Stochastic re-reserving in multi-year internal models –
An approach based on simulations

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1. Introduction

Increasing natural catastrophes, difficult capital market environment and fundamental changes in supervisory requirements (Solvency II for European Union member countries) have placed increasing challenges on management strategy in insurance companies. Management is faced with the challenge of allocating and managing capital resources efficiently. This requires a suitable insurance portfolio structure in combination with an adequate asset allocation towards the insurance cash flows. As a result most companies are developing modern management techniques such as value and risk-based management. In this context the appropriate use of diversification effects plays an important role.

Internal risk models can play an important part in management decisions in value and risk-based management strategy. Insurers with suitable internal models are in a position to calculate the risk capital to be put up by the company at large as well as strategic sub-segments on an individual basis according to the risk structure of the company. This enables companies to address issues affecting risk-bearing capability and profitability in the company at large as well as sub-portfolios down to individual product level. So companies are able to assess the amount of risk that can be taken in individual company units and the returns that can be reached from a specified risk position. Increasing transparency in the risk situation, identification of high-risk factors, and identification of segments that generate or decrease shareholder value are essential ingredients in generating a strategic value and risk-based company management approach aimed towards a long-term and sustainable increase in

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1 Provinzial NordWest Holding AG, Germany
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shareholder value. So these models should also be used in a company’s “Own Risk and Solvency Assessment” (ORSA).\(^2\)

Although non-life insurance contracts have short periods in contrast to life insurance, risk-return strategy should follow calculations that span several years. In the actual literature several questions concerning the use of internal models in a multi-year management context are not answered to date.\(^3\) The aim of this paper is to present a multi-year model approach, which allows the quantification of one-year risk capital of the first simulated year as required by Solvency II. Moreover we calculate a “multi-year” risk capital, to have a higher probability of settling all losses that occur within the entire period of simulated \(n\) years without needing external capital sources (Section 2).

Only internal models optimally reflecting the risk situation facing the company allow insurers to assess the level of risk capital required according to the corporate risk structure. This importantly involves measuring and evaluating all relevant risks the insurers are exposed to. With this paper we want to make a contribution to risk modeling of two parts of the insurance risks: The one-year reserve risk (Section 3) and parameter risk in premium risk (Section 4).

In literature there is a wide variety of methods for stochastic reserving such as the Mack method, Bootstrap method,\(^4\) regression approaches,\(^5\) Bayesian methods,\(^6\) etc. All these approaches are based on an ultimo view, so that the uncertainty of full run-off of the liabilities is quantified. The same holds for premium risk. In contrast Solvency II requires the quantification of the one-year reserve risk. In addition the investment results are also based on a one-year view, which means that actually many internal models show an ultimo view for insurance results and the one-year view for investment results. So at the moment there is a discussion in academic literature and in insurance practice, how this one-year reserve risk can be quantified.\(^7\) In Section 3.1 we present the idea of re-reserving which can be applied in modeling reserve risk and premium risk. Based on this approach we can quantify one-year risk capital and multi-year risk capital as defined in Section 2. We compare the results of the stochastic re-reserving method with the results of the analytic approach shown in Merz and Wüthrich (2008). In Section 3.2 we present a case study to show the effects of multi-year stochastic re-reserving.

Prediction risk is a major risk that needs to be quantified using internal models, which can be divided into parameter risk and process risk. While the standard formula (Solvency II) and internal models both usually take account of process and parameter risks in modeling reserve risk, parameter risk is often omitted in premium risk, thus only taking process risk into account, although there have been discussed some methods of modeling parameter uncertainty in literature.\(^8\) In Section 4 we will be quantifying the effects of including parameter uncertainty in premium risk with two different modeling approaches (bootstrap method and a Bayesian approach) using example data. With this study we want to raise the awareness of the importance of these risks.

