_x^{(\kappa)}} \right)^{-\frac{1}{\psi}} \quad \text{and} \quad c_t^* = \lambda^{-\frac{1}{\gamma}} \; e^{\frac{1}{\gamma}(r-\rho)t}. \tag{13}$$

- Equality in the 'average' budget constraint (i) determines the Lagrangian multiplier λ. The equation is

$$\lambda^{-\frac{1}{\psi}}(1 - r\bar{a}_x^{(r)}) \left(\frac{1 - r\bar{a}_x^{(r)}}{1 - \kappa\bar{a}_x^{(\kappa)}} \right)^{-\frac{1}{\psi}} + \lambda^{-\frac{1}{\gamma}} \; \bar{a}_x^{(r_0)} = y \; \bar{a}^{(r)}_{x:\bar{n}|}. \tag{14}$$

- Notice that with life insurance included, the optimal consumption and the pension payments become smaller than without life insurance present.

- This just tells us the obvious: When some resources are bound to be set aside for the beneficiaries, less can be consumed while alive.

- The optimal amount in life insurance is an increasing function in income y, and depends on the interest rate r, the pension age n, the insured's relative risk aversion γ as well as his impatience rate ρ.

- Further, the bequest relative risk aversion ψ and the corresponding impatience rate κ, the insured's age x when initializing the pension and insurance contracts, and the insured's life time distribution through the actuarial formulas.

- Comparative statics in the parameters are not straightforward, and numerical technics are necessary.

- As an example, when $\psi = \gamma$, it can be seen that the optimal amount of life insurance $z^*(\kappa)$ as a function of the bequest impatience rate κ is increasing for $\kappa \leq \kappa_0$ for some $\kappa_0 > 0$, and decreasing in κ for $\kappa > \kappa_0$.

- For reasonable values of κ this means that more impatience with respect to life insurance means a higher amount z^* of life insurance.

- When $u = v$ and $\kappa = \rho$, equality in the "average" budget constraint determines the Lagrangian multiplier λ:

- $$\lambda^{-\frac{1}{\gamma}} = y \, \frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_0)}_x + (1 - r\bar{a}^{(r)}_x)\left(\frac{1-r\bar{a}^{(r)}_x}{1-\rho\bar{a}^{(\rho)}_x}\right)^{-\frac{1}{\gamma}}}$$

- The optimal life insured amount is seen to depend on income, the interest rate, the insured's EIS-coefficient $\alpha = \frac{1}{\gamma}$ and his impatience rate ρ. $\square$

- The above results deviate rather much from the standard actuarial formulas, which is to be expected since the two approaches are indeed different:

- The actuarial theory is primarily based on the principle of equivalence and risk neutrality. This is problematic, since risk neutral insurance customers would simply not demand any form of insurance.

- Therefore we assume that the individuals are risk averse, unlike what is done in actuarial theory, and use expected utility as our optimization criterion.

- Going back to the actuarial relationship (10), the three quantities p, b and z representing the premium, the pension benefit and the insured amount respectively could be any non-negative numbers satisfying this relationship.

- In the above example, all these quantities are in addition derived so that expected utility is optimized.

- The optimal contracts still maintain the actuarial logic represented by the principle of equivalence, which in our case corresponds to the budget constraint (i) on the 'average'.

- The present analogue to the relationship (10) is:

$$\int_0^n (y - c_t^*) \frac{l_{x+t}}{l_x} e^{-rt} dt = \int_n^\tau c_t^* \frac{l_{x+t}}{l_x} e^{-rt} dt + z^*(1 - r\bar{a}_x^{(r)}), \quad (15)$$

- Here the constant premium p corresponds to the time varying $p_t = (y - c_t^*)$ for $0 \leq t \leq n$, the constant pension benefit b corresponds to the optimal c_t^* for $n \leq t \leq \tau$, and the number z corresponds to z^* found in (13), where also the optimal pension c_t^* is given.

- So far the insured amount is still a deterministic quantity, albeit endogenously derived. The reason for the non-randomness in z^* in the present situation is that only biometric risk is considered.

When uncertainty in the financial market is also taken into account, we shall demonstrate that the optimal insured amount becomes state dependent, and the same is true for c_t^*.

- Both real and nominal amounts are then of interest when comparing the results with insurance theory and practice.

- Including risky securities in a financial market is our next topic.

4. A Financial Market including Uncertainty

- We consider a consumer/insurance customer who has access to a securities market, and pension and life insurance.

- The securities market can be described by a price vector $X' = (X^{(0)}, X^{(1)}, \cdots, X^{(N)})$, where (prime means transpose here)

$$dX_t^{(n)} = \mu_n X_t^{(n)}\,dt + X_t^{(n)}\sigma^{(n)}\,dB_t, \qquad X_0^{(n)} > 0, \qquad t \in [0, T], \tag{16}$$

- Here $\sigma^{(n)}$ is the n-th row of a matrix σ consisting of constants in $R^{N\times N}$ with linearly independent rows, and where μ_n is a constant. Here N is also the dimension of the Brownian motion B.

- Underlying there is a probability space $(\Omega, \mathcal{F}, P)$ and an increasing information filtration $\mathcal{F}_t$ generated by the d-dimensional Brownian motion.

- Each price process $X_t^{(n)}$ is a geometric Brownian motion.

- We suppose that $\sigma^{(0)} = 0$, so that $r = \mu_0$ is the risk free interest rate. T is the finite horizon of the economy.

- The state price deflator π is given by

$$\pi_t = \xi_t e^{-rt}. \tag{17}$$

- The density process ξ has the representation

$$\xi_t = \exp(-\eta' \cdot B_t - \frac{t}{2}\eta' \cdot \eta). \tag{18}$$

- Here η is the market-price-of-risk for the discounted price process $X_t e^{-rt}$, defined by

$$\sigma\eta = \nu. \tag{19}$$

- ν is the vector with n-th component $(\mu_n - r)$, the excess rate of return on security n, $n = 1, 2, \cdots, N$.

- From Ito's lemma it follows from (18) that

$$d\xi_t = -\xi\ \eta' \cdot dB_t, \tag{20}$$

i.e., the density ξ_t is a martingale.

The Consumption/Investment/Pension Problem

- The consumer's problem is, for each initial wealth w, to solve

$$\sup_{(c,\varphi)} U(c) \tag{21}$$

subject to an intertemporal budget constraint

$$dW_t = \big(W_t(\varphi_t' \cdot \nu + r) - c_t\big)dt + W_t\varphi_t' \cdot \sigma dB_t, \qquad W_0 = w. \tag{22}$$

- Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ are the fractions of total wealth held in the risky securities.

- The first order condition for the problem is given by the Bellman equation. It is

- Bellman equation:

$$\sup_{(c,\varphi)} \left\{ \mathcal{D}^{(c,\varphi)} J(w,t) - \mu_x(t) J(w,t) + u(c,t) \right\} = 0, \tag{23}$$

with boundary condition

$$E J(w, T_x) = 0, \quad w > 0. \tag{24}$$

- The function $J(w,t)$ is the indirect utility function of the consumer at time t when the wealth $W_t = w$, and the differential operator $\mathcal{D}^{(c,\varphi)}$ is given by

$$\begin{aligned} \mathcal{D}^{(c,\varphi)} J(w,t) &= J_w(w,t)(w\varphi \cdot \nu + rw - c) + J_t(w,t) \\ &+ \frac{w^2}{2} \varphi' \cdot (\sigma \cdot \sigma') \cdot \varphi \, J_{ww}(w,t). \end{aligned} \tag{25}$$

- This is a non-standard dynamic programing problem, a so called non autonomous problem.

- Instead of solving this problem directly, we solve an equivalent one.

- As is well known (e.g., Cox and Huang (1989) or Pliska (1987)), when the market is complete, the dynamic program (21) - (25) has the same solution as the following simpler, yet more general problem:

<u>The Alternative Problem Formulation</u>

Find

$$\sup_{c \in L} U(c), \tag{26}$$

subject to

$$E\left\{ \int_0^{T_x} \pi_t c_t \, dt \right\} \leq E\left\{ \int_0^{T_x} \pi_t e_t \, dt \right\} := w \tag{27}$$

- Here e is the endowment process of the individual. It is assumed that e_t is $\mathcal{F}_t$ measurable for all t.

