Solvency II and Nested Simulations – A Least-Squares Monte Carlo Approach

Daniela Bergmann
with Daniel Bauer and Andreas Reuß

Ulm University, Georgia State University, ifa Ulm
Agenda

• Motivation

• Solvency II Requirements

• Numerical Approaches
  – Nested Simulations Approach
  – Least-Squares Monte Carlo Approach

• Results

• Summary and Outlook
Motivation

- **Solvency II**: New regulatory framework for insurance companies in the European Union

- **Key aspect**: Determine required risk capital (SCR) for a one-year time horizon based on a market-consistent valuation of assets and liabilities

- **Standard model**: Approximation of SCR via square-root formula

  → Various deficiencies (cf. Pfeifer/Strassburger (2008), Sandström (2007)).
Motivation

- **Alternative:** Multivariate approach based on stochastic model for the insurance company (Internal Model).

- **Problems:**
  - Valuation of life insurance contracts in closed form not possible (due to embedded options and guarantees)
  - Unsolved numerical and computational problems

→ This paper provides a mathematical framework for the calculation of the SCR and discusses different approaches for the numerical implementation.
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Definitions

Assessment of solvency position can be split into two components:

1. **Available Capital** $AC_0$:
   - Amount of financial resources available at time $t=0$ which can serve as a buffer against risks and absorb financial losses
   
   $$AC_0 := MVA_0 - MVL_0 \approx MCEV_0$$
   
   - $MCEV_0$ denotes the market consistent embedded value, i.e. $MCEV_0 = ANAV_0 + PVFP_0 - CoC_0$, where
     - $ANAV_0$ is derived from statutory shareholders‘ equity,
     - $PVFP_0$ is the present value of post-taxation shareholder cash flows from the assets backing (statutory) liabilities
     - $CoC_0$ is the Cost-of-Capital charge (not discussed further here)
   
   - Main computational issue: Calculation of PVFP
Definitions

Assessment of solvency position can be split into two components:

2. Solvency Capital Requirement (SCR):

- SCR is based on the Available Capital at t=1, where
  \[ AC_1 \approx MCEV_1 + X_1, \]  
  where \( X_1 \) denotes shareholder cash flows at t=1.

- Intuition: An insurance company is considered solvent under Solvency II if its Available Capital at t=1 is positive with a probability of at least \( \alpha = 99.5\% \).

- Therefore consider loss function \( L := AC_0 - AC_1 / (1+i) \) where \( i \) denotes the one-year risk-free rate at t=0.

\[
SCR := \arg\min_x \{ P(AC_0 - AC_1 / (1+i) > x) \leq 0.5\% \} = \text{VaR}_{99.5\%}(L)
\]

- Main computational issue: calculation of 99.5%-quantile of -\( AC_1 \).
Valuation at t=0

- **Target:** \( AC_0 = \underbrace{ANAV_0}_{\text{from statutory balance sheet}} + \mathbb{E}^Q\left[ \sum_{t=1}^{T} \exp(-\int_0^t r_u \, du) X_t \right] \)

- **Problem:** No closed-form solution for \( V_0 \)

- **Monte Carlo Simulation:** \( \tilde{V}_0(K_0) = \frac{1}{K_0} \sum_{k=1}^{K_0} \sum_{t=1}^{T} \exp\left(-\int_0^t r_u^{(k)} \, du\right) X_t^{(k)} \)
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Valuation at $t=1$

- **Target:** Distr. of \( AC_1 = \text{ANAV}_1 + \mathbb{E}^Q \left[ \sum_{t=2}^{T} \exp(- \int_{1}^{t} r_u \, du) X_t \mid (Y_1, D_1) \right] + X_1 \sim F \)

  \[ =: V_1 \]

  - Simulate $N$ first-year paths “under $\mathcal{P}$” \( (Y_1^{(i)}, D_1^{(i)}) \)
  - Simulate $K_1$ paths “under $\mathcal{Q}$” starting in \( (Y_1^{(i)}, D_1^{(i)}) \) to estimate $V_1^{(i)}$
  - Total of $N \cdot K_1$ sample paths
Estimation of SCR