\(^2\) See CEIOPS (2008).
\(^3\) In Diers (2008a) a model approach of a multi-year model is presented.
\(^4\) See for example England / Verrall (2002).
\(^5\) See for example Christofides (1990).
\(^7\) See Merz / Wüthrich (2008), Diers (2007b) and Ohlsson / Lauzeningks (2009).
\(^8\) See for example Cairns (2000).
2. Results by calendar year – Multi-year model approach

We have based the model design on an internal simulation model in non-life insurance, modeling strategic insurance segments and asset classes based on economic principles and simulating the results considering suitable dependencies.\(^9\) The economical result, \(EcRes_t\), prediction for a future year \(t\), can be expressed by the change in economic capital \(EcCap\) within the respective year:\(^10\)

\[
EcRes_t = EcCap_t - EcCap_{t-1} = EcResLiab_t + EcResAs_t - O_t - A_t,
\]

where

- \(EcResLiab_t\) = net insurance result at time \(t\),
- \(EcResAs_t\) = investment result at time \(t\),
- \(O_t\) = result from operational risk at time \(t\),
- \(A_t\) = tax at time \(t\)\(^{11}\).

The net insurance result is calculated using underwriting result (2.1) and claims development result (2.2):

\[
(2) \quad EcResLiab_t
\]

\[
= P_t - C_t - U_t - F_t + F_{t-1},
\]

whereas:

- \(P_t\) = premiums earned at time \(t\),
- \(C_t\) = costs (acquisition costs, administration costs, internal settlement costs) at time \(t\),
- \(U_t\) = prediction for the ultimate claim losses of the simulated accident year at time \(t\),
- \(F_t\) = prediction for the ultimate claim losses of former accident years (<\(t\)) at time \(t\),
- \(F_{t-1}\) = prediction for the ultimate claim losses of former accident years (<\(t\)) at time \(t-1\).

All insurance-related figures such as premiums, ultimate losses, and costs are to be seen as net figures, i.e. after reinsurance, and external settlement costs are modelled together with the claims. In the multi-year model not only claims but also premiums are modelled stochastically.

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\(^9\) See Diers (2007a) for details. Refer to Diers (2008a) in the multi-year model design.

\(^{10}\) Economical capital is defined as the difference between the market value of assets and market value of technical provisions (best estimate plus market value margin). We have selected a somewhat simplified representation such as by ignoring other assets and liabilities, while focussing on the result before earnings distribution to shareholders.

\(^{11}\) See Diers (2007a) for tax modeling.
in order to represent the effects of premium cycles.\textsuperscript{12}

Internal models begin by modeling gross insurance results\textsuperscript{13} and the reinsurance results are matched to individual agreements. This enables evaluation of the gross insurance results in return and risk aspects, and assessment of individual reinsurance results or alternative reinsurance structures for efficiency.

The ultimate claim loss prediction in the simulated accident year ($U_t$) and former years ($F_t$) in time $t$ may be calculated using methods such as re-reserving from the ultimate model. Stochastic cash flows are essential for this purpose (see Section 3 for details). The one-year reserve risk is calculated using the simulated empirical distribution of $F_0 - F_1$, where we condition on all observations up to time $t=0$.

For modeling the economic investment results we use capital market scenarios that arise from real-world models. In the asset model management rules (e.g. sales priorities, rebalancing rules) have to be taken into account. Like the insurance result that can be calculated according to insurance segments, the economic investment result can also be modeled down to the most detailed asset level (asset class) depending on the model’s depth.\textsuperscript{14}

Usually, management requires that substantial risks (natural catastrophes, development results in long-tail-business) be viewed from an underwriting context of several years in order to address the following issues:

- How many years of catastrophe risks or other major events, such as negative capital market development, can the company economically withstand at a certain confidence level without needing external capital sources?
- How much risk capital does a company currently provide to maintain a certain confidence level to ensure its status as a going concern for another five years, i.e. taking five future underwriting years into account, without needing external capital sources?