- The pension insurance element secures the consumer a consumption stream as long as needed, but only if it is needed. This makes it possible to compound risk-free payments at a higher rate of interest than r.

- The optimal wealth process W_t associated with a solution c^* to the problem (26)-(27) can be implemented by some adapted and allowed trading strategy φ^*, since the marketed subspace M is equal to L.

- Without mortality this is well-known, and by introducing the new random variable T_x we claim it still holds.

- In principal mortality corresponds to a new state of the economy, which should "be priced", but the insurer can diversify this risk away by pooling over the agents, so the corresponding Arrow-Debreu state price is equal to $\exp\{-\int_0^t \mu_x(u)du\}$.

- Since this quantity is non-stochastic, adding the pension insurance contract in an otherwise complete model has no implications for the state price π, and thus the model is still complete.

The Optimal Consumption/Pension

- The constrained optimization problem (26)-(27) can be solved by Kuhn-Tucker and a variational argument.

- The Lagrangian of the problem is

$$\mathcal{L}(c;\lambda) = E\Big\{ \int_0^{T_x} \big(u(c_t,t) - \lambda(\pi_t(c_t - e_t))\big)\, dt\Big\}, \tag{28}$$

- By our assumptions the optimal solution c^* to the problem (26)-(27) satisfies $c_t^* > 0$ a.s. for a.a. $t \in [0, T_x)$.

- There exists of a Lagrange multiplyer, λ, such that c^* maximizes $\mathcal{L}(c;\lambda)$ and complementary slackness holds.

- Denoting the directional derivative of $\mathcal{L}(c;\lambda)$ in the "direction" $h \in L$ by $\bigtriangledown\mathcal{L}(c;\lambda;h)$, the first order condition of this unconstrained problem becomes

$$\bigtriangledown\mathcal{L}(c^*;\lambda;c) = 0 \qquad \text{for all } c \in L \tag{29}$$

- This is equivalent to

$$E\left\{\int_0^{\mathcal{T}} \Big(\big(u'(c_t^*)e^{-\rho t} - \lambda\pi_t\big)c(t)\Big)P(T_x > t)dt\right\} = 0, \qquad \text{for all } c \in L, \tag{30}$$

where the survival probability $P(T_x > t) = \frac{l(x+t)}{l(x)}$.

- In order for (30) to hold true for all processes $c \in L$, the first order condition is

$$u'(c_t^*) = \lambda e^{-\rho t}\pi_t = \lambda e^{-(r-\rho)t}\xi_t \qquad \text{a.s.,} \qquad t \geq 0 \tag{31}$$

- The optimal consumption process is accordingly

$$c_t^* = u'^{-1}\Big(\lambda e^{-(r-\rho)t}\xi_t\Big) \qquad \text{a.s.,} \qquad t \geq 0, \tag{32}$$

where the function $u'^{-1}(\cdot)$ inverts the function $u'(\cdot)$.

- Comparing the first order condition to the one where only biometric risk is included, we notice that the difference is the state price density ξ_t in (31). Still mortality does not enter this latter condition.

- Differentiation (31) in t along the optimal path c_t^*, by the use of Ito's lemma and diffusion invariance the following stochastic differential equation for c_t^* is obtained

$$dc_t^* = \Big((r-\rho)T(c_t^*) + \frac{1}{2}T^3(c_t^*)\,\frac{u'''(c_t^*)}{u'(c_t^*)}\,\eta' \cdot \eta\Big)\,dt + T(c_t^*)\ \eta' \cdot dB_t \tag{33}$$

where $T(\cdot)$ is the risk tolerance function defined earlier.

- Comparing with the corresponding differential equation for c_t^* with only biometric risk present, it is seen that including market risk means that the dynamic behavior of the optimal consumption is not so crucially dependent upon whether $r < \rho$ or not.

- This follows since, first, there is an additional term in the drift, and, second, there is a diffusion term present under market risk. The definition of what impatience means will also change with market risk present, as we shall see.

- Notice that when the market-price-of-risk $\eta = 0$, the two equations coincide. We consider an example:

Example 4: The CEIS-consumer:

- In this case the optimal consumption takes the form
$$c_t^* = (\lambda \pi_t e^{\rho t})^{-\frac{1}{\gamma}}.$$
The budget constraint determines the Lagrange multiplier λ.

- Let
$$r_1 = \frac{\rho}{\gamma} + (1 - \frac{1}{\gamma})(r + \frac{1}{2}\,\frac{1}{\gamma}\,\eta' \cdot \eta),$$

- The optimal consumption ($t \in [0, n]$) and the optimal pension ($t \in [n, \tau]$) are both given by the expression
$$c_t^* = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_1)}_{x}}\,e^{\frac{1}{\gamma}(r-\rho)\,t}\,\xi_t^{-\frac{1}{\gamma}} \quad \text{for all } t \geq 0. \tag{34}$$

- To be compared to the corresponding process with only mortality risk present. Notice that this latter formula follows from (34) by setting $\eta = 0$, in which case $\xi_t = 1$ for all t (a.s.) and $r_1 = r_0$.

- The expected value of the optimal consumption is given by

$$E(c_t^*) = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_1)}_x}\exp\left\{\frac{1}{\gamma}\Big(r + \frac{1}{2}\eta' \cdot \eta(1 + \frac{1}{\gamma}) - \rho\Big)t\right\}, \tag{35}$$

  which is seen to grow with time t already when $r > \rho - \frac{1}{2}\eta' \cdot \eta(1 + \frac{1}{\gamma})$.

- When the opposite inequality holds, this expectation decreases with time. In terms of expectations, the crucial border value for the impatience rate ρ is no longer r but $(r + \frac{1}{2}\eta' \cdot \eta(1 + \frac{1}{\gamma}))$ when a stock market is present.

- As an alternative derivation of c_t^*, the stochastic differential equation (33) for the optimal consumption process is

$$\frac{dc_t^*}{c_t^*} = \Big(\frac{r-\rho}{\gamma} + \frac{1}{2}\,\frac{1}{\gamma}\,(1+\frac{1}{\gamma})\,\eta'\cdot\eta\Big)\,dt + \frac{1}{\gamma}\eta'\cdot dB_t, \qquad (36)$$

from which it follows that c_t^* is a geometric Brownian motion.

- The risk tolerance function $T(c) = \frac{c}{\gamma}$. The "solution" to this stochastic differential equation is

$$c_t^* = c_0 e^{\frac{1}{\gamma}[(r-\rho+\frac{1}{2}\eta'\cdot\eta)t+\eta'\cdot B_t]}, \qquad t \geq 0.$$

- The initial value c_0 is finally determined by the budget constraint, and (34) again results. The dynamics of c_t^* will be used later. $\square$

- Notice that when stock market uncertainty is present, the above solution tells us that when state prices π_t are low, optimal consumer is high, and vice versa.

- State prices reflect what the representative consumer is willing to pay for an extra unit of consumption. In particular is π_t is high in times of crises, and low in good times.

- In real terms the result for pensions is as for optimal consumption in society at large: In times of crises the pensions are lower than in good times.

- This only explains the obvious: society can only pay the pensioners that the economy can manage at each time. Insurance companies pay the pensions from funds, which in bad times are lower than in good times.

- The government is similarly affected. Since pensions are, presumably, paid out to the whole generation of people above a certain age, it is in principle not possible to insure the entire society against crises and bad times.

- A single individual can of course find a strategy to hedge against low income in certain periods, and so can an insurance company by proper use of risk management, but this types of hedging will not work for the entire population, by *the mutuality principle*:

- In equilibrium everyone holds a non-decreasing function of aggregate consumption. If aggregate consumption in society is down, everyone is in principle worse off.

Pensions in nominal terms

- Pensions (and insurance payments) are usually not made in real, but in nominal terms. There exist index-linked contracts, but these are still more the exception than the rule. In nominal terms the optimal consumption is $c_t^* \pi_t$.

