MODEL FOR CALCULATION OF LIABILITY VALUE ARISING FROM LIFE INSURANCE

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Abstract:
In this paper we propose the fair value model based on the risk neutral calculations methodology corresponding to the hypothetical secondary market of policies. Similarly to the real insurers way of thinking and despite of the most theoretical approaches we do not consider policyholders being financially rational and therefore we do not use option pricing models to evaluate policyholders and embedded options (typically the surrender and paid-up option). After general introduction of the approach, we show possible way of liability value calculation using Gaussian short term interest rates model and incorporating different investment strategies and profit-sharing systems.

Keywords:
Interest rate model, Life Insurance, Liability Evaluation, Surrender Rates

1 INTRODUCTION

Since the introduction of the risk neutral framework for pricing the instruments traded on financial markets there were attempts to prepare similar framework for the life insurance products. However, the insurance community did not accept outcomes of such attempts and have reported the value creation using the embedded value or the accounting principles based on deferral and matching of income and costs (like US GAAP standards for insurance).

We see two main problems of the advanced approaches to the life insurance liability evaluation suggested by the theoreticians.

First problem are the assumptions on “financially rational” behavior of the policyholders (usually represented by pricing the policyholders options to surrender contract similar way as the financial options are priced). In fact it means assumption of arbitrage-free primary market of the life insurance contracts and at the end it leads to the assuming the worst scenarios from the insurer’s viewpoint. However, the everyday practice shows that this assumption is too far from reality and the insurers frequently count on the policyholder’s “irrational” behavior.

The second problem is modeling of the investment strategy and profit sharing system on the liability value. For example the profit sharing rates are often determined by accounting yields, which may, of course, differ from the market interests, or the profit sharing rates might be based on the total profit of the book of the policies. The above-mentioned option-pricing models are incapable to incorporate such effects - otherwise we have to appraise large group of the early-exercisable path-dependant instruments (and the contracts might be moreover mutually connected through the common yield on the provisions or other profit sharing rules). Even though there exist advanced approaches to appraise early-exercisable path-dependant instruments, it is probably impossible to apply them on the real book of policies without neglecting important product features.

On the other hand the insurers identified the need of appropriate evaluation of their financial guarantees and sometimes also try to incorporate the dependence of the policyholder behavior on the financial market situation. However, the surrender rates development rules are usually based on actuarial judgment without (exact) knowledge of the surrender rates distribution inside the model.

Our main target is offering a way to fill the gap between above-mentioned two groups of the liability appraisal models. We introduce the framework to create fair value models for life insurance, which are applicable in practice and allow to model policyholder's behavior

according to the development of the global environment. We believe that the suggested framework is a real base for evaluation of almost all life insurance products – contrary to the option-pricing approaches, which are usually focused on one special simplified product.
Then we show the concrete models based on the above-mentioned framework with the realistic calculation of the profit sharing amounts according to the accounting and profit sharing rules used by many companies. Especially we introduce representation of the bonds portfolio backing life reserves allowing calculation of the financial revenues through the amortization of the bond and show the simulation results for one contract.

2 FRAMEWORK FOR THE FAIR VALUE MODELS

We assume that all the cash flows arising from the book of contracts are composed of the cash flows arising from individual contracts, so we can focus on the fair value of liabilities arising from particular contracts. The fair value of the liability arising from the portfolio is then calculated by summing up the fair values of the individual contracts. In practice it may be necessary to model whole book of with-profit policies together to set up realistic profit sharing rates, but the required changes are not crucial for the below introduced model.
At first we will formulate the very general framework allowing the stochastic consideration of all effects with impact on the policy cash flow. Our model will be a generalized version of the multidecremental model with m causes of decrement described (in its traditional form) by [Gerber]. In subsequent we will assume that the decrements represent all premature policy termination causes (i.e. also the termination on policyholder's demand).
Let $\left(\Omega, \mathrm{G}_\infty, \{\mathrm{G}_t \,|\, 0 \le t\}, \tilde{P}\right)$ be the probability space with filtration $\{\mathrm{G}_t \,|\, 0 \le t\}$ and let t denote the time from the valuation date. We will introduce following random variables and processes

- $\tilde{T}$ ... time of the decrement,
- $\tilde{J}$ ... cause of the decrement ($\tilde{J} : \Omega \to \{1, 2, \ldots, m\}$),
- $C_j(t)$ ... claim payment in case of the decrement by the j-the cause at the time t,
- $E^M(t)$ ... instantaneous rate of maintenance expenses at the time t (or more generally instantaneous rate of all cash flows arising from the contract in force except the premium and annuity payments),

- $E_j^C(t)$ ... claim processing expenses in case of the decrement by the *j*-th cause at the time *t* (or more generally value of all cash flows arising from the contract in such case except the claim payment – for example the claw back of the commission),
- $\Pi(t)$ ... instantaneous rate of premium and annuity payments at the time *t*,
- $r(t)$ ... the instantaneous rate of interest at the time t.

In case that there is defined policy term *N* (from the evaluation date), let *C*(*N*) denote the value of the payment in case of the contract maturity. All the above-defined processes are assumed as $\{\mathrm{G}_t | 0 \le t\}$ adapted, $\sigma\{(\tilde{T}, \tilde{J}) | \tilde{T} \le t\} \subseteq \mathrm{G}_t$ for any nonnegative *t* and $C(N)$ is G_N measurable random variable. Let *B*(*t*) denote the value of bank account with initial amount of one unit, i.e. $B(t) = \exp\left\{\int_0^t r(s)\,\mathrm{d}s\right\}$

Let $\tilde{Q}$ be the risk neutral probability measure (also called equivalent martingale measure). As we have stated above, we do not reckon the primary life-insurance market as arbitrage free. We will use the risk neutral probability measure corresponding to the hypothetical secondary insurance market, i.e. measure implied by the IAS definition of fair value as *the amount for which an asset could be exchanged, or a liability settled, between knowledgeable, willing parties in an arm's length transaction*.

We will also assume that the third party has to adopt the profit-sharing policy of the original insurer and takeover the assets portfolio underlying the life reserves - in other words we assume preservation of the original profit-sharing system without any inconsistencies. We consider all future payments from the profit sharing system as a liability of the insurer regardless of their legal enforceability since we consider all expected future profit shares as constructive obligation of the insurer.

Because we want to calculate under the natural probability measure, we have to use the deflator technique described by [Mandl] or by [Duffie]. We can simply say that the deflator is the stochastic discounting factor defined (in the continuous case under some technical restrictions) as

$$\tilde{D}(t) = \mathrm{E}\left[\frac{d\tilde{Q}}{d\tilde{P}}\middle| \mathrm{G}_t\right] B^{-1}(t).$$

Now we can write general formulae for the random variable representing "fair" present value of the future cash flows from the insurance contract

$$Value = \left[C_J(\tilde{T}) + E_J^C(\tilde{T})\right]\tilde{D}(\tilde{T}) - \int_0^{\tilde{T}} (\Pi(t) - E^m(t))\tilde{D}(t)\,\mathrm{d}t, \tag{2.1}$$

or

$$\begin{aligned} Value &= \left[C_J(\tilde{T}) + E_J^C(\tilde{T})\right]\tilde{D}(\tilde{T}) - \int_0^{\tilde{T}} (\Pi(t) - E^M(t))\tilde{D}(t)\,\mathrm{d}t \quad \tilde{T} < N, \\ Value &= C(N)\,\tilde{D}(N) - \int_0^{N} (\Pi(t) - E^M(t))\tilde{D}(t)\,\mathrm{d}t \quad \tilde{T} \geq N \end{aligned} \tag{2.2}$$

in case that there is the policy term N specified in the contact. The mean of the above defined random variable is the fair value of the liabilities arising from the contract, i.e.

$$FairValue = \mathrm{E}_{\tilde{P}}\, Value\,. \tag{2.3}$$

Above introduced framework is very general and allows to implement wide range of specific models including the models with the similar approach to the policy-holder options as the approach used by financial markets in the case of american options. However, due to the above mentioned reasons we will narrow our framework by specifying effects relevant for pricing on the hypothetical secondary market of insurance contracts.