To address issues of this nature, we can use our multi-year model to calculate a “multi-year” risk-capital taking into account $n, n \in \mathbb{IN}$, future accident years. We define the random $MaxLoss$ variable as follows:

\begin{equation}
MaxLoss(n) = \operatorname{MAXIMUM} \left\{ \text{CumLoss}_t \right\}_{1 \leq t \leq n}, \quad \text{with}
\end{equation}

\begin{equation}
\text{CumLoss}_1 = -\text{EcRes}_1 \quad \text{and}
\end{equation}

\begin{equation}
\text{CumLoss}_t = \text{CumLoss}_{t-1} - \text{EcRes}_t, \quad 1 \leq t \leq n.
\end{equation}

$MaxLoss$ is to be provided for each simulation path at $t=0$ to enable the insurance company to settle all losses that occur within the entire period simulated of $n$ years without needing external capital sources.

\textsuperscript{12} See for example Cummins and Outreville (1987) for an analysis on premium cycles. Dependencies between premium cycles and various market indexes should be modeled in an appropriate way.

\textsuperscript{13} Modeling on net values would not be an option due to the changing reinsurance structures alone.

\textsuperscript{14} See for example Diers (2007a) for modeling investment results.
The selected risk measure, $\rho$, can now be applied to the $\text{MaxLoss}$: $\Omega \rightarrow \text{IR}$ in order to determine the multi-year risk-capital requirement e.g. for tail value at risk (TVaR)\(^{15}\) at percentile $(1-\alpha n)$:

$$\rho(\text{MaxLoss}(n)) = \text{TVaR}_{\alpha n} (\text{MaxLoss}(n)).$$

The confidence level $\alpha n$ may decrease with increasing values of $n$. By definition, the multi-year risk capital is always at least as high as the one-year risk capital for values of $\alpha n = \alpha 1 = \alpha$. If the insurance company can cover its multi-year risk capital with its own economic capital at $t=0$, $\text{EcCap}_0$, the following will apply:

$$\text{EcCap}_0 \geq \rho(\text{MaxLoss}(n)).$$

The company can therefore cover all losses that may incur over the simulation period without external capital supply at a probability of more than $1-\alpha n$.\(^{16}\)

So the multi-year risk-capital concept may take on the role of a strict constraint in internal models in addressing strategic issues, which can be used for “Own Risk and Solvency Assessment” (ORSA).

3. Modeling one-year reserve risk using the re-reserving method

3.1 Methodology

In this section we model one-year and multi-year development results in internal models using the re-reserving method.

In order to get a better understanding of the difference between the ultimate and the one-year risk view we start with the ultimate development result, which is defined as:

$$\text{DevResUlt} = F_0 - F_{\omega} = R_0 - R_{\omega},$$

where $F_{\omega}$ refers to the sum of cash flows from previous accident years, which are simulated up to the final settlement of claims at $t=\omega$, where we condition on all observations up to time $t=0$. $F_0$ refers to the associated best estimate. $R_0$ and $R_{\omega}$ refer to the corresponding reserves.

The ultimate reserve risk can be calculated using the simulated empirical distribution of $\text{DevResUlt}$. The literature describes a variety of approaches for modeling the ultimate reserve

\(^{15}\) TVaR$_a(L) = E[L \mid L \geq \text{VaR}_a(L)] = \text{VaR}_a(L) + E[L - \text{VaR}_a(L) \mid L \geq \text{VaR}_a(L)]$, where $E$ refers to the expected value, which is conditional in this case. VaR refers to value at risk. Tail value at risk is a coherent risk measure for random variables with continuous distribution. A discussion of advantages and disadvantages of different risk measures such as VaR or TVaR is necessary, but exceeds the purpose of this paper, see for example Rootzén / Klüppelberg (1999).

\(^{16}\) See Diers (2008a) for the multi-year risk capital concept. Note that the optimal level of economic capital should be assessed according to optimal business management strategy in a shareholder value calculation; see Gründl / Schmeiser (2002).
In contrast, Solvency II requires a one-year risk-perspective, where the change in reserves and therefore also the development result for the next year is to be calculated. In addition the investment results, which have to be added to insurance results, are also based on a one-year view, which means that actually many internal models show an ultimo view for insurance results and the one-year view for investment results. So a consistent view of investment results and investment results is required.