- For the preferences of Example 4, the nominal consumption/pension is given by the
$$c_t^* \pi_t = \left(\lambda e^{\rho t}\right)^{-\frac{1}{\gamma}} \pi_t^{(1-\frac{1}{\gamma})}$$

- Here $\gamma = \epsilon = 1$ is seen to be a border value of the relative risk aversion in the sense that for $\gamma > 1$ ($\epsilon < 1$) both optimal consumption and pensions in nominal terms are countercyclical.

- This can give rise to an *illusion* of being insured against times of crises.

- People with $\gamma < 1$ ($\epsilon > 1$) experience no such illusion, since nominal amounts behaves as real amounts with respect to cycles in the economy.

- In the situation when $0 < \gamma < 1$ the agent is sometimes called *risk tolerant*.

- This phenomenon is connected to the alternative interpretation of γ, where the quantity $\epsilon = 1/\gamma$ is the *elasticity of intertemporal substitution*.

- If $\epsilon < 1$ will an increase in income of 1% lead to a higher increase in consumption today than tomorrow.

- If $\epsilon > 1$ the substitution effect will dominate, and consumption tomorrow increases the most.

- If $\epsilon = \gamma = 1$ the income and the substitution effect will cancel out, and the consumption at the two time points will increase equally much.

- Most people seem to have $\gamma > 1$, and a value larger than 2 is found in most experimental situations. This could, perhaps, explain the impression that some people have (in particular state employees), namely that in good times, everyone else "seem better off".

- First, this person's nominal consumption is low, second a larger part of the increased income will be consumed today, than the part invested for consumption later on.

- This is probably a reasonable description of how many people act, although the model is, admittedly, very simplistic.

- The risk tolerant individual, of which there are fewer in society, are not subject to this distorted perception: In good times both his real and nominal consumption are high, and since $\epsilon > 1$ the substitution effect will dominate, and a larger part of income increases is invested rather than consumed right away.

- One would, perhaps, think that the investment of income for later consumption is consistent with risk averse behavior, and thus be stronger when $\gamma > 1$, but this is not so according to the standard model.

- A better description of this latter issue may, perhaps, be obtained if the elasticity of intertemporal substitution could be *separated* from the individual's risk aversion.

- There are several representations of preferences that accommodate this, like recursive utility, habit formation, Epstein-Zin utility, etc.

- See e.g., Duffie and Skiadas (1994), Duffie and Epstein (1992a-b) and Epstein and Zin (1989). Kreps and Porteus (1978) provide the axiomatic framework for the representation of recursive utility.

- The issue of when uncertainty is resolved is an important one it this kind of analysis.

- We now give an introduction to this subject relevant for continuous time framework of the present exposition. Despite the apparent complexity of the recursive framework, we are able to derive some strikingly simple results.

Recursive utility

- As we have seen it can be difficult to separate the effects of risk aversion (γ) and consumption substitution (ϵ) on optimal consumption and optimal pension insurance in the standard model.

- In order for insurance customers to demand insurance, ϵ can not be too low (preferably $\epsilon \geq 1$), but then the relative risk aversion γ will be too low for the individual to be interested in insurance purchases.

- Instead the agent will self-insure by investing in the market portfolio. Since life and pension insurance do exist, we seek out alternative explanations.

- In this section we introduce recursive utility, which is a way of representing preferences where these two effects can be separated.

- This problem is connected to the field of forward-backwards stochastic differential equations through the stochastic maximum principle.

- We start with a felicity function $f(c_t, V_t)$. The idea is that utility must depend on more than just consumption. Here V_t is thought of as current, remaining utility at time t.

- Utility of c for the felicity f is then $U(c) = V_0$.

- By the general theory we then have the representation (see Duffie and Epstein (1992))

$$V_t = E_t\Big(\int_t^{T_x} f(c_s, V_s)ds \Big), \quad t \in [0, T]. \tag{37}$$

- In order to find the first order condition for the consumer's problem, we proceed in principle as in Section 4.3 above.

- In maximizing the Lagrangian of the problem, we calculate the directional derivative $\bigtriangledown U(c^*; h) = (\bigtriangledown U(c^*))(h_t)$ in the "direction" h_t.

- Here $\bigtriangledown U(c^*)$ is the gradient of U at c^*.

- Since U is continuously differentiable, this gradient is a linear and continuous functional, and thus, by the Riesz representation theorem, it has a Riesz representation.

- The first order condition is that the directional derivative of the Lagrangian is zero at the optimal c_t^* in all directions $h \in L$:

$$(\bigtriangledown U(c^*))(h_t) - \lambda E\Big(\int_0^{T_x} \pi_t h_t \, dt \Big) = 0 \quad \text{for all} \quad h \in L. \tag{38}$$

- The result is that for the Riesz representation of the gradient of U to be equal to the state price deflator π_t it is necessary and sufficient that

$$\lambda \pi_t = Y_t \, \frac{\partial f}{\partial c}(c_t^*, V_t) \quad a.s. \tag{39}$$

- where

$$Y_t = \exp\Big(\int_0^t \frac{\partial f}{\partial V}(c_s^*, V_s)\, ds \Big) \quad a.s. \tag{40}$$

Here (39) corresponds to the first order condition (31).

- Again we have made use of the complete market. The felicity index f that we have in mind, called Kreps-Porteuse utility, is the following

$$f(c, V) = \frac{b}{a} \, \frac{c^a - (\alpha V)^{\frac{a}{\alpha}}}{(\alpha V)^{\frac{a}{\alpha} - 1}}, \tag{41}$$

- Here the constants satisfy $0 \neq a \leq 1,\ 0 \leq b,\ 0 \neq \alpha \leq 1$. Here $\frac{1}{1-a}$ corresponds to ϵ and α corresponds to $(1-\gamma)$, while b corresponds to ρ.

- Risk aversion represented by α is separated from consumption substitution represented by the parameter a.

- Risk aversion of the intertemporal ordering increases if α falls.

- In terms of the utility function U, for $\alpha_2 \leq \alpha_1$, then for any deterministic consumption process $\bar{c}$

$$U_{\alpha_1}(c) \leq U_{\alpha_1}(\bar{c}) \Rightarrow U_{\alpha_2}(c) \leq U_{\alpha_2}(\bar{c}). \tag{42}$$

  i.e., U_{α_2} is more risk averse than U_{α_1} in the sense that any consumption process c rejected by U_{α_1} in favor of some deterministic process $\bar{c}$ would also be rejected by U_{α_2}.

- From the above we have the following equations

$$\pi_t = Y_t\, f_c(c_t^*, V_t) \tag{43}$$

$$Y_t = \exp\Big(\int_0^t f_V(c_s^*, V_s) ds \Big) \tag{44}$$

$$V_t = E_t\Big(\int_t^T f(c_s^*, V_s) ds \Big) \tag{45}$$

The equation for Y_t can be written

$$dY_t = Y_t\, f_V(c_t^*, V_t)\, dt. \tag{46}$$

Solving (43) for c_t we get

$$c_t^* = \Big(\frac{\lambda \pi_t (\alpha V_t)^{\frac{a}{\alpha}-1}}{b\, Y_t} \Big)^{\frac{1}{a-1}}. \tag{47}$$

- We substitute this into (46) and (47) and get

$$dY_t = h(t, Y_t, V_t)\,dt; \quad 0 \le t \le T \tag{48}$$

$$Y_0 = 1 \quad \text{(forward SDE)} \tag{49}$$

and

$$dV_t = -g(t, Y_t, V_t)\,dt + K_t dB_t; \quad 0 \le t \le T \tag{50}$$

$$V_T = 0 \quad \text{(backward SDE)} \tag{51}$$

where

$$h(t, Y_t, V_t) = Y_t f_V\Big(\Big(\frac{\lambda \pi_t (\alpha V_t)^{\frac{a}{\alpha}-1}}{b\,Y_t}\Big)^{\frac{1}{a-1}}, V_t\Big) \tag{52}$$

and

$$g(t, Y_t, V_t) = f\Big(\Big(\frac{\lambda \pi_t (\alpha V_t)^{\frac{a}{\alpha}-1}}{b\,Y_t}\Big)^{\frac{1}{a-1}}, V_t\Big). \tag{53}$$

- The system (48) - (51) is a coupled system of forward-backward stochastic differential equations. This topic constitutes an active research field.