We will assume that the sides from the IAS Framework definition of the fair value do not consider stochastic development of factors that are specific for single policyholders and we will focus only on the factors common for all policyholders in the assessed book of policies (referred as non-specific factors in subsequent). For example we will not consider development of surrender probability (or occurrence) for each individual contract, but will consider the development of surrender rates for whole book of contracts (similarly for mortality assumptions etc). This approach corresponds to the practice of life insurers when assessing the liabilities from the insurance contracts. Stochastic consideration of non-specific variables does also mean that we assess all embedded derivatives according to the definition of [Engeländer]. Examples of the non-specific factor are the state of financial market, the expected future profit-shares or population mortality development.

Let F_t denote the history of the non-specific factors relevant for the assessment in the time interval $\langle 0, t\rangle$. Then $(\mathrm{F}_t | 0 \leq t)$ is filtration and for any nonnegative t holds $\mathrm{F}_t \subseteq \mathrm{G}_t$. We will move from the probability space $\left(\Omega, \mathrm{G}_\infty, \{\mathrm{G}_t | 0 \leq t\}, \tilde{P}\right)$ to the space $\left(\Omega, \mathrm{F}_\infty, \{\mathrm{F}_t | 0 \leq t\}, P\right)$ where P is the measure identical with the $\tilde{P}$ on F_∞. Similarly Q will be the restricted risk neutral measure, i.e. Q is the probability measure on $\left(\Omega, \mathrm{F}_\infty\right)$ identical with $\tilde{Q}$ on F_∞.

For simplicity we assume that the insurance company determines the attributed profit shares according to the non-specific factors and that the maintenance expenses are considered being common for the whole book of contracts, i.e. that the $C_j(t), E^M(t), E_j^C(t), \Pi(t), r(t)$ are F_t adapted random processes (and $C(N)$ is F_N measurable in case of the existing policy term N).

We will also specify the model of policyholder behavior according to the development considered in F_t. We assume that

$$\tilde{Q}\left(\tilde{T} \geq t \,\middle|\, \mathrm{F}_t\right) = \exp\left\{-\int_o^t (\mu_1(s) + \ldots + \mu_m(s))\,\mathrm{d}\,s\right\} \tag{2.4}$$

$$\lim_{h \to 0+} h^{-1}\,\tilde{Q}\left(\tilde{J} = j, t \leq \tilde{T} \leq t+h \,\middle|\, \mathrm{F}_t\right) = \mu_j(t) \exp\left\{-\int_o^t (\mu_1(s) + \ldots + \mu_m(s))\,\mathrm{d}\,s\right\} \tag{2.5}$$

where $\mu_1(s), \ldots, \mu_m(s)$ are continuous, F_t adapted random processes. The relation between the non-specific factors and the demographic development represented by $(\tilde{T}, \tilde{J})$ is fully given by $\mu_1(s), \ldots, \mu_m(s)$, i.e.

$$\left(\sigma\{\tilde{T}, \tilde{J}\} \cap \mathrm{F}_t\right) \subseteq \sigma\{\mu_1(s), \ldots, \mu_m(s) \mid s \leq t\}.$$

For subsequent calculations we will need process of insured sum in force at time t

$$K(t) = \exp\left\{-\int_o^t (\mu_1(s) + \ldots + \mu_m(s))\,\mathrm{d}\,s\right\} \tag{2.6}$$

and deflator restricted to F_t

$$D(t) = \mathrm{E}\left[\tilde{D}(t) \,\middle|\, \mathrm{F}_t\right] = \mathrm{E}\left[\frac{d\tilde{Q}}{d\tilde{P}} \,\middle|\, \mathrm{F}_t\right] B^{-1}(t).$$

Now we can move to the valuation in the space $\left(\Omega, \mathrm{F}_\infty, (\mathrm{F}_t \mid 0 \leq t), P\right)$ by introducing random variable *ValueF*

$$ValueF = \mathrm{E}\left[Value \,\middle|\, \mathrm{F}_\infty\right]$$

$$\text{or} \quad ValueF = \mathrm{E}\left[Value \,\middle|\, \mathrm{F}_N\right]$$

in case that there is the policy term N.

By substitution of *Value* and calculation we come to the expression

$$ValueF = \sum_{j=1}^{m} \int_o^\infty \left[C_j(t) + E_j^C(t)\right] \mu_j(t) K(t) D(t)\,\mathrm{d}\,t - \int_0^\infty \left[\Pi(t) - E^M(t)\right] K(t) D(t)\,\mathrm{d}\,t$$

or

$$ValueF = \sum_{j=1}^{m} \int_o^N \left[C_j(t) + E_j^C(t)\right] \mu_j(t)\, K(t)\, D(t)\,\mathrm{d}\,t + C(N)\, K(N)\, D(N) - \int_0^N \left[\Pi(t) - E^M(t)\right] K(t) D(t)\,\mathrm{d}\,t \tag{2.7}$$

in case that there is the policy term *N*.

Instead of the calculation according to (2.3) we define fair value as

$$FairValue = \mathrm{E}_P\left[ValueF\right]. \tag{2.8}$$

We came to the practicable fair-value definition. However there is only a general framework described above. In the subsequent section we show models that allow implementing the dependence between financial market situation and policyholder's behavior and also faithful modeling of the profit-sharing development.

3 FAIR VALUE MODELS

Now we focus on the fair value of the with-profits contracts. First of all we will specify causes of decrements, assumed cash flows and the relevant non-specific factors. To simplify the subsequent formulae we will consider only the payments between insurer and policyholder (i.e. premium, claim payments and surrender payments). The only considered non-specific factor will be the development of the financial market.

Only two causes of decrement are assumed further – the death of the insured and the surrender. We denote $\mu^D(t)$ and $\mu^S(t)$ corresponding instantaneous rates from (2.4) and (2.5) where the $\mu^D(t)$ will be the deterministic function of time and $\mu^S(t)$ process dependent on the financial market development.