- We now have:

  1. $\hat{AC}_0(K_0) = ANAV_0 + \tilde{V}_0(K_0)$

  2. $\hat{AC}_1^{(i)}(K_1) := ANAV^{(i)}_1 + \tilde{V}^{(i)}_1(K_1) + X^{(i)}_1, 1 \leq i \leq N.$

- Hence, we can estimate SCR by

$$\hat{SCR} = \hat{AC}_0 + \frac{Z(m)}{1 + i}$$

$Z(m)$ is the $m^{th}$ order statistic of $-\hat{AC}_1^{(i)}$ and $m = \lceil N \cdot 0.995 + 0.5 \rceil$

- We have the following sources of error:
  - Estimation of $AC_0$ with only $K_0$ sample paths
  - Estimation of the quantile with only $N$ real-world scenarios
  - Estimation of $AC_1^{(i)}$ with only $K_1$ sample paths

- Analysis of different errors in the estimation of SCR
Variance-Bias Tradeoff

- **Idea**: Minimize Mean-Square-Error (MSE)

\[
MSE = \mathbb{E} \left[ (\hat{SCR} - SCR)^2 \right] = Var(\hat{SCR}) + \left[ \frac{\mathbb{E}(\hat{SCR}) - SCR}{\text{bias}} \right]^2
\]

- Similar to Gordy/Juneja (2008), we obtain the following optimization problem:

\[
\frac{\sigma^2_0}{K_0} + \frac{\alpha(1 - \alpha)}{(N + 2)f^2(SCR)} + \frac{\theta^2_\alpha}{K_1^2 \cdot f^2(SCR)} \to \min
\]

with the budget restriction \( \Gamma = N \cdot K_1 + K_0 \).

- Systematic overestimation of SCR due to positive bias
- Immense computational effort to obtain small bias
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Least-Squares Algorithm

- Based on LSM approach by Longstaff/Schwartz (2001) for the valuation of non-European options

- Algorithm:
  - Simulate N scenarios (first year $\mathcal{P}$, other years $\mathcal{Q}$)
    
    $$ PV_1^{(i)}(\omega_i) := \sum_{t=2}^{T} \exp \left\{ - \int_{1}^{t} r_s(\omega_i) \, ds \right\} X_t(\omega_i) = \mathbb{E}^{\mathcal{Q}} \left[ PV_1^{(i)} \mid \mathcal{F}_1 \right] + \epsilon_i, \quad 1 \leq i \leq N $$

  - 1$^{st}$ step: Approximate $V_1$ by finite sum of appropriate basis functions
    
    $$ V_1 = \mathbb{E}^{\mathcal{Q}} \left[ \sum_{t=2}^{T} \exp (- \int_{1}^{t} r_u \, du) X_t \left| (Y_1, D_1) \right. \right] \approx \hat{V}_1^{(\text{M})}(Y_1, D_1) = \sum_{k=1}^{M} \alpha_k \cdot e_k(Y_1, D_1) $$

  - 2$^{nd}$ step: Estimate unknown parameter vector $\alpha$ via regression:
    
    $$ \hat{\alpha}^{(\text{N})} = \arg\min_{\alpha \in \mathbb{R}^M} \left\{ \sum_{i=1}^{N} \left[ PV_1^{(i)} - \sum_{k=1}^{M} \alpha_k \cdot e_k \left( Y_1^{(i)}, D_1^{(i)} \right) \right]^2 \right\} $$

  - Estimate Available Capital:
    
    $$ \hat{AC}_1^{(i)} = \text{ANAV}_1^{(i)} + \sum_{k=1}^{M} \hat{\alpha}_k^{(\text{N})} \cdot e_k(Y_1^{(i)}, D_1^{(i)}) + X_1^{(i)}, \quad 1 \leq i \leq N $$
LSM approach: Does it work?