The dynamics of the optimal consumption

- At this stage we shall demonstrate what effect recursive utility has on the dynamics of the optimal consumption c_t^* of equation (47). We use the notation for the volatility K_t:

$$K_t = (\alpha V_t)\,\sigma_V$$

- Here σ_V is a vector of volatilities of the growth rate of V_t. Using Ito's lemma for the variables π_t, V_t and Y_t, and recognizing that the latter is of bounded variation, we obtain the following:

$$\frac{dc_t^*}{c_t^*} = \mu_c\,dt + \sigma_c\,dB_t \tag{54}$$

where

$$\mu_c = \epsilon\,(r-b) + \frac{1}{2}\,\epsilon\,(1+\epsilon)\,\eta'\eta + (a-\alpha)\,\epsilon\,\eta'\,\sigma_V + \frac{1}{2}\,(a-\alpha)^2\epsilon\,||\sigma_V||^2 \tag{55}$$

and

$$\sigma_c = \epsilon \left(\eta' + (\alpha - a)\, \sigma_V\right). \tag{56}$$

- Here $\epsilon = 1/(1-a)$. Because the state price π_t is a geometric Brownian motion generating the uncertainty of the model, due to Ito's lemma it is safe to conjecture that σ_V is a vector of constants.

- Regardless whether this conjecture is true or not, we can infer the following from these expressions:

- The two first terms in the drift (55) and the first term in the volatility (56) match to the terms in (36) exactly, when ϵ is interpreted as $1/\gamma$, and ρ as b.

The volatility

$$\sigma_c = \epsilon \left(\eta' + (\alpha - a)\, \sigma_V \right) :$$

- We notice that when $\alpha < a$, the volatility in (55) is smaller that the classical one. This would be the situation where $\gamma > \frac{1}{\epsilon}$, which we think is reasonable.

- For example could $\epsilon \approx 1$ and $\gamma > 2$, an impossible combination in the classical interpretation. Here it leads to a smoother consumption trajectory than predicted by the standard model.

- This is one important aspect of this extension of utility in life and pension insurance: That the insurance customer's risk aversion tend to reduce the covariation of the optimal pension insurance with the business cycle.

- This points in the direction of defined benefit pension schemes rather than defined contribution: The optimal pension is still of the defined contribution type, but less risky than predicted by the classical model.

<u>The growth rate of consumption</u>

$$\mu_C = \epsilon\,(r-b) + \frac{1}{2}\epsilon\,(1+\epsilon)\,\eta'\eta + (a-\alpha)\,\epsilon\,\eta'\,\sigma_V + \frac{1}{2}(a-\alpha)^2\epsilon\,||\sigma_V||^2 :$$

- Risk aversion is seen to only enters the last two terms and can have a negative or a positive effect depending upon the size of the risk aversion compared to the size of the substitution effect.

- The interpretation is that the two first terms on the right hand side are really dictated by consumption substitution rather than risk aversion.

- When $\alpha < a$ the last two terms in the drift are positive:

- So the impact of a large enough risk aversion results in an increased growth rate of consumption.

Implications for the representative consumer: Equilibrium

- If the representative agent is of this type in an equilibrium context, this could be used to explain the low consumption volatility observed in the market.

- The consumption variable c is typically excluding durable goods, and it is known that the aggregate consumption of nondurables does not vary too much, with an annualized standard deviation of the order of 3.57%.

- Put more simply, people need food regardless of the investment climate, which is reflected in observations, but not well explained in the standard model.

- This is better understood on the basis of the recursive type of preferences that we here consider.

- We can also derive the equilibrium risk premium, and the short term spot interest rate in equilibrium using the present set up for the representative agent.

- The results point in the right direction both when it comes to on explaining the equity premium puzzle, as well as the risk free rate puzzle.

- To illustrate, for $\epsilon = 1$ and $\gamma = 4$ we can explain an equity premium for the last century (the data of Mehra and Prescott (1985 in the US-economy) of 6% for a value of $\sigma_V = 0.11$.

- The equilibrium spot interest rate $r = \rho + 0.005$ or 1.5% (if $\rho = 0.01$).

- This can be a solution to both the *Equity Premium Puzzle*, as well as *The Risk Free rate Puzzle.*

The connection to actuarial theory and insurance practice

- We may compare our results to (i) Actuarial theory; (ii) Insurance practice.

- (i) In actuarial theory the nominal pension is nonrandom. This is only consistent with $\gamma = 1$.

- In addition this theory commonly uses the principle of equivalence to price insurance contracts, where the state price density $\xi_t \equiv 1$.

- This implies that the agent is really risk neutral, so $\gamma = 0$ should follow. There seems to be an inconsistency inherent in this theory.

- (ii) Insurance practice usually offer two different types of contracts: (a) defined benefits, (ytelsesbasert) and (b) defined contributions (inskud-dsbasert)

- (a) Defined benefits: Before profit sharing the nominal value is a constant. After profit sharing nominal value becomes random.

- (b) Defined contributions: Nominal as well as real value is random: The situation where $\gamma > 1$, is the "usual" case for this quantity. This contract is also closest to our development.

- In neither case does a guaranteed return rate enter the optimal contract: Such a guarantee will affect the insurance company's optimal investment plan. (Introduced in Norway more or less through an oversight (inadvertently)).

- Typically, due to the nature of the guarantees and regulatory constraints, the companies are led to sell when the market goes down, and buy when the market rises, which is just the opposite of what is optimal.

Pensions versus ordinary consumption

- We demonstrate that with pension insurance allowed, the actual consumption at each time t in the life of the consumer is at least as large as the corresponding consumption when the possibility of "gambling" on own life length is not allowed, provided the value of life time consumption w is fixed. This demonstrates a very concrete effect of pooling.

- Question: *How is consumption affected by the presence of pensions?*

- To this end, consider the random, remaining life time T_x of an x-year old as we have worked with all along, and for comparison, the deterministic life length T, where $T = E(T_x) = \bar{e}_x$ is the expected remaining life time of an x-year old pension insurance customer.

- Let $E\left\{ \int_0^{T_x} \pi_s c_s^* ds \right\} = w =$ initial wealth.

- Referring to Example 4, we use $\gamma = 1$ for simplicity.

- This is then equivalent to

$$c_t^* = \frac{e^{-\rho t}}{\lambda^* \pi_t}, \qquad \text{where} \quad \lambda^* = \frac{\bar{a}_x^{(\rho)}}{w}.$$

- Suppose instead that the time horizon T is deterministic:

- Let $T = \bar{e}_x := E(T_x)$. The optimal consumption is then

$$c_t^T = \frac{e^{-\rho t}}{\lambda^T \pi_t}, \qquad \text{where} \quad \lambda^T = \frac{\bar{a}_{\overline{T}|}^{(\rho)}}{w}.$$

Here $\bar{a}_{\overline{T}|}^{(\rho)} = \int_0^T e^{-\rho t} dt = \frac{1}{\rho}(1 - e^{-\rho T})$ is a convex function.

- It follows that $\bar{a}_x^{(\rho)} = E\Big\{ \int_0^{T_x} e^{-\rho t} dt \Big\} = E(\bar{a}_{\overline{T_x}|}^{(\rho)}) < \bar{a}_{\overline{T}|}^{(\rho)}$ by Jensen's inequality.

- This means that $\lambda^T > \lambda^*$ and, consequently, for each time t

$$c_t^* > c_t^T, \qquad \forall \quad w.$$

- With pension insurance the individual obtains a higher consumption at each time t that he/she is alive, due to pooling.