In 2007 Association Internationale des Sociétés d’Assurance Mutuelle (AISAM) and Association of European Cooperative and Mutual Insurers (ACME) initiated an international study which aims at clarifying how the reserve risk should be calculated over one-year horizon (AISAM / ACME 2007). Wüthrich, Merz and Lysenko (2008) presented an analytical approach towards calculating the claims development result for the next calendar year and its prediction uncertainty based on the chain ladder model. However, the analytical approach is often not sufficient in internal models, because (simulated) cash flows are needed for future claim settlements.

As shown in Section 2, multi-year internal models yield the following simulation results for the development results per calendar year $t$:

\[
\text{DevRes}_t = F_{t-1} - F_t = R_{t-1} - I_t - R_t,
\]

whereas $R_{t-1}$ refers to the opening reserve estimate at the beginning and $R_t$ refers to the closing reserve estimate at the end of the simulated year $t$. $I_t$ refers to the sum of payments (cash flow) during the year $t$.

So for $t = 1$ we can write:

\[
\text{DevRes}_1 = R_0^D - I_1^D - R_1^D,
\]

where all observations up to time $t=0$ are given by $D_j$ and we condition on the “observed” part of the claim triangle.

In the following for simplicity of the presentation we only consider former accident years up to $t=0$. So in the example given in Fig. 1 in the years 2007ff we simulate the claim development of accident years up to 2006.

The idea of re-reserving is considered in Ohlsson and Lauzeningks (2009) and Diers (2007b). In the following we want to describe this modeling approach based on re-reserving using three steps.

**Step 1: Calculating the opening reserve estimate $R_0^D$.**

First, a claim reserving model is set as a base for the ultimate stochastic reserving process. This involves stochastic reserving methods that yield simulated future cash flows, such as

\[
\text{There is a wide variety of methods for stochastic reserving such as the Mack method, Bootstrap method (see for example England / Verrall (2002)), regression approaches (see for example Christofides (1990)), Bayesian methods (see for example England / Verrall (2002) and England / Verrall (2006)), etc.}
bootstrap approaches or Bayesian methods.\textsuperscript{18}

The first step is to assess the opening reserve estimate $R_0 = R^{D_1}$, which can be calculated from the underlying reserving model and should agree with the actuary’s best estimate for outstanding claims in $t=0$.

**Step 2: Calculating the payments $I^{D_1}$**

The second step is to simulate payments $I^{D_1}$ for the next calendar year $t=1$ for each former accident year using the underlying ultimate stochastic reserving process. So we (only) simulate the next diagonal in the development triangle (see Fig. 1). The sum of claim payments on the simulated diagonal equals to $I_t$. This level of knowledge matches the “actuary in the box” at the end of year $t=1$. So using simulation methods we get an empirical distribution of $I^{D_1}$, which – with growing simulation number – converges to the theoretical distribution $I^{D_1}$.

**Step 3: Calculating the closing reserve estimates $R^{D_{t+1}}$**

Step 2 is used as a basis for the third step, which is to carry out a best estimate for the closing reserve $R^{D_{t+1}}$ in each simulation path according to the underlying reserving model which is assumed to be set. This process is called *re-reserving*. So we obtain $\text{DevRes}_{t}$.

Fig. 1: Re-reserving method (conditional on $t=0$)

For calculating multi-year risk capital as defined in (4) we need the cumulative development result $\text{CumDevRes}_{t}$ of the simulated year $t$, which is defined as

\begin{equation}
\text{CumDevRes}_{t} = F_0 - F_t = R_0 - \sum_{l=1}^{t} I_l - R_t. 
\end{equation}

Note that we condition on $t=0$. Moreover the cumulative development result is a part of $\text{CumLoss}_{t}$ as defined in (3), which is used to define multi-year risk capital for the company.

To simulate $\text{CumDevRes}_2$ in each simulation $R_0$ represents the opening reserve estimate. We simulate payments $I_1$ for the first simulated future calendar year as described above. The payments $I_2$ for the second future calendar year $t=2$ are also simulated according to the underlying stochastic reserving method and we use the re-reserving-method again to achieve $R_2$, etc. So re-reserving is a method based on the underlying stochastic ultimate reserving model. We have not entered into a discussion on possibilities and limitations of the “actuary in the box” at this point; however, this discussion is necessary.