Without any real loss of generality we will assume that there is always the policy term *N*. We will also introduce the processes representing present value of particular cash flows. Let $FV_P(t)$ denote present value of premium paid in the time interval (0,*t*), $FV_D(t)$ and $FV_S(t)$ denote present value of death claims and surrender payments in the same interval. According to (2.7) we can write

$$\mathrm{d}\,FV_P(t) = K(t)\,D(t)\,\Pi(t)\,\mathrm{d}\,t,$$

$$\mathrm{d}\,FV_D(t) = K(t)\,C_D(t)\,\mu^D(t)\,D(t)\,\mathrm{d}\,t,$$

$$\mathrm{d}\,FV_S(t) = K(t)\,C_S(t)\;\mu^S(t)\,D(t)\,\mathrm{d}\,t,$$

where $C_D(t)$ and $C_S(t)$ are payments in case of death and surrender. According to the definition $FV_P(0) = FV_D(0) = FV_S(0) = 0.$

In case that the FV_M denotes present value of maturity payment $FV_M = K(N)\,C(N)\,D(N)$, formulae (2.7) can be rewritten to the form

$$ValueF = FV_M + FV_D(N) + FV_S(N) - FV_P(N).$$

3.1 Market processes models

We will dominantly focus on the interest rate models, because the insurance companies do primarily invest into debt instruments as bonds and treasuries. For the faithful modeling of profit shares we need models that allow easy calculation of bond price at any time. For this reason and also to keep the calculations simple we have decided to use instantaneous interest rate diffusion models with normal distribution of instantaneous interest rate $r(t)$. We will briefly introduce three models ordered according to their complexity. Let $P(t,T)$ denote the price of zero-coupon bond maturing at T at time t and $f(t,T)$ denote forward interest rate at t.

3.1.1 Hull-White model

The simplest model we assume is a generalized form of the famous Vasicek model. The SDE of the $r(t)$ is

$$\mathrm{d}\,r(t) = \big(\theta(t) - a\,r(t)\big)\mathrm{d}\,t + \sigma\,\mathrm{d}\tilde{W}(t), \tag{3.1}$$

where $\tilde{W}(t)$ is Q Wiener process, a,σ are constant positive parameters and $\theta(t)$ is the function calibrating the model yield curve at time zero to the curve observed on the market. This model provides explicit formulae for $\theta(t)$, bond prices and forward interests.

We will use this model under the natural probability measure. Therefore we have to rewrite (3.1) to the form

$$\mathrm{d}\,r(t) = \big(\theta(t) + \lambda\sigma - a\,r(t)\big)\mathrm{d}\,t + \sigma\,\mathrm{d}W(t), \tag{3.2}$$

where $W(t)$ is the P-Wiener process and λ is the risk price corresponding to the process $W(t)$. The deflator $D(t)$ is then defined as

$$D(t) = \exp\left\{-\int_0^t \big(r(z) + 0{,}5\,\lambda^2\big)\mathrm{d}\,z - \lambda W(t)\right\}.$$

The disadvantage of this model is closely related to its simplicity. When we look at the forward interest SDE

$$\mathrm{d}\, f(t,T) = \sigma\, e^{-a(T-t)}\, \mathrm{d}\widetilde{W}(t) + \frac{\sigma^2}{a}(1 - e^{-a(T-t)})\, e^{-a(T-t)}\, \mathrm{d}t$$

we see that the yield curve development is very simple, especially that the forwards for the different maturities are perfectly correlated and that the long-term forwards may be almost stable (for higher values of a) or that the volatility of $r(t)$ may grow into unrealistic heights (for lower a). We will therefore introduce two models that help to overcome this problem.

3.1.2 Simple two-factor gaussian model

The possible solution for the above-mentioned disadvantage is to introduce a model where the forward rates are driven by two (or more) Wiener processes with different impact on the different parts of the yield curve. At first we will introduce model inspired by [Baxter, Rennie]. The instantaneous interest rate is defined as

$$r(t) = \varphi(t) + x(t) + y(t), \tag{3.3}$$

where $\varphi(t)$ is deterministic function allowing calibration of the model to the initial yield curve and $x(t)$ and $y(t)$ are stochastic processes with initial value equal to zero and have stochastic differential equations under the risk-neutral probability measure

$$\mathrm{d}\, x(t) = \sigma_1\, \mathrm{d}\widetilde{W}_1(t) \text{ and}$$

$$\mathrm{d}\, y(t) = -a\, y(t)\, \mathrm{d}t + \sigma_2\, \mathrm{d}\widetilde{W}_2(t).$$

The forward rates development is given by

$$\mathrm{d}\, f(t,T) = \sigma_1\, d\widetilde{W}_1(t) + \sigma_2\, e^{-a(T-t)}\, \mathrm{d}\widetilde{W}_2(t) + \\ + \left[\sigma_1^2 (T-t) + \frac{\sigma_2^2}{a}(1 - e^{-a(T-t)})\, e^{-a(T-t)} \right] \mathrm{d}t.$$

We observe that the shocks caused by the process $\widetilde{W}_1(t)$ have the same impact on all points of the yield curve while the shocks caused by $\widetilde{W}_2(t)$ have much stronger influence on the left side (i.e. on the forwards with a short time to maturity).

Again we have to rewrite stochastic equations using Wiener processes under the natural probability P $(W_1(t), W_2(t))$

$$\mathrm{d}\, x(t) = \lambda_1 \sigma_1\, \mathrm{d}t + \sigma_1\, \mathrm{d}W_1(t) \text{ and}$$

$$\mathrm{d}\, y(t) = \lambda_2 \sigma_2\, \mathrm{d}t - a\, y(t)\, \mathrm{d}t + \sigma_2\, \mathrm{d}W_2(t),$$

where λ_1, λ_2 are corresponding risk prices. The deflator is

$$D(t) = \exp\left\{-\int_0^t \left(r(z) + 0{,}5(\lambda_1^2 + \lambda_2^2)\right)\mathrm{d}z - \lambda_1 W_1(t) - \lambda_2 W_2(t)\right\}$$

3.1.3 General two-factor gaussian model

This model is the most general model we consider. It is intimately described by [Brigo, Mercurio]. Similarly to the previously defined simple version the interest rate is

$$r(t) = \varphi(t) + x(t) + y(t),$$

where $\varphi(t)$ is the deterministic function allowing calibration of the model to the initial yield curve and $x(t)$ and $y(t)$ are stochastic processes with initial value equal to zero and have stochastic differential equations under the risk-neutral probability measure

$$\mathrm{d}x(t) = -a_1 x(t)\,\mathrm{d}t + \sigma_1\,\mathrm{d}\tilde{W}_1(t) \text{ and}$$

$$\mathrm{d}y(t) = -a_2 y(t)\,\mathrm{d}t + \sigma_2 \rho\,\mathrm{d}\tilde{W}_1(t) + \sigma_2 \sqrt{1-\rho^2}\,\mathrm{d}\tilde{W}_2(t).$$

This model allows various shapes of the volatility curve (especially allows humped volatility curve that is often observed on the derivative markets) and therefore the model is recommended as a simple model suitable to calculate fair prices of interest derivatives.

Under our assumptions of constant risk prices λ_1, λ_2 the processes $x(t)$ and $y(t)$ develop according to equations

$$\mathrm{d}x(t) = \left(\lambda_1\sigma_1 - a_1 x(t)\right)\mathrm{d}t + \sigma_1\,\mathrm{d}W_1(t) \text{ and}$$

$$\mathrm{d}y(t) = \left(\lambda_1 \sigma_2 \rho + \lambda_2 \sigma_2 \sqrt{1-\rho^2} - a_2 y(t)\right)\mathrm{d}t + \sigma_2 \rho\,\mathrm{d}W_1(t) + \sigma_2 \sqrt{1-\rho^2}\,\mathrm{d}W_2(t),$$

where $\left(W_1(t), W_2(t)\right)$ is a two dimensional Wiener process and the deflator formula is the same as in the case of simple two-factor model.