Including Life Insurance

- We are now in position to analyze life insurance in the problem formulation we use. We assume that the felicity index u and the utility function v are as in Example 3. The problem can then be formulated as follows:

$$\max_{z,c\geq 0} E\Big\{ \int_0^{T_x} e^{-\rho t}\frac{1}{1-\gamma}c_t^{1-\gamma}dt + e^{-\kappa T_x}\frac{1}{1-\psi}z^{1-\psi}\Big\}$$

subject to

$$E\Big\{e^{-rT_x}W(T_x)\Big\} \geq E\Big\{\pi_{T_x}z\Big\},$$

where z is the amount of life insurance, here a random decision variable.

- The Lagrangian of the problem is:

$$\mathcal{L}(c, z; \lambda) = E\Big\{ \int_0^\tau e^{-\rho t} \frac{1}{1-\gamma} c_t^{1-\gamma} \frac{l_{x+t}}{l_x} dt + e^{-\kappa T_x} \frac{1}{1-\psi} z^{1-\psi}$$
$$- \lambda \Big[\pi_{T_x} z - \int_0^\tau (e_t - c_t) \frac{l_{x+t}}{l_x} dt \Big] \Big\}.$$

- The first order condition in c is:

$$\nabla_c \mathcal{L}(c^*, z^*; \lambda; c) = 0, \quad \forall c \in L_+$$

- This is equivalent to

$$E\Big\{ \int_0^\tau \big((c_t^*)^{-\gamma} e^{-\rho t} - \lambda \pi_t \big) c_t \frac{l_{x+t}}{l_x} dt \Big\} = 0, \quad \forall c \in L_+$$

- The optimal consumption/pension

$$c_t^* = \big(\lambda e^{\rho t} \pi_t \big)^{-\frac{1}{\gamma}} \quad \text{a.s. } t \geq 0$$

as we have seen before.

- The first order condition in the amount of life insurance z is:

$$\nabla_z \mathcal{L}(c^*, z^*; \lambda; z) = 0, \quad \forall z \in L_+$$

- This is equivalent to

$$E\Big\{\big((z^*)^{-\psi} e^{-\kappa T_x} - \lambda \pi_{T_x}\big) z\Big\} = 0, \quad \forall z \in L_+ \tag{57}$$

- Notice that both z^* and z are $\mathcal{F} \vee \sigma(T_x)$ - measurable. For (57) to hold true, it must be the case that

$$z^* = \big(\lambda e^{\kappa T_x} \pi(T_x)\big)^{-\frac{1}{\psi}} \quad \text{a.s.}, \tag{58}$$

- This shows that the optimal amount of life insurance z^* is a state dependent $\mathcal{F}_{T_x}$ - measurable quantity.

- If the state is relatively good at the time of death, the state price π_{T_x} is then low and $(\pi_{T_x})^{-\frac{1}{\psi}}$ is relatively high (when $\psi > 0$).

- Thus this life insurance contract covaries positively with the business cycle. In practice this could be implemented by linking the payment z^* to an equity index.

- One can of course wonder how desirable this positive correlation with the economy is. With pensions we found it quite natural, but life insurance is something else. This product possess many of the characteristics of an ordinary, (non-life) insurance contact.

- In some cases it may seem reasonable that a life insurance contract is countercyclical to the economy, thereby providing real insurance in time of need.

- For this to be the result, however, the function v must be convex, corresponding to *risk proclivity* which here means that $\psi < 0$, but risk loving people do not buy insurance.

- We may find $E(z^*)$, as well as finding the Lagrange multiplier from the budget constraints (see the paper).

- Here we just show the case when $\kappa = \rho$ and $\psi = \gamma$ so that $u = v$. This gives

$$\lambda^{-\frac{1}{\gamma}} = \frac{y\bar{a}^{(r)}_{x:\bar{n}|}}{(1 + (1 - r_1)\bar{a}^{(r_1)}_x)}.$$

It is at this point that pooling takes place in the contract.

- In this situation the optimal consumption/pension is given by

$$c_t^* = \frac{y\bar{a}^{(r)}_{x:\bar{n}|}}{(1 + (1 - r_1)\bar{a}_x^{(r_1)})} e^{((r-\rho)/\gamma)t} \xi_t^{-\frac{1}{\gamma}}, \tag{59}$$

- The optimal amount of life insurance at time T_x of death of the insured is

$$z^* = \frac{y\bar{a}^{(r)}_{x:\bar{n}|}}{(1 + (1 - r_1)\bar{a}_x^{(r_1)})} e^{((r-\rho)/\gamma)T_x} \xi_{T_x}^{-\frac{1}{\gamma}}. \tag{60}$$

- One could, perhaps, say that these contracts represent an "innovation" in life insurance theory.

- One objection to the optimal solutions (58) and (60) is that the amount payable has not been subject to ”enough pooling” over the individuals.

- The pooling element is present, since it is used in the budget constraints, but the amount payable is here crucially dependent on the actual time of death T_x of the insured, which is unusual in life insurance theory.

- One alternative approach is to integrate out mortality in the first order condition (57). Notice that this is strictly speaking not the correct solution to the optimization problem, but must instead be considered as a suboptimal pooling approximation.

- This results in the following approximative first order condition ($\kappa = \rho$ and $\psi = \gamma$):

$$E_{z,z^*}\Big\{\Big((z^*)^{-\gamma}(1-\rho\bar{a}_x^{(\rho)}) - \lambda^* \int_0^\tau \pi_t f_x(t)dt\Big)z\Big\} = 0, \quad \forall z.$$

- The optimal amount of life insurance is also now a random variable. It is given by

$$z^* = \Big(\frac{\lambda^* \int_0^\tau \xi_t e^{-rt} f_x(t)dt}{1-\rho\bar{a}_x^{(\rho)}}\Big)^{-\frac{1}{\gamma}} \qquad \text{a.s.}$$

- However, this contract is seen to depend on the state of the economy from time 0 when the insured is in age x, to the end of the insured's horizon τ. At time of death T_x $(< \tau)$ this quantity is not known. (When $\xi_t = E(\xi_t) = 1$, then the solutions coincide.)

- Ignoring this problem for the moment, using the budget constraint, the Lagrange multiplyer λ^* is found as

$$(\lambda^*)^{-\frac{1}{\gamma}} = \frac{y\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(r_1)}_x + \frac{E[(\int_0^\tau \xi_t e^{-rt} f_x(t)dt)^{(1-\frac{1}{\gamma})}]}{(1-\rho\bar{a}^{(\rho)}_x)^{-\frac{1}{\gamma}}}}$$

- From this the optimal consumption/pension is given by

$$c_t^* = (\lambda^*)^{-\frac{1}{\gamma}} \, e^{\frac{r-\rho}{\gamma}t} \, \xi_t^{-\frac{1}{\gamma}} \qquad \text{a.s.},$$

and

$$z^* = (\lambda^*)^{-\frac{1}{\gamma}} \left(\frac{\int_0^\tau \xi_t e^{-rt} f_x(t)dt}{1-\rho\bar{a}^{(\rho)}_x}\right)^{-\frac{1}{\gamma}} \qquad \text{a.s.}$$

- When stock market uncertainty goes to zero, i.e. $\xi_t \to 1$ a.s., this optimal life insurance contract converges to the one containing only biometric risk (which is likely to be the one that would be implemented).

- An insured amount z^{**} consistent with the information available at time of death of the insured, could then be computed as

$$z^{**} := E\{z^* | \mathcal{F}_{T_x}\}.$$

- This is a random variable at the time when the life insurance contract is initialized, and an observable quantity at the time of death of the insured.

- Its advantage is that it takes into account pooling over life contingencies, at all stages of the analysis. Furthermore it is consistent with the standard analysis with "no market risk in the limit".

- Note that such a contract would benefit a young family in the case of early death of the provider, since those who die early are subsidized by those who live long when the insured sum is subject to enough averaging.

5. The investment problem

- In the present setting it can be solved together with the optimal consumption/pension problem. For this we need the agent's net wealth W_t at time t.

- Illustrating with the CRRA case, it is given by

$$W_t = \frac{1}{\pi_t} E_t \Big\{ \int_t^{T_x} \pi_s c_s^* ds \Big\} = c_t^* \, \bar{a}_{x+t}^{(r_1)}.$$

- Here E_t means conditional expectation given the information filtration $\mathcal{F}_t \vee (T_x > t)$, i.e., given the financial information available at time t and the fact that the individual is alive then.