Fig. 2 shows cumulative cash flows referring to a motor third-party segment generated from 100,000 simulations from one accident year (2005) in the ultimate (left) and one-calendar-year view (right). The example cash flows highlighted in white in the left and right parts of the diagram show results from the same simulation path. The simulated payments for the third development year 2007, which correspond to the first simulated year, agree in both views. The payments in 2005 and 2006 are known, and are therefore deterministic. Since the view on the left-hand side represents the ultimate view, further payments (from 2008 onwards) are simulated up to the final claim settlement using Mack bootstrap.\(^\text{19}\) The one-calendar-year view on the right-hand part shows the best estimate calculated at the end of 2007 in each simulation path according to the underlying reserving model (“actuary in a box”).

The cash flow highlighted in white, for example, shows a very high payment volume for the fifth development year that has been underestimated in the one-year view in the right-hand part. This leads to a reserve estimate that is “too low” in this simulation path.

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\(^\text{19}\) See England and Verrall (2006).
be applied to generate a one-calendar-year view using this ultimate reserving method as a basis. The standard deviation in the calendar-year view is reduced to around 71%.

So we use the probability distribution of $DevResUlt$ to calculate risk capital in the ultimate view and $DevRes$ for the one-year reserve risk. With regard to value at risk at a confidence level of 99.5% according to Solvency II, which corresponds to a 200-year event, the risk capital level amounts €8.2 million in the ultimate view and €6.2 million in the calendar-year view. This is a reduction in risk capital of 24.4%. If a company calculates risk capital at a higher internal risk level, for example using tail value at risk at 99.8%, the average of the two hundred worst simulation results will be taken for risk capital calculation at a hundred thousand simulations. This will result in a risk capital level of €7.6 million in the calendar-year view, which is also significantly lower than the corresponding value of €9.9 million in the ultimate view (reduction by 24%).

For comparison we applied the re-reserving method to a claim payment triangle (where the first accident year has been completely settled) presented by Wüthrich / Merz (2008) and compared our results obtained by re-reserving to those presented in this paper based on the analytical approach. If we simulate claims to ultimate based on Mack bootstrap (using Mack’s bias correction and normal distribution for process risk) and use a re-reserving approach based on deterministic Chain-Ladder to investigate variability of the run-off reserve after one year we obtain nearly the same prediction error as presented by Wüthrich / Merz (2008) in their example shown in Section 4.

Analogue calculations are essential in assessing underwriting results, which can be transferred to development results in multi-year models such as by simulating the first payment $NI_t$ of the new accident year and estimating the related closing reserve $NR_t$ per simulation path as described above from the underlying reserving model. So formula (2.1) can be written as:

\[
(10) \quad P_t - C_t - U_t = P_t - C_t - NI_t - NR_t.
\]

---

20 See Ohlsson and Lauzeningks (2009) for modeling one-year underwriting risk.
3.2 Application in the multi-year context – Case Study

In this Section we present a case study in order to give an application of the multi-year re-reserving method. We show the development of risk capital in a multi-year context up to the final claim settlement. We compare the results with the one-year risk capital used for Solvency II purposes and the ultimate risk capital.\(^{21}\)

The results presented in the following are based on 200,000 simulations. For quantifying risk capital we use the two risk measures value-at-risk \(\text{VaR}\) and tail-value-at-risk \(\text{TVaR}\). Both risk measures are often used in practice.

For our simulation case study we use the claim development triangle of paid claims presented in Mack (1993) and in England and Verrall (2006), see Figure 4.