• Issues to consider:
  – Suitability of regression approach
  – Convergence of the algorithm
  – Bias
    • Finite number of basis functions
    • Estimation of regression parameters
  – Choice of regression function

• Ultimate test: How well does it perform in a somewhat realistic framework?
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A Participating Life Insurance Contract

• Term-fix insurance contract with minimum interest rate guarantee
• Bonus distribution models obligatory payments to the policyholder (MUST-case from Bauer et al. (2006))
• No mortality → no biometric risk

• Dividends $d_t$ are paid to the shareholders
• Company obtains additional contribution $c_t$ from its shareholders in case of a shortfall
• Asset model: Extended Black-Scholes model with stochastic interest rates (see Bauer/Zaglauer (2008))
Bias in Nested Simulations
N=100,000

- Choice of $K_1$ significantly influences estimation of SCR
- Optimization of parameters based on pilot simulation:
  $K_0 \approx 1,500,000$; $N\approx320,000$ and $K_1\approx300$
Comparison of different parameter choices

<table>
<thead>
<tr>
<th>$N$</th>
<th>$K_1$</th>
<th>Mean (SCR)</th>
<th>Empirical Variance</th>
<th>Estimated Bias</th>
<th>Estimated MSE</th>
<th>Corrected Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>160.000</td>
<td>600</td>
<td>1247.7</td>
<td>24.6</td>
<td>1.4</td>
<td>26.6</td>
<td>1246.3</td>
</tr>
<tr>
<td>320.000</td>
<td>300</td>
<td><strong>1249.3</strong></td>
<td><strong>15.8</strong></td>
<td><strong>2.9</strong></td>
<td><strong>24.0</strong></td>
<td><strong>1246.4</strong></td>
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<tr>
<td>640.000</td>
<td>150</td>
<td>1251.3</td>
<td>7.9</td>
<td>5.7</td>
<td>40.6</td>
<td>1245.6</td>
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<tr>
<td>1.280.000</td>
<td>75</td>
<td>1257.4</td>
<td>4.2</td>
<td>11.4</td>
<td>133.1</td>
<td>1246.1</td>
</tr>
</tbody>
</table>
### Choice of the regression function in the LSM approach

<table>
<thead>
<tr>
<th>#</th>
<th>Regression Function</th>
<th>Mean (SCR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1$</td>
<td>1007.3</td>
</tr>
<tr>
<td>2</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2$</td>
<td>1165.5</td>
</tr>
<tr>
<td>3</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1$</td>
<td>1272.6</td>
</tr>
<tr>
<td>4</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2$</td>
<td>1276.5</td>
</tr>
<tr>
<td>5</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1$</td>
<td>1233.2</td>
</tr>
<tr>
<td>6</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1$</td>
<td>1233.9</td>
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<tr>
<td>7</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$</td>
<td>1241.3</td>
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<tr>
<td>8</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$</td>
<td>1244.5</td>
</tr>
<tr>
<td>9</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$</td>
<td>1245.9</td>
</tr>
</tbody>
</table>

- Influence of basis function is quite pronounced
- For good choices the results are close to those from the Nested Simulations approach
- Significantly faster than Nested Simulations Approach
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Summary

• Nested Simulations:
  – Inadequate choice of $K_0, N$, and $K_1$ may yield erroneous outcomes.
  – Immense computational effort to achieve accurate results.

• LSM:
  – Fast approach to achieve relatively accurate results.
  – Results are similarly positive when calculating SCR for longer time horizons ("richer sigma field").
  – Care is required in choice of regression function even though simple algorithms yield good results in our applications.
  – Open question: theoretical results regarding validity of approximation.
Further research

• Improvement of the Nested Simulations Approach by
  – Variance reduction techniques
  – QMC
  – Screening procedures

• Use of statistical methods to determine the regression function.

• Analysis of other risk measures, such as TVaR.
Contact Details

Daniela Bergmann
Ulm University
Helmholtzstr. 18
89069 Ulm
Germany

daniela.bergmann@uni-ulm.de

Thanks for your attention!