- Using the dynamics for c_t^* given in (36), by Ito's lemma we obtain the following dynamic representation for the wealth W_t:

$$dW_t = \mu_W(t)dt + \frac{1}{\gamma} W_t \eta' \cdot dB_t,$$

  for some drift term $\mu_W(t)$.

- Comparing this to the intertemporal budget constraint of Section 4, we may apply diffusion invariance to determine the the optimal fractions $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ of total wealth held in the risky securities at each time t.

- By equating the two diffusion terms, we obtain that

$$\frac{1}{\gamma}\eta' = \varphi_t \cdot \sigma.$$

- Recalling that $\sigma\eta = \nu$, it follows from this that the optimal investment fractions are

$$\varphi = \frac{1}{\gamma}(\sigma\sigma')^{-1}\nu, \tag{61}$$

where ν, with components $\nu_n = \mu_n - r$, $n = 1, 2, \cdots N$, is the vector of risk premiums for the N risky securities.

- These ratios are all seen to be constants, meaning that they do not depend upon the age $(x + t)$ of the investor, the state of the economy π, or on the investor's death intensity μ_{x+t}.

- This result is the same as the one found by Mossin (1968), Samuelson (1969) and Merton (1971) without pension insurance present. A random time horizon simply does not alter this result.

- The formula (61) basically tells us that when prices of stocks increase, it is optimal to sell, and when prices fall it is optimal to buy.

- From an insurance perspective companies are often led to do the opposite, as we have mentioned before, which is of course unsatisfactory.

- One immediate objection to this investment result is that the optimal strategy does not depend upon the investor's horizon, or put differently, is independent of the investor's age $(x + t)$ at the time of investment.

- This is, however, against empirical evidence, and also against the typical recommendations of portfolio managers. The typical advice is that as the horizon gets shorter, the investor should gradually go out of equities, and thus take on less financial risk.

- One of the reasons for the advice that younger people should hold a higher fraction in equities is the tendency for stocks to outperform bonds or bills over the long run, despite the higher stock market volatility.

- This should not be mistaken as a ”time diversification” advice, which is a different but related issue, typically arising after each down-turn in the stock market (e.g., Delong (2008), Bodie (2009)).

- For example, following the 2008/09 market crash it is evident that many people around the world have lost their pensions, partly or entirely. For many old people it seems obvious that they have too short remaining lifetimes to regain what has been lost.

- Paul A. Samuelson has explained, in many articles over the years, what is wrong with time diversification.

- In Samuelson (1989) for example, he demonstrates that under the standard assumptions of the financial market, the optimal portfolio strategy based on maximizing expected utility of consumption over the investor's lifetime, beats various buy-and-hold strategies by clear margins.

- The standard assumptions are: 1) asset returns are i.i.d., 2) agents have additively separable constant relative risk aversion (CRRA) utility, 3) agents have no non-tradeable assets, and 4) markets are frictionless and complete.

- If portfolio choice is going to depend on age and/or on wealth, then one or more of these standard assumptions must be relaxed.

- Aase (2009) has discussed this problem by a slight reformulation of assumption 2), which we discuss next.

<u>The horizon problem</u>

- In this part we examine the effect of horizon and wealth on portfolio choice. We assume that the felicity index $u(x,t)$ satisfies the following

Assumption 1

$$u(x,t) = \begin{cases} \frac{1}{1-\gamma(t)} x^{(1-\gamma(t))} e^{-\rho t}, & \text{if } \gamma(t) \neq 1; \\ \ln(x) e^{-\rho t}, & \text{if } \gamma(t) = 1. \end{cases} \tag{62}$$

where $\gamma : [0,\tau) \rightarrow R_+$ is a continuous and strictly positive function of time.

- Notice that in this case $u(x,t)$ is not time and state separable, but this is the only relaxation of the standard assumptions 1) - 4) that is done.

- Using this assumption, Aase (2009) shows that under Assumption 1 the optimal fractions in the risky assets are

$$\varphi(t) = \frac{1}{\gamma(\tilde{t}_t)}(\sigma\sigma')^{-1}\nu, \tag{63}$$

where $\tilde{t}_t$ is an $\mathcal{F}_t$−measurable random time satisfying $\tilde{t}_t \in (t, \tau)$.

- It is determined at each time t by the equation

$$\gamma(\tilde{t}_t) = \frac{\int_t^{\tau} g(s,t)ds}{\int_t^{\tau} g(s,t)\frac{1}{\gamma(s)}ds} := \frac{W(t)}{Y(t)}. \tag{64}$$

- Here $W(t)$ is the agent's optimal wealth at time t, given by equation

$$W_t = \int_t^{\tau} \left(\lambda e^{\rho s}\right)^{-\frac{1}{\gamma(s)}} \pi_t^{-\frac{1}{\gamma(s)}}$$
$$\exp\Big\{ -\Big(r + \frac{1}{2}\frac{1}{\gamma(s)}\eta' \cdot \eta\Big)\Big(1 - \frac{1}{\gamma(s)}\Big)\big(s - t\big)\Big\}\frac{l(x+s)}{l(x+t)}ds. \quad (65)$$

  Notice that when the function $\gamma(t) \equiv \gamma$, then the wealth in this equation becomes the same as the wealth we have sn before, as the case should be.

- Clearly the quantity $Y(t)$ can be computed from the expression for W_t in (65) and the function $\gamma(t)$.

- The consequences of this result are several, and the above reference gives the details.

- Here we only point out that if the risk aversion function $\gamma(t)$ is increasing with time, then this result implies that individuals should invest more in the risky asset when they have a longer horizon, i.e., when they are young, and gradually move into bonds as they grow older.

- This is then in agreement with both advice from investment professionals, and with empirical studies of actual behavior.

- It seems natural, with this assumption, that the investor should pick some average time in the remaining horizon when deciding on today's portfolio choice.

A second investment puzzle

- In connection with the optimal investment result (61), there is also another empirical puzzle.

- Mehra and Prescott (1985) studied consumption and market data in the US-economy for the period of 1889-1978. The data are summarized in Table 1. Newer data are of course somewhat different, but the main conclusions remain.

- Using the data of Table 1, for a relative risk aversion of around two, the optimal fraction in equity is 132% based on the standard, first term in (66) (when $el_W(c_t^*) = 1$).

	Expectation	Standard deviation
Consumption growth	1.83%	3.57%
Return S&P500	6.98%	16.67%
Government bonds	0.80%	5,67%
Equity premium	6.18%	16.54%

Table 1: Key numbers for the time period 1889-1978

- In contrast, depending upon estimates, the typical household holds between 6% to 20% in equity. Conditional on participating in the stock market, this number increases to about 40% in financial assets.

- Based on an equilibrium model with production and capital stock (K), and a non-linear production function, Aase (2011) shows, among other things, that the equilibrium demand for the risky asset is given by

$$\varphi_t = \Big(- \frac{u_c(c_t^*)}{u_{cc}(c_t^*)c_t^*} \Big) \frac{1}{el_W(c_t^*)} \frac{\mu_R - r}{\sigma_R \sigma_R} - \frac{el_K(c_t^*)}{el_W(c_t^*)} \frac{\sigma_R \sigma_K}{\sigma_R \sigma_R}, \tag{66}$$

- Here σ_K is the volatility of capital, σ_R is the volatility of the return on equity, μ_R is the expeced rate of return on equity, and $el_W(c_t^*)$ and $el_K(c_t^*)$ are partial consumption elasticities with respect to wealth and capital stock, respectively.

- The first term is seen to be the solution corresponding to (61), reformulated to the present situation, in the case when $el_W(c_t^*) = 1$.

- Implied by results in Aase (2011), the observed risk premium $(\mu_R - r) = 6.18$ and the observed value for the short rate $r = 0.0080$ of the last century follow from the production model for a value of the relative risk aversion of $\gamma = 2.27$, provided the investors only use the first term in (66), with $el_W = 1$.

- The last term in (66) reflects the investor's demand for the risky asset to hedge against unfavorable changes in technology.