3.1.4 Other assets categories

According to the principle of parsimony we do not expect the insurance companies to model the yield development for high number of different assets (or asset classes). We support modeling of only few "not-interest" asset categories (and only in case that such categories represent a significant part of the assets backing the life reserves). The suggested approach is to consider only bonds and shares yields (and sometimes also property and foreign currencies yields).

Let S_i denote price of i-th category unit. We assume that the price development is given by

$$\mathrm{d}\,S_i(t) = S_i(t)\left[r(t)\,\mathrm{d}\,t + \omega_i\,\mathrm{d}\,\tilde{Z}_i(t)\right],$$

where ω_i is the volatility corresponding to the i-th category and $\tilde{Z}_i(t)$ is a Wiener process under the risk-neutral measure Q. We can assume fixed correlations between the interest rate and the considered asset category $\mathrm{d}\,\tilde{Z}_i(t)\,\mathrm{d}\tilde{W}_j(t) = \hat{\rho}_{ij}\,\mathrm{d}\,t$ and correlations between different asset categories $\mathrm{d}\,\tilde{Z}_i(t)\,\mathrm{d}\,\tilde{Z}_j(t) = \overline{\rho}_{ij}\,\mathrm{d}\,t$.

Because the dominant part of assets backing life reserves consists of the debt instruments as bonds and bank deposits, we will consider only one non-debt asset category with the unit price $Z(t)$. For simplicity we will talk about this category as about shares.

3.2 Modeling of policyholder's behavior

Now we will suggest a simple method to model changes in policyholder's behavior expressed by $\mu_1(t),\ldots,\mu_m(t)$ due to the development of global processes which history defines F_t.

Because we assume that only the financial market processes are the relevant global processes, we will assume deterministic mortality $\mu^D(t)$. We have to point out that $\mu^D(t)$ is not the expected mortality, but it is already adjusted by the risk margins according to (2.4) and (2.5).

We assume that there is a connection between the state of the financial market and the policyholder's tendency to surrender policy. The hardest task during creation of the fair value model is to set-up appropriate relation between the surrender rates and the financial market development (and the expected future profit-shares).

In the subsequent we will suggest very simple dependence – the big advantage of the below-introduced approach is the easy analytical tractability of the surrender rate development allowing exactly controlling its behavior and simple interpretation of the dependence.

It is clear that the main input for the policyholder's decision is the yield curve development (since the history of shares prices provides no information about a future development). We will assume that the policyholder has higher motivation to continue in case of lower interest rates because the minimum interest guarantee is more valuable than in case of higher interest rates.

Let $\nu(t)$ denote the expected force of surrender adjusted by the risk margins (corresponding to the policyholder specific risk of surrender). We require the probability of the "contract survival" defined by deterministic functions $\nu(t)$ and $\mu^D(t)$ to be equal to the expected "contract survival" probability arising from $\mu^S(t)$ and $\mu^D(t)$, i.e.

$$\exp\left\{-\int_0^t \left(\mu^D(z)+\nu(z)\right)\mathrm{d}\,z\right\} = \mathrm{E}\exp\left\{-\int_0^t \left(\mu^D(z)+\mu^S(z)\right)\mathrm{d}\,z\right\}. \tag{3.4}$$

3.2.1 Surrender development for Hull-White model

A simple possibility to define $\mu^S(t)$ development is to introduce stochastic equation

$$\mathrm{d}\,\mu^S(t) = \left(\eta(t) - a\,\mu^S(t)\right)\mathrm{d}\,t + k\,\sigma\,\mathrm{d}W(t), \tag{3.5}$$

where $W(t)$ is the same *P*-Wiener process as in (3.2). Also the parameters a, σ are the same as in the case of interest rate. $k > 0$ is the ratio between the volatility of the interest rate *r*(*t*) and $\mu^S(t)$. The deterministic function $\eta(t)$ is determined by (3.4). Of course $\mu^S(0) = \nu(0)$.

When we compare expressions

$$r(t) = r(0)e^{-at} + e^{-at}\int_0^t e^{az}(\theta(z)+\lambda\sigma)\,\mathrm{d}\,z + \sigma e^{-at}\int_0^t e^{az}\,\mathrm{d}W(z),$$

$$\mu^S(t) = \nu(0)e^{-at} + e^{-at}\int_0^t e^{az}\eta(z)\,\mathrm{d}\,z + k\sigma e^{-at}\int_0^t e^{az}\,\mathrm{d}W(z)$$

we see that the stochastic parts are equal except the multiplication by *k*. It does mean that both *r*(*t*) and $\mu^S(t)$ are sums of a deterministic function and a markov process with zero expected value and that the markov process is same for *r*(*t*) and $\mu^S(t)$ except the multiplication by *k*.

The disadvantage of this simple approach is the possibility of negative force of surrender. In case of low $\nu(t)$ for some *t* it is necessary to (at least) introduce a non-constant parameter $\sigma(t)$ in (3.5) (or even parameters $\sigma(t)$ and $a(t)$). Unluckily we are losing the simple interpretation of the relation between *r*(*t*) and $\mu^S(t)$. We can, of course, use completely different way to define $\mu^S(t)$ (for example linked to the ratio between market and guaranteed interest rate or to expected future profit shares), but we will probably get much more complicated and analytically untractable surrender rate process.

3.2.2 Surrender development for simple two-factor model

Similarly to the previous case we will adopt a simple approach based on the multiplication of the processes $x(t)$ and $y(t)$ by constants k_1 and k_2. The force of surrender is then

$$\mu^S(t) = \psi(t) + k_1\, x(t) + k_2\; y(t),$$

where the function $\psi(t)$ guarantees validity of (3.4).

As we can see, the relation between $r(t)$ and $\mu^S(t)$ is similar as in the Hull-White model. The advantage is that the parameters k_1 and k_2 do not only determine the policyholder's sensitivity on the development of the interest rate, but also his sensitivity on the movements of different parts of the yield curve. For example k_1 much higher than k_2 means that the policyholder decisions depend mainly on the long-term yields.

Also in this case we face the problem of negative force of surrender. Again it can be partially solved by replacing k_1 and k_2 by functions $k_1(t)$ and $k_2(t)$ or by complete change of the $r(t)$ and $\mu^S(t)$ relation.

3.2.3 Surrender development for general two-factor model

The situation is nearly the same as in case of the simple two-factor model. Therefore we use the same expression for $\mu^S(t)$

$$\mu^S(t) = \psi(t) + k_1\, x(t) + k_2\; y(t),$$

where the specifications of k_1, k_2 and $\psi(t)$ are the same as in the section 3.2.2.

3.3 Profit sharing systems

One of the hardest (and often underestimated) problems of life portfolio projections is the corresponding modeling of provided profit shares according to the market development. Especially in case of traditional with-profits contracts the relation between the market yields and profit sharing rates is really complicated due to the insurer's internal accounting and profit sharing rules and also sometimes due to the discretionary features of the profit-sharing system - it is even often that the management determines the profit-sharing rates without explicit connection to a (however defined) part of the company profit.

For modeling purposes we have to replace the discretionary features by some deterministic rules derived mainly from the past practice of the company. However, the companies do not usually base their decisions on pure market yields and use accounting yields instead. Especially the profit-sharing policies usually recognize the revenues from bonds using the amortization instead of the bond market price development (what significantly decreases the volatility of the return from investments and subsequently decreases the possibility that the interest rate guarantees provided to clients will be in-the-money).