<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Development Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>357,848</td>
</tr>
<tr>
<td>1</td>
<td>352,118</td>
</tr>
<tr>
<td>3</td>
<td>310,608</td>
</tr>
<tr>
<td>4</td>
<td>443,160</td>
</tr>
<tr>
<td>5</td>
<td>396,132</td>
</tr>
<tr>
<td>6</td>
<td>440,832</td>
</tr>
<tr>
<td>7</td>
<td>359,480</td>
</tr>
<tr>
<td>8</td>
<td>376,686</td>
</tr>
<tr>
<td>9</td>
<td>344,014</td>
</tr>
</tbody>
</table>

CL Factors: 3.4906, 1.7473, 1.4574, 1.1739, 1.1038, 1.0863, 1.0539, 1.0766, 1.0177

Fig. 4: Claims Development Runoff Triangle (Accumulated Figures)

<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Chain Ladder Reserve Estimate</th>
<th>Prediction Error (Mack)</th>
<th>Prediction Error (Mack) in % of Reserve Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>79.82%</td>
</tr>
<tr>
<td>1</td>
<td>94,634</td>
<td>75,535</td>
<td>79.82%</td>
</tr>
<tr>
<td>2</td>
<td>469,889</td>
<td>411,010</td>
<td>28.96%</td>
</tr>
<tr>
<td>3</td>
<td>709,638</td>
<td>558,317</td>
<td>25.64%</td>
</tr>
<tr>
<td>4</td>
<td>998,899</td>
<td>875,328</td>
<td>22.33%</td>
</tr>
<tr>
<td>5</td>
<td>1,419,439</td>
<td>971,258</td>
<td>22.70%</td>
</tr>
<tr>
<td>6</td>
<td>2,177,641</td>
<td>1,363,155</td>
<td>29.47%</td>
</tr>
<tr>
<td>7</td>
<td>3,920,301</td>
<td>2,447,095</td>
<td>13.10%</td>
</tr>
<tr>
<td>8</td>
<td>4,278,972</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4,625,811</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>18,680,856</td>
<td>2,447,095</td>
<td>13.10%</td>
</tr>
</tbody>
</table>

Fig. 5: Estimated Reserve and Prediction Errors

For simplicity we assume, that all claims are settled after a development period of ten years and thus do not use a tail for the development periods greater than ten years.

\(^{21}\) The simulation model described in the case study is modelled using the simulation software EMB IGLOO™ Extreme.
Figure 5 shows the chain ladder reserve estimates which are calculated using the deterministic chain ladder algorithm. The standard error according to Mack (1993) is estimated using Mack’s formula and presented in the second column of the Figure 5.

In the following we use the idea of multi-year stochastic re-reserving presented in Section 3.1. In order to simulate the next \( n \) diagonals we used bootstrap methods and bayesian methods using markov chain monte carlo techniques based on the classic Mack model (1993) as presented in England and Verrall (2006). Because both methods led to similar results we concentrate on bootstrap methodology in the following.

For the bootstrap method we first have to calculate the pearson residuals (see England and Verrall 1999, 2006). With the pearson residuals we can generate pseudo development factors and pseudo data for measuring parameter uncertainty using resampling. For modeling the process error we additionally need an assumption on the underlying frequency distribution. In our application we use the normal distribution. With the help of the normal distribution as process distribution we can generate the future payments per simulation path.

200,000 repetitions of this process lead to 200,000 different claims development triangles. For the multi-year stochastic re-reserving based on bootstrap techniques we use the cash flows for the next \( n \) calendar years generated with bootstrap techniques as the next \( n \) diagonals in our run off triangle. Hereby we get 200,000 new triangles, differing only in the last \( n \) diagonals. We use the deterministic chain ladder method on each of these new triangles. This means per simulation path we complete the triangles to quadrangles, which is the base for estimating the claim reserves per accident year \( i \).

We repeat the re-reserving process until the final settlement of all claims as described in Section 3.1. We assume that after ten development years all claims are finally settled and so we repeat the re-reserving process nine times.

As a result, Figure 6 shows the empirical frequency distribution of the simulated multi-year cumulative claims development results \( \text{CumDevRes}_n \), for \( 1 \leq n \leq 9 \). Because the runoff triangle is completely settled after nine years, we have \( \text{CumDevRes}_9 = \text{CumDevRes}_{10} \). This means that the ultimate claims development results equals to the nine-year one, which is achieved by repeating the re-reserving method for nine future development years.