- If the investors take into account also information conveyed by the real economy, they find that the stock market may have appeared more risky for the consumers than it really was.

- The consumption based capital asset pricing model still holds in the production economy, which implies that the risk premium should have been closer to 1% than to the observed 6% of the last century for a reasonable value of the relative risk aversion γ.

- If this explanation holds, from the first term alone, φ is down to 20% in equities, for $\gamma = 2.27$, $\mu_R - r = 0.013$ when $el_W = 1$. The last term in (66) further adjusts this number in the right direction.

- Conditional on these results, also the second investment puzzle can be explained.

- Of course, the above mentioned paper aims at solving two other celebrated puzzles;

- The *equity premium puzzle*, and the *risk free rate puzzle*.

- If we accept the results of that paper, all these puzzles are more or less explained.

<u>6. Discussion and extensions</u>

- The life cycle model was analyzed. When there is only biometric risk, the optimal consumption paths are crucially dependent on the impatience rate.

- The impatient consumer ($\rho > r$) must always look forward to an ever decreasing optimal consumption ($dc_t^*/dt = (r - \rho)T(c_t^*)$)

- The patient agent ($\rho < r$) can look forward to an ever increasing optimal consumption.

- While this gives an interesting and intuitive interpretation of the impatience rate, or the subjective interest rate ρ, it is not likely to give reliable predictions.

- When a securities market is allowed, these results are not so strikingly sharp anymore.

- This opens up for interpersonal comparisons of consumption behavior at the same time in agents' life cycles.

- Impatience is more naturally discussed in terms of expectations when a stock market is present, in which case both the market-price-of-risk and the relative risk aversion must be taken into account when characterizing this property.

- The optimal pensions in these two situations differ only by a random factor with a stock market included.

- This factor is reciprocal to the state price density, a fact which was found to have several interesting implications.

- In particular the optimal pensions are found to be positively correlated with the economy in the sense that when stock prices are high, the pensions are also high, and vice versa.

- This is a quite natural property, in particular for the aggregate economy, since such a consumption pattern is consistent with what the economy can deliver.

- The discussion of nominal pensions revealed a weakness with the additive and separable framework of preference representations that this paper is build on.

- The reason is that the parameter γ has two different, and sometimes conflicting interpretations.

- We have a strikingly simple demonstration of the advantages of pooling with regard to pensions. It is shown that, with the same economic resources, the optimal yearly pension is strictly larger with pooling, than without.

- This shows the mutuality idea is still fruitful, a fact that is worth a reminder, in particular since we live in a time of individualism, seemingly picturing a world in which we are solely responsible for our own successes and failures.

- Optimal life insurance, where the insured amount is endogenously determined, is analyzed. Like pension insurance, also the insured amounts in life insurance are co-cyclical with the economy.

- It should be pointed out that we know little about the specification of the function v, when it serves a bequest motive, as compared to u.

- Life insurance is after all only a financial tool for controlling interpersonal transfers, which necessitates references to the theory of transfers (like e.g., Bernheim, Schleifer and Summers (1985)).

- We show that if the insured amount is to be countercyclical to the economy, and thus be a bona fide insurance against tough times for the beneficiaries, this requires *risk proclivity* of the bequest function v.

- This effectively rules out this possibility. A countercyclical insured amount does not appear directly irrational in a finance setting, but risk proclivity does.

- The results are compared to both actuarial theory, and to insurance practice.

- Defined contribution plans without guarantees of any kind are closest to our optimal pension/life insurance results, when the relative risk aversion $\gamma > 1$.

- Finally the paper discusses optimal investment strategies. This culminates with the formula (61), characterizing the optimal investment plan.

- As with all simple formulas, there are pros and cons. The advantage is the simple logic this formula conveys, the drawback is that it is framed in a very simple model of a complete, frictionless financial market, which is, perhaps often taken too literally.

- One particular assumption about this market is that the investment opportunity set is constant. When this is not the case, as in the real world, other state variables must be taken into account.

- We refer to an article where this was done indirectly through a production economy, instead of using only a pure exchange economy that is most common when analyzing such questions.

- When the state variables are capital and labor, the message is that provided the information about these quantities is being utilized, the picture changes and the above formula does not capture all the key elements of the risks.

- The investors have in reality hedging possibilities related to the "real" economy, and when these are properly evaluated, the stock market may not appear quite so risky as it was perceived to be during the last century.

- These insights are then used to explain an investment puzzle, as well as the equity premium puzzle.

- Another weakness with the optimal investment theory is related to the "horizon problem". Here we make the only deviation from the additive and separable preference representation:

- We relax the separability of state and time in the felicity index $u(x, t)$.

- This can be used to explain observed behavior, namely that as investors grow older, they invest a larger proportion of their wealth in government bonds.

- We have considered private pension insurance, or pension insurance underwritten by a company on behalf of its workers.

- Longevity and cohorts risk.

  What about (i) *longevity risk*? (people living longer), or *cohort risk*? (some years have large numbers of retired people)

- Longevity risk the most important of the two: It is structural. Cohort risk is not systematic: It is temporal, or ”seasonal”.

- In some countries the actuarial tables are modified every year, like in Canada, in other countries the same tables as were constructed in 1963 were still used in 2009, like in Norway.

- The theory in this paper assumes that the tables capture the real mortality risk, and pooling works so that there is no economic risk premium associated with mortality.

- As long as the proper measures have been taken regarding reserving for longevity risk, there should be few problems for the private insurance industry with respect to either of these two types of risk.

- For government welfare programs, the situation is of course different. Many developed countries have a social security system that pays a basic pension to its citizens.

- This is usually independent of what the individuals have arranged in terms of pensions from the insurance industry.

- In Norway, for example, this latter component is determined by the principle of "pay as you go":

- In the parliament (Stortinget) each year the politicians determine a basic amount (called one G = NoK 75.641 as of 1.5 - 2010) from which most pensions are based. (This corresponds to AUD 13.088 per year.)

- First; the incentives seem right: The daughters and sons of the beneficiaries determine the benefits. The pensioners obtain what the nation can afford at each time.

- A fund-based system is not likely to be superior: Just recall the current financial/economic crisis. Many people around the world have lost parts of, or their entire pensions.

- Second, what about economic sustainability? Since all pensions are determined from the basic amount G, by making this amount *state dependent*, matters can be arranged such that the nation each year pays the pensions it can afford.

- In practice, to set G lower one year than the previous year may require a great deal of political determination and courage, which means that the system represents no guarantee that the nation will not consume beyond its means. Here rules rather than discretion may be the solution.

- So the problem is really a *policital* one: Do the politicians have the courage to make G state dependent? Rules vs. discretion:

- By increasing the pensionable age a few years, the projected increase in the states' pension expenses may be mitigated, a recent (Norwegian) report shows.

- This report claims that by increasing the pension age by to years, this increases the state's income of about four per cent of GDP.

- For an average working age of 40 years, an increase of two years means that the total work effort in society has increased by five per cent. Society can become five per cent richer if people work two more years.

- However, this is of course very simple, and also controversial. For once, it "assumes away" unemployment, which is not negligible in most western countries. That it is politically difficult, we know from Greece, Ireland, France, Portugal, Spain, etc.

- The problems with longevity and cohort risk are thus seen to have both macro, public, and political economic perspectives.

7. Summary

- In the paper we analyzed the ”life cycle model” in a modern setting, by first introducing a credit market with biometric risk only, and then market risk via risky securities.

- This approach enabled us to investigate pension and life insurance step by step. We found optimal pension plans and life insurance contracts where the benefits were state dependent. We compared these solutions both to the ones of standard actuarial theory, and to policies offered in practice.

- For the standard model with additive and separable utility we demonstrated how risk aversion and intertemporal elasticity of substitution in consumption sometimes play conflicting roles in life and pension insurance.

- We proposed to look at a wider class of utility functions, called recursive utility, to sort out some of these problems. Here we obtained several new insights of relevance for optimal life and pension insurance.

- In the paper we only discussed some of the implications, and certainly there is more to be done on this topic. This analysis constitutes the most novel and original part of this paper.