For that reason (and also because the bonds represent the dominant part of the assets backing life reserves) we will focus mainly on the profit-sharing behavior in case that the statutory reserves are backed by bonds, which price is recognized by amortization.

3.3.1 General assumptions about the profit sharing system

We will assume that the insurer does distribute a part of the financial profit to the shareholders. We will also assume that the company uses the bank account profit-sharing system (i.e. the credited profit shares are held on a separate account and are paid together with the claim or surrender payment from the primary insurance).

Let $V(t)$ denote statutory reserve of the original contract. $V(t)$ is determined by Thiele differential equation and the contractual premium $\Pi(t)$, the claim payment $c(t)$, the technical interest rate δ and the 1st order mortality. $V(t)$ is the base to calculate profit shares and the surrender payment.

Deliberately we have decided that all below described profit sharing systems do meet the condition that *PSRatio* of the financial profit is allocated to the policyholders profit shares (or are marketed this way at least). As we will see, there are vast differences between the liability value arising from the systems despite of this unifying condition.

Let us denote $PS(t)$ the value of the credited profit shares corresponding to one unit of the sum insured and we assume that $PS(t)$ fulfils equation (except the case of deferred crediting)

$$\mathrm{d}\,PS(t) = \left[PSRatio\left(V(t) + PS(t)\right)\left(RatePS(t) - \delta\right)_+ + PS(t)\,\delta\right]\mathrm{d}\,t, \qquad (3.6)$$

where $RatePS(t)$ is the rate corresponding to the accounting interest return on financial assets backing the life reserves (and $(X)_+$ is the maximum of 0 and X).

For simplicity we specify the formulae for the claim and surrender payments

$$\begin{aligned} C_D(t) &= c(t) + PS(t), \\ C_S(t) &= \left(1 - \gamma(t)\right)V(t) + PS(t), \\ C(N) &= V(N) + PS(N), \end{aligned} \qquad (3.7)$$

where $\gamma(t)$ determines the surrender penalty at time *t*. Now we will focus on the faithful modeling of the *RatePS*(*t*) according to the market state and the investment strategy of the insurer.

3.3.2 Reserves backed by the bank account

The simplest approach is to assume that the insurer invests only in the bank account. Under any accounting system the interest yield is then the instantaneous interest rate *RatePS*(*t*) = *r*(*t*) and from (3.7) we get

$$\mathrm{d}\,PS(t) = \left[(V(t) + PS(t))(r(t) - \delta)_+ + PS(t)\,\delta\right]\mathrm{d}\,t.$$

Assumption of this investment strategy is obviously unrealistic but we suppose it closer to the reality than the sometimes-used assumption of all investments in the shares-like assets.

3.3.3 Reserves backed by the shares

Even though we do not find it corresponding to the real situation, we will now define the system based on the assumption of all reserves invested into shares. However we will use the introduced approach when we will assume the investing concurrently into the bonds and shares.

It is not possible to define continuous increase of credited profit shares as in (3.7) in this situation. We have to assume that the financial profit is calculated periodically in the equidistant times and that the insurer determines the increase of *PS*(*t*) at such times. Let denote *h* the considered period. The development of *PS*(*t*) is then determined by

$$PS(kh) = PSRatio\left(V((k-1)h) + PS((k-1)h)\right)\left(\frac{S(kh)}{S((k-1)h)} - e^{\delta h}\right)_+ + PS((k-1)h)\,e^{\delta h}$$

for all $k = 1, 2, \ldots$ when for all $t > 0$

$$PS(t) = PS([h^{-1}t]h).$$

It does mean that the process *PS*(*t*) has jumps in the multiples of *h* and is constant elsewhere.

3.3.4 Reserves backed by the bonds maturing at the end of policy

As we have said, the life insurers invest primarily into the long-term bonds. The simplest way to implement such strategy is to assume that the insurer holds only the zero-coupon bonds maturing at the time *N* (i.e. the time of the policy maturity). For any time *t* between 0 and *N* we require that the bond portfolio has the accounting value at least equal to the statutory reserve $K(t)\big(V(t)+PS(t)\big)$ and also that the insurer does not sell the bonds in case that the portfolio accounting value is higher than the statutory reserve.

We assume that $K(t)\big(V(t)+PS(t)\big)$ has continuous trajectories with existing right derivatives. Let *BookV*(*t*) denote the accounting value of the bond portfolio at the time *t* and *FaceV*(*t*) the corresponding face value. We will also denote *AF*(*t*) the amortization rate corresponding to the portfolio at the time *t*

$$AF(t) = \frac{1}{N-t}\ln\left(\frac{FaceV(t)}{BookV(t)}\right).$$

No new bonds are bought in case that the accounting value of the portfolio is higher than the statutory reserve, i.e.

$$\text{if } BookV(t) > K(t)(V(t)+PS(t)) \text{ then}$$

$$\mathrm{d}\,FaceV(t) = 0 \text{ and}$$

$$\mathrm{d}\,BookV(t) = AF(t)\,BookV(t)\,\mathrm{d}\,t.$$

Otherwise company buys new bonds to keep equality between the statutory reserve and the accounting value of the bond portfolio, i.e.

$$\text{if } BookV(t) = K(t)(V(t)+PS(t)) \text{ then}$$

$$\mathrm{d}\,FaceV(t) = P^{-1}(t,N)\left(\frac{\partial\big[K(t)\big(V(t)+PS(t)\big)\big]}{\partial_+ t} - AF(t)\,BookV(t)\right)_+ \mathrm{d}t \text{ and}$$

$$\mathrm{d}\,BookV(t) = \left[\left(\frac{\partial\big[K(t)\big(V(t)+PS(t)\big)\big]}{\partial_+ t} - AF(t)\,BookV(t)\right)_+ + AF(t)\,BookV(t)\right]\mathrm{d}t.$$

The above considered investment strategy allows building up a relatively simple model, but in practice we will observe significant differences between the model and the reality even at the beginning because the real bond portfolio will be probably quite different from the modeled one.

There might be also problems in case of modeling asset portfolio backing reserves for whole book of contracts, which (of course) includes contracts with different maturities. For that reason it would be more precise to adapt the general model of the bond portfolio described in the next subsections.

3.3.5 General model of the bond portfolio

For the purposes of the calculation we set following representation of the bond portfolio:

- *FaceV*(t,s) denotes the total face value of (zero coupon) bonds held at time t with maturity at time s or earlier,
- *BookV*(t,s) represents then the total book value at time t of (zero coupon) bond with face value 1 and maturity at time s. The appropriate amortization factor is then

$$AF(t,s) = \frac{1}{s-t}\ln\left(\frac{1}{BookV(t,s)}\right).$$

The basic requirement is simple – at any time t between 0 and N the book value of the bond portfolio has to be equal to the statutory reserve at the same time, i.e.

$$K(t)\big(V(t) + PS(t)\big) = \int_t^{\infty} BookV(t,s)\ \mathrm{d}_s\ FaceV(t,s), \tag{3.8}$$

We assume (of course) validity of (3.8) for $t = 0$. More generally we can assume that the statutory reserve has to be equal or lower than the book value of bonds and prevent possible sales of bonds from the portfolio but we will neglect this effect since it is not so important according to our calculations.

To implement the model in practice we have to specify the investment strategy and find out equations for *FaceV*(t,s) and *BookV*(t,s) assuring (3.8). It is necessary being cautious about the complexity of the strategy to keep the model computable. Therefore we have developed it for the simple investment strategy that should be sufficient in most cases.