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\(^{22}\) For the estimation of \( \hat{\sigma}_n^2 \) we used the simplified extrapolation formula \( \hat{\sigma}_n^2 = \min\{ \hat{\sigma}_n^2, \hat{\sigma}_s^2, \sigma_s \} \), which differs from the form used in Mack (1993).

\(^{23}\) We consider the normal distribution as process distribution in this application only for illustrative purposes and for comparability reason since this makes our results comparable to those presented in Merz and Wüthrich (2007). Relying on the normal distribution, however, has some disadvantages, e.g., it allows negative cumulative claim payments, which is not adequate. Furthermore, the symmetric form of the normal distribution is not adequate for the right-skewed claim distributions in non-life insurance. As a different process distribution the lognormal or the gamma distribution can be used (see, e.g., England and Verrall, 2002, 2006; Bjoerckwall et al., 2009).
By means of the risk measures VaR and TVaR we now calculate the economic capital that is needed in order to survive. For this purpose we use the empirical frequency distribution shown in Figure 6 where we have the negative multi-year CumDevRes\(_n\) as a random variable of losses \(L\) for the respective future development year \(n\).

The results, i.e., the development of risk capital over time from one-year risk capital up to the ultimate risk capital after nine years are shown in Table 7. Here you can see that the one-year
risk capital measures around 70\% of the ultimate risk capital. The higher the confidence level the higher the demand for risk-based capital. Furthermore, the use of TVaR instead of VaR leads to a higher demand for risk-based capital.

Next to this line of reasoning, we argue that management is faced with the risk of running out of capital before the end of time period \( n \). Hence, we have to take into account the fact, that the negative multi-year claims development result (\( \text{CumDevRes}_n \)) for time period \( n \) can be lower than any negative multi-year \( \text{CumDevRes} \) before. Thus, we have to take the maximum loss of all negative multi-year claims development results (\( \text{CumDevRes}_n \)) as a random variable of losses \( L \) for the respective future development years \( \{1, \ldots, n\} \). This leads to a different empirical frequency distribution and therefore to a different need for risk-based capital.

In Figure 8 we illustrate the different development of the simulated negative multi-year \( \text{CumDevRes}_n \) for one single simulation path. Here you can see that the maximum function leads to a different negative multi-year \( \text{CumDevRes}_n \) for the development years 4, 6, 7, 8 and 9.

![Figure 8: MaxLoss\(_n\) and -CumDevRes\(_n\)](image)

Finally, in Figure 9 we illustrate the development of multi-year risk-based capital using the random variable \( \text{MaxLoss}(n) = \text{MAXIMUM}_{1 \leq i \leq n} \{\text{CumLoss}_i\} \).

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\(^{24}\) The dataset used in our example corresponds to a runoff triangle for a long-tailed line of business. We have also used runoff triangles for a short-tailed line of business. Here you can see that the one-year risk-based capital measures around 90\% of the ultimate risk measure.
4. Parameter risk in premium risk – A quantitative study

In the following we come to another part of insurance risks – parameter risk in premium risk which is often ignored in internal models. We start with a short description of the sources and character of parameter risk.

Modeling in internal models is based on a kind of prediction process. That means for example that we try to predict future claim losses based on observations from previous years. This may give rise to different sources of uncertainty: model uncertainty, prediction uncertainty, which can be divided into parameter uncertainty and process uncertainty. Process uncertainty describes the uncertainty from the actual random process. Parameter uncertainty, on the other hand, results from the uncertainty in estimating the parameters from the model.

While modeling reserve risk, for example, process and parameter risk are considered in both the standard formula (Solvency II) and in stochastic modeling in internal models (Mack’s model, Bayesian methods, bootstrap methods etc.). By contrast in premium risk parameter risk is often ignored, leaving only the process risk modeled.

Some methods of modeling parameter uncertainty have been covered in the literature (see for example Cairns (2000) and Mata (2000)). We will be referring to two of these in the following – bootstrap and Bayesian, as described by Borowicz/Norman (2006a) – in modeling premium risk and give a quantification of parameter risk for one line of business.