- We also discussed two related investment puzzles in the light of recent research, one is the horizon problem, the other is related to the aggregate market data of the last century where theory and practice diverge, and suggested resolutions to these problems.

- Finally we presented some comments on longevity risk and cohort risk, and fond that these problems are, perhaps, best analyzed in the perspective of macro, public, and political economics.

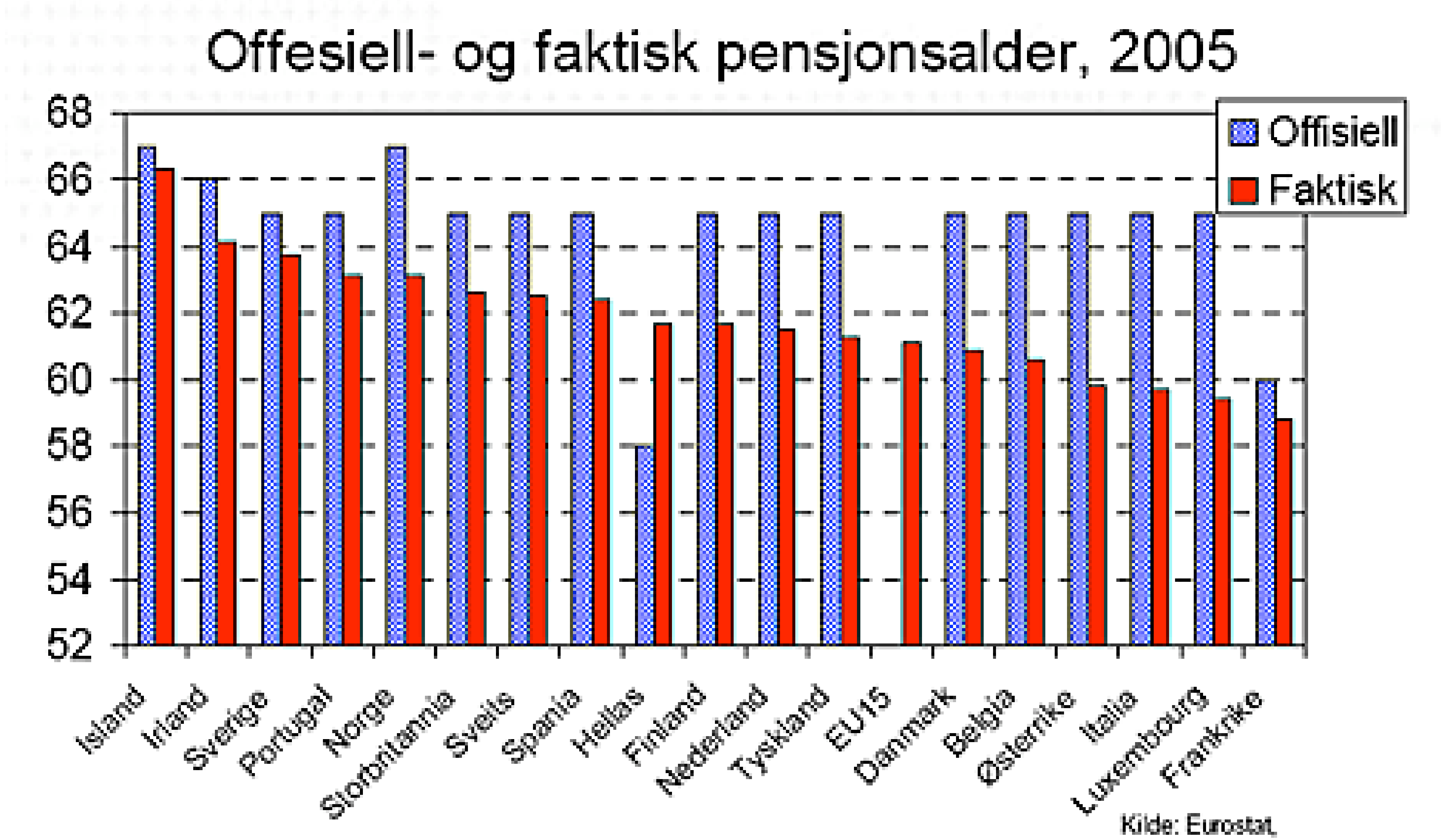

Figure 1: Official and real pension age

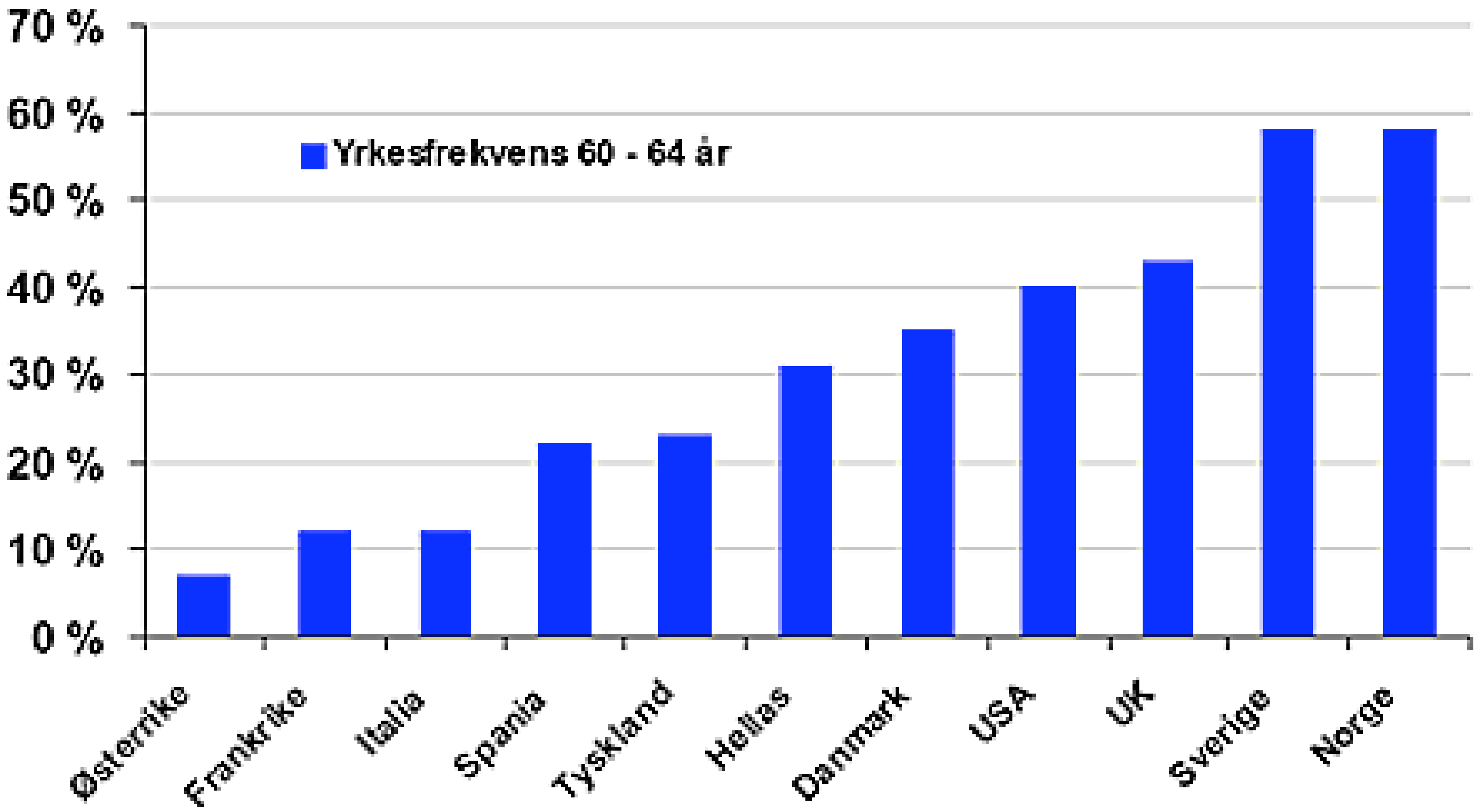

Figure 2: Employment frequency 60-64 years