3.3.6 Purchase of bonds with constant time to maturity

Let's assume that at any time t the insurer buys (zero coupon) bonds with time to maturity D, i.e. maturing at time $t+D$. For simplicity we also assume that there are no bonds with maturity at time D or later in the portfolio at the beginning of the calculation (this assumption is just technical and can be eliminated by splitting the portfolio into the part bought before the calculation date and the rest).

Then it is simple to define the development of *BookV*(t,s)

$$\begin{aligned} &\frac{\partial BookV}{\partial t}(t,s) = AF(t,s)\,BookV(t,s) \quad s < t + D, \\ &BookV(t,s) = P(s-D,s) \quad s \ge t + D. \end{aligned} \tag{3.9}$$

Let's assume that *FaceV*(*t*,*s*) is continuous function on $<0,T>\times<0,T+D>$ with *t*-derivatives and right *s*-derivatives. (3.8) is valid in case that

$$\begin{aligned}
&\frac{\partial FaceV}{\partial t}(t,z)=-\frac{\partial FaceV}{\partial_+ s}(t,t) \quad z<t+D,\\
&\frac{\partial FaceV}{\partial t}(t,z)=\left[\frac{\partial\big(K(t)\big(V(t)+PS(t)\big)\big)}{\partial t}-\int_t^{t+D}\frac{\partial BookV}{\partial t}(t,y)\,\mathrm{d}_y\,FaceV(t,y)+\right. \qquad (3.10)\\
&\left.+\big(1-P(t,t+D)\big)\frac{\partial FaceV}{\partial_+ s}(t,t)\right]\frac{1}{P(t,t+D)} \qquad z\ge t+D
\end{aligned}$$

when the statutory reserve $K(t)\big(V(t)+PS(t)\big)$ is continuous. This condition is satisfied in case that *PS*(*t*) is continuous (for example in case that *PS*(*t*) develops according to (3.6)).

However, in practice we will sometimes need a model allowing jumps in the bond distribution represented by *FaceV*(*t*,*s*) – for example in case that we will consider shares in the asset portfolio or to incorporate the starting portfolio precisely. Therefore we split *FaceV*(*t*,*s*) in two functions - $FaceV^{Cont}(t,s)$ and $FaceV^{Discr}(t,s)$.

$$FaceV(t,z)=FaceV^{Discr}(t,z)+FaceV^{Cont}(t,z).$$

$FaceV^{Cont}(t,s)$ matches up to conditions for *FaceV*(*t*,*s*) in previous "continuous" situation (i.e. is continuous and the required derivatives exists). For any *t* $FaceV^{Discr}(t,s)$ is constant relative to *s* except the finite number of discontinuity points, where $FaceV^{Discr}(t,s)$ is right-continuous. Condition (3.8) is satisfied if $FaceV^{Cont}(t,s)$ develops according to

$$\begin{aligned}
&\frac{\partial FaceV^{Cont}}{\partial t}(t,z)=-\frac{\partial FaceV^{Cont}}{\partial_+ s}(t,t) \qquad \text{for } z<t+D,\\
&\frac{\partial FaceV^{Cont}}{\partial t}(t,z)=\frac{1}{P(t,t+D)}\left[\frac{\partial\big(K(t)\big(V(t)+PS(t)\big)\big)}{\partial_+ t}-\right.\\
&-\int_t^{t+D}\frac{\partial BookV}{\partial t}(t,y)\,\mathrm{d}_y\,FaceV^{Cont}(t,y)- \qquad (3.11)\\
&\left.-\int_t^{t+D}\frac{\partial BookV}{\partial t}(t,y)\,\mathrm{d}_y\,FaceV^{Discr}(t,y)+\big(1-P(t,t+D)\big)\frac{\partial FaceV^{Cont}}{\partial_+ s}(t,t)\right]\\
&\qquad \text{for } z\ge t+D
\end{aligned}$$

and $FaceV^{Discr}(t,s)$ develops according to

$$FaceV^{Discr}(t,z) = \lim_{h\to o+}\left(FaceV^{Discr}(t-h,z) - FaceV^{Discr}(t-h,t)\right)$$
$$\text{for } t \le z < t+D,$$
$$FaceV^{Discr}(t,z) = \lim_{h\to o+} FaceV^{Discr}(t-h,z) + \tag{3.12}$$
$$+P^{-1}(t,t+D)\left[K(t)\left(V(t)+PS(t)\right) - \lim_{h\to o+} K(t-h)\left(V(t-h)+PS(t-h)\right) + \right.$$
$$\left. + \left(1-P(t,t+D)\right)\lim_{h\to o+} FaceV^{Discr}(t-h,t)\,\right] \qquad \text{for } z \ge t+D.$$

The rate for profit sharing is

$$RatePS(t) = \frac{\int_t^{t+D} AF(t,s)\, BookV(t,s)\,\mathrm{d}_s\, FaceV(t,s)}{\int_t^{t+D} BookV(t,s)\,\mathrm{d}_s\, FaceV(t,s)} \tag{3.13}$$

and the credited profit shares development is calculated by (3.6).

3.3.7 Purchase of bonds with constant time to maturity and shares

Now we can combine the profit sharing systems described in the subsections 3.3.3 and 3.3.6. Let's assume that the company holds α of its assets in the shares and the rest in bonds (and that the insurer buys the new bonds with the time to maturity D as in 3.3.6).

For the reasons presented in 3.3.3 we assume the period of shares evaluation with length $h > 0$. This assumption will cause jumps in the value of asset portfolio and (subsequently) in the value of reserve. Therefore we will need to use the $FaceV^{Cont}(t,s)$ and $FaceV^{Discr}(t,s)$ representation of the bond portfolio. We will denote $RateBond(t)$ the instantaneous rate of the appreciation of bonds

$$RateBond(t) = \frac{\int_t^{t+D} AF(t,s)\, BookV(t,s)\,\mathrm{d}_s\, FaceV(t,s)}{\int_t^{t+D} BookV(t,s)\,\mathrm{d}_s\, FaceV(t,s)}. \tag{3.14}$$

To have the correct representation of the portfolio we have to change the condition (3.8) to

$$(1-\alpha)\,K(t)\left(V(t)+PS(t)\right) = \int_t^{\infty} BookV(t,s)\ \mathrm{d}_s\, FaceV(t,s).$$

The simplest way of defining a profit sharing system is to assume continuous crediting of the profit from the bond portfolio with jumps in the credited profit shares corresponding to the appreciation of shares

$$
\begin{aligned}
&\mathrm{d}\,PS(t) = \left[PSRatio\,(1-\alpha)(V(t)+PS(t))(RateBond(t)-\delta)_+ + PS(t)\,\delta\right]\mathrm{d}\,t \\
&\qquad\qquad \text{for} \quad t \notin \{kh \mid k = 0,1,\ldots\} \\
&PS(kh) = PSRatio\,\alpha\,(V((k-1)h)+PS((k-1)h))\left(\frac{S(kh)}{S((k-1)h)} - e^{\delta h}\right)_+ + \\
&\qquad + \lim_{l\to 0_+} PS(kh-l) \qquad\qquad \text{for} \quad k = 1,2,\ldots
\end{aligned}
\tag{3.15}
$$

Unluckily the (3.15) system separates sharing of profits from bonds and from shares. The insurer bears full loss from one asset category even though there is profit on the second. This is, of course, not realistic. Therefore we need to implement the deferred crediting of profit shares, which allows sharing part of total financial profit from longer period.

3.3.8 Deferred crediting of profit shares

The idea of the deferred crediting is simple - the insurer accumulates all the instantaneous financial profits and losses (more precisely the part belonging to policyholders) into a special fund *BRes*(t) and at the end of a year credits a part of this fund to the policyholder. In case that the *BRes*(t) is negative at the end of the year the insurer resets its value to 0. Assuming the same investment strategy as described above (i.e. α in shares, rest in bonds) we get

$$
\begin{aligned}
&\mathrm{d}\,BRes(t) = PSRatio\,K(t)(V(t)+PS(t))\left[((1-\alpha)RateBond(t)-\delta)\,\mathrm{d}\,t + \alpha\,\mathrm{d}(\ln S(t))\right] \quad 0 < t \notin \{1,2,\ldots\}, \\
&BRes(k) = \max\left\{0; \lim_{h\to 0+}\left[BRes(k-h) - K(k)(PS(k)-PS(k-h))\right]\right\} \qquad k \in \{1,2,\ldots\},
\end{aligned}
$$

The main question is how the insurer determines the credited profit-shares at the end of the year.

Yearly crediting of the profit shares

This is the simplest considered system. Insurer credits to policyholder whole amount accumulated in *BRes*(t). However such system corresponds to real situation since many insurers determine the credited profit shares from the yearly financial profit. In such case

$$
\begin{aligned}
&\mathrm{d}\,PS(t) = \delta\,PS(t)\,\mathrm{d}\,t \quad \text{for} \quad 0 < t \notin \{1,2,\ldots\} \\
&PS(k) = \lim_{h\to 0+} PS(k-h) + PSRatio\,\frac{\max\left\{0; \lim_{h\to 0+} BRes(k-h)\right\}}{K(k)} \qquad \text{for} \quad k \in \{1,2,\ldots\}.
\end{aligned}
$$

There are insurers that do not distribute all *BRes*(t) at the end of the year and keep some buffer to co-finance possible future financial losses and to smooth the profit sharing rates. Two subsequent systems allow such buffer creation.

Crediting constant percent of *BRes*

The first one assumes that at the end of the year the insurer credits $\Phi BRes(t)$ to the policyholders ($0 < \Phi < 1$). $PS(t)$ develops according to

$$\mathrm{d}\,PS(t) = \delta\, PS(t)\,\mathrm{d}t \quad \text{for} \quad 0 < t \notin \{1,2,\ldots\}$$

$$PS(k) = \lim_{h\to 0+} PS(k-h) + \frac{\left\{0; \Phi \lim_{h\to 0+} BRes(k-h)\right\}}{K(k)} \quad \text{for} \quad k \in \{1,2,\ldots\}.$$

Keeping constant ratio between *BRes* and statutory reserves

This system is inspired by [Grosen, Jørgensen] who assumed that the company tries to keep the *BRes* at a minimum level of $\Gamma\, K(t)\big(PS(t)+V(t)\big)$ at the end of the year ($\Gamma > 0$). The system is then described by

$$\mathrm{d}\,PS(t) = \delta\, PS(t)\,\mathrm{d}t \quad \text{for} \quad 0 < t \notin \{1,2,\ldots\}$$

$$PS(k) = \lim_{h\to 0+} PS(k-h) + \frac{\max\left\{0; \lim_{h\to 0+} BRes(k-h) - \Gamma\, K(k)\big(V(k)+PS(k)\big)\right\}}{K(k)} \qquad (3.20)$$

$$\text{for} \quad k \in \{1,2,\ldots\}.$$

4 SIMULATIONS RESULTS

The simulations were prepared for one endowment policy with the term of 20 years. The insured person was assumed to be a 40 years old male at issue of the policy. The input force of surrender $\nu(t)$ was assumed to be high at the beginning of the contract and decreasing to the level of approx. 6% and the surrender penalty (defined in % of the statutory reserve) was linearly decreasing form 15% at the policy issue to zero at the end of the policy term.

In this paper we present three result tables corresponding to the scenario with the constant yield curve equal to 4.5 % p.a. and to the general two-factor model described in section 3.1.3. Parameters of the interest-rate process was $\sigma_1 = 0.004$, $a_1 = 0.03$, $\sigma_2 = 0.008$, $a_2 = 0.2$ and $\rho = 0.3$.

Each table includes results for different profit sharing systems. Table 1 corresponds to the best-estimate approach, table 2 includes results of stochastic simulation of the interest rates with the deterministic surrender rates and finally the table 3 represents the results considering the stochastic surrender rates (k_1=1, k_2=1).

At first we show impacts of different investment strategies as described above, neglecting the possibility of investing all asset into the shares. In case of buying the new bonds with constant time to maturity we assume 4 different values of the time to maturity – 1, 2, 4 and 8 years to show impact of investing into the instruments with different durations. Finally the effect of the delayed crediting is shown under the 3 above described set-ups (using yearly crediting of the profit-shares).

As we can see (and is well known) the best estimate leads to the misleading value of the liability from the policy. (The investment strategy does not influence the value of the liability in our case due to the flat yield curve input.) Only the effect of the profit-sharing deferral is shown – especially we can see that the buffer relative to the statutory reserve creation sharply decreases the insurer's liability value. The above described surrender penalty is the reason of the positive margin in case the guaranteed and the interest rate equality.

Profit sharing system	New bonds time to maturity	Technical interest rate					
		5.0%	4.5%	4.0%	3.5%	3.0%	2.5%
Instantaneous interest rate	X	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
Bonds to maturity of policy	20 - *t*	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
Bonds – new bonds with	1	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
constant time to maturity	2	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
	4	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
	8	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
Shares and bonds	1	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
shares evaluated monthly	2	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
bonds const. time to mat.	4	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
(shares ratio α 40%)	8	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
Bonds – new bonds with	1	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
Time to maturity equal to	2	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
minimum of constant	4	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
And time to matur. of policy	8	**4.3**	-76.8	-96.6	-118.3	-141.8	-167.3
Deferred crediting	1	**4.3**	-76.8	-98.0	-121.0	-146.0	-173.0
100% credited at the end	2	**4.3**	-76.8	-98.0	-121.0	-146.0	-173.0
of year	4	**4.3**	-76.8	-98.0	-121.0	-146.0	-173.0
(shares ratio α 40%)	8	**4.3**	-76.8	-98.0	-121.0	-146.0	-173.0
Deferred crediting	1	**4.3**	-76.8	-99.9	-125.1	-152.3	-181.8
75% credited at the end	2	**4.3**	-76.8	-99.9	-125.1	-152.3	-181.8
of year	4	**4.3**	-76.8	-99.9	-125.1	-152.3	-181.8
(shares ratio α 40%)	8	**4.3**	-76.8	-99.9	-125.1	-152.3	-181.8
Deferred crediting	1	**4.3**	-76.8	-127.6	-153.3	-179.8	-208.3
keeping 2% of statutory	2	**4.3**	-76.8	-127.6	-153.3	-179.8	-208.3
reserve in the buffer	4	**4.3**	-76.8	-127.6	-153.3	-179.8	-208.3
(shares ratio α 40%)	8	**4.3**	-76.8	-127.6	-153.3	-179.8	-208.3

Table 1 Deterministic projection – liability value corresponding to insured sum 10 000

Profit sharing system	New bonds time to maturity	Technical interest rate 5.0%	4.5%	4.0%	3.5%	3.0%	2.5%
Instantaneous interest rate	X	**63.8**	**12.7**	-34.4	-77.4	-117.0	-154.0
Bonds to maturity of policy	20 - *t*	**0.4**	-61.2	-100.3	-125.7	-149.7	-175.5
Bonds – new bonds with	1	**55.1**	**3.5**	-43.3	-85.4	-123.5	-159.0
constant time to maturity	2	**47.5**	-4.8	-51.2	-92.1	-128.8	-162.9
	4	**35.5**	-18.0	-63.8	-102.4	-136.2	-167.8
	8	**19.6**	-36.4	-80.5	-114.7	-144.3	-172.5
Shares and bonds	1	**959.7**	**927.9**	**893.8**	**857.3**	**818.2**	**776.6**
shares evaluated monthly	2	**959.1**	**927.3**	**893.1**	**856.6**	**817.6**	**775.9**
bonds const. time to mat.	4	**958.0**	**926.2**	**892.0**	**855.4**	**816.4**	**774.7**
(shares ratio α 40%)	8	**956.7**	**924.8**	**890.6**	**854.0**	**814.9**	**773.3**
Bonds – new bonds with	1	**55.1**	**3.5**	-43.3	-85.4	-123.5	-159.0
time to maturity equal to	2	**47.6**	-4.6	-51.1	-92.0	-128.7	-162.8
minimum of constant	4	**35.9**	-17.6	-63.4	-102.1	-136.0	-167.7
and time to matur. of policy	8	**20.7**	-35.2	-79.4	-114.0	-143.8	-172.4
Deferred crediting	1	**218.0**	**172.8**	**126.9**	**80.7**	**34.1**	-12.9
100% credited at the end	2	**216.3**	**170.9**	**125.0**	**78.9**	**32.4**	-14.6
of year	4	**213.4**	**167.9**	**122.1**	**75.9**	**29.4**	-17.4
(shares ratio α 40%)	8	**209.7**	**164.2**	**118.3**	**72.1**	**25.7**	-20.8
Deferred crediting	1	**186.0**	**137.9**	**89.4**	**40.8**	-7.9	-57.2
75% credited at the end	2	**184.2**	**135.9**	**87.6**	**39.0**	-9.7	-58.8
of year	4	**181.3**	**132.9**	**84.4**	**35.9**	-12.8	-61.7
(shares ratio α 40%)	8	**177.4**	**129.0**	**80.5**	**31.9**	-16.7	-65.3
Deferred crediting	1	**139.1**	**90.4**	**42.0**	-5.8	-53.0	-99.6
keeping 2% of statutory	2	**137.0**	**88.2**	**39.8**	-7.9	-55.0	-101.4
reserve in the buffer	4	**133.4**	**84.5**	**36.0**	-11.5	-58.3	-104.5
(shares ratio α 40%)	8	**128.6**	**79.6**	**31.2**	-16.3	-62.8	-108.6

Table 2 Stochastic interest rates – liability value corresponding to insured sum 10 000

Profit sharing system	New bonds time to maturity	Technical interest rate 5.0%	4.5%	4.0%	3.5%	3.0%	2.5%
Instantaneous interest rate	X	**83.9**	**30.2**	-19.6	-65.1	-106.8	-145.6
Bonds to maturity of policy	20 - *t*	**26.2**	-40.2	-84.7	-113.1	-137.5	-162.9
Bonds – new bonds with	1	**75.4**	**20.9**	-28.7	-73.4	-113.8	-151.1
constant time to maturity	2	**68.0**	**12.7**	-36.8	-80.5	-119.5	-155.3
	4	**56.4**	-0.5	-49.8	-91.4	-127.6	-160.9
	8	**41.5**	-18.7	-67.0	-104.7	-136.4	-166.0
Shares and bonds	1	**996.9**	**964.6**	**930.0**	**893.0**	**853.5**	**811.3**
shares evaluated monthly	2	**996.6**	**964.3**	**929.7**	**892.7**	**853.1**	**810.9**
bonds const. time to mat.	4	**996.0**	**963.7**	**929.1**	**892.0**	**852.4**	**810.2**
(shares ratio α 40%)	8	**995.5**	**963.2**	**928.5**	**891.5**	**851.9**	**809.7**
Bonds – new bonds with	1	**75.4**	**20.9**	-28.7	-73.4	-113.8	-151.1
time to maturity equal to	2	**68.1**	**12.8**	-36.7	-80.4	-119.4	-155.3
minimum of constant	4	**56.7**	-0.3	-49.6	-91.2	-127.5	-161.0
and time to matur. of policy	8	**42.3**	-17.9	-66.4	-104.4	-136.6	-166.6
Deferred crediting	1	**240.4**	**194.0**	**146.9**	**99.5**	**51.8**	**3.6**
100% credited at the end	2	**238.7**	**192.2**	**145.1**	**97.7**	**50.1**	**1.9**
of year	4	**236.0**	**189.3**	**142.2**	**94.9**	**47.2**	-0.8
(shares ratio α 40%)	8	**232.6**	**185.8**	**138.8**	**91.4**	**43.8**	-3.8
Deferred crediting	1	**208.8**	**159.3**	**109.6**	**59.7**	**9.7**	-40.7
75% credited at the end	2	**207.0**	**157.5**	**107.8**	**57.9**	**7.9**	-42.4
of year	4	**204.1**	**154.5**	**104.7**	**54.9**	**4.9**	-45.2
(shares ratio α 40%)	8	**200.5**	**150.8**	**101.0**	**51.2**	**1.2**	-48.6
Deferred crediting	1	**161.3**	**111.0**	**61.1**	**11.8**	-36.9	-84.8
keeping 2% of statutory	2	**159.2**	**108.8**	**58.9**	**9.7**	-38.9	-86.8
reserve in the buffer	4	**155.6**	**105.1**	**55.1**	**6.0**	-42.2	-89.9
(shares ratio α 40%)	8	**150.9**	**100.3**	**50.4**	**1.3**	-46.7	-93.9

Table 3 Stochastic interest rates and force of surrender – liab. value corresponding to insured sum 10 000

Results of stochastic calculations show that the more volatile is the accounting interest from the asset portfolio the bigger is the insurer's liability value. In fact we can see that the strategy of buying long-term bonds and amortizing the purchase price leads to the lowest liability value at all. It explains why many life insurers use this strategy.

Comparing the results with deterministic and stochastic force of surrender we observe that the liability has increased by the amount approximately corresponding to one monthly premium. However the increase in the liability value is diametrically lower than in the case of replacing the best estimate with stochastic interest rates, even this change is definitely significant from the life insurer's perspective.

5 CONCLUSION

We have developed framework for the fair value that can be used in practice, shown specific models originating from the model and especially we have developed the representation of the bond portfolio allowing the projection corresponding to the real practice of many life insurers. The framework and above-shown models were inspired by the approach of the insurance practitioners and therefore it is not surprising that some of its features may resemble the recently developed models used in practice. However, most of the practically used models are based on approximate assumptions and their architecture does often not allow stochastic and dynamic modeling of all the important features.

Unluckily, the calibration of above described models allowing stochastic development of surrender rates requires actuarial judgment since it would be nearly impossible to reliably estimate the parameters from the past.

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