Sources of parameter risk

This section will begin with a brief explanation of parameter risk and its origin. In internal models parametric distributions are fitted to historical claims severity and frequency distributions for prediction of future claims. Assume $Y$ as random variable and consider $y$ as realisations of $Y$. Assume that the distribution class is already known, that is, that the distribution of $Y$ has been fully specified except for an unknown parameter. If this class is referred to as $\Gamma = \{F_\theta : \theta \in \mathbb{R}^d\}$, element $F_{\theta_0}$ remains to be determined in the distribution family as corresponds to the distribution of $Y$. Since parameter $\theta_0$ is unknown, it will have to be estimated from the finite set of observations. Traditionally, the parameter is estimated as a
deterministic value, for example using the method of moments or maximum likelihood estimation. But if you take two finite samples from the identical distribution, the maximum likelihood estimates for the parameters will differ. So the estimator itself is only a realisation of a random variable. This results in parameter risk.

**Modeling parameter risk using bootstrap methods**

First we use the bootstrap method to quantify parameter risk. The bootstrap method is based on the idea of “sampling with replacement” from the original sample, which is known in discrete probability theory in connection with basic urn models. The assumption is that the observations are independent and identically distributed.

Our aim is to explain the parameter risk calculation using an example for large claim modeling.\(^{25}\) We start with modeling claim numbers of large claims. Fig. 10 shows an example record for ultimate claim numbers of large claims from the last \(n=10\) years in an example segment.\(^{26}\)

<table>
<thead>
<tr>
<th>Year</th>
<th>1998</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Large Claims</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

Fig. 10: Claim frequencies observed

Applying the bootstrap method we obtain a data set \(y^s = (y_1^s, ..., y_n^s)\) by sampling with replacement from the observed data. Repeating this procedure can create a multitude of data sets (e.g. 10,000 scenarios \(s\)).

For modeling the number of claims here we use the Poisson distribution as the process distribution.\(^{27}\) Parameter \(\lambda^s\) of the Poisson distribution is estimated from each simulated data record \(y^s\). This approach yields the empirical bootstrap distribution (see Fig. 5).

For generating predictions for claim numbers we have to include process risk. So we draw a single claim number \(z^s\) from a Poisson distribution with parameter \(\lambda^s\) for each scenario \(s\) (e.g. 10,000) and obtain the empirical claim frequency distribution including parameter and process uncertainty.

Applying the bootstrap process is highly “time-intensive.” Another disadvantage is that the parameters can only be simulated within a limited range set by observations, because we sample from the observed data.

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\(^{25}\) See also *McLennan/Murphy* (2004).

\(^{26}\) €250,000 has been set as threshold for large claims.

\(^{27}\) We will desist from using statistical methods for selecting a suitable model for modeling claim frequencies here (e.g. Poisson or negative binomial).
**Modeling parameter risk using a Bayesian approach**

One alternative in modeling parameter risk is the use of a Bayesian approach as described in Borowicz/Norman (2006a).

Consider \(y\) as observations from random variable \(Y\) with unknown parameters. Let \(\theta = (\theta_1, \ldots, \theta_m)\) be the random parameter vector of the distribution of \(Y\). With some initial knowledge about the distribution of the unknown parameters \(p(\theta)\), namely the likelihood and the prior distribution, one can produce a posterior distribution \(p(\theta | y)\) using Bayes’ Theorem:

\[
p(\theta | y) = \text{constant} \cdot \text{likelihood} \cdot \text{prior}.
\]

Under uniform prior distributions for parameter vector \(\theta\) the posterior distribution (based on observations \(y\)) is simply proportional to the likelihood:\(^{28}\)

\[
p(\theta | y) \propto L(\theta | y).
\]

We return to the example segment with the observed claim numbers \(y = (y_1, \ldots, y_{10})\) from Fig. 4. Assume that as above the Poisson distribution is the underlying process distribution. In this case, the posterior distribution of the parameter \(\theta = \lambda\) is exactly solvable. With uniform prior it can be shown that the density of the corresponding posterior distribution can be represented using the density from gamma distribution:\(^{29}\)

\[
p(\theta | y) = f(\theta; \beta; \gamma) = \frac{\theta^{\beta-1} \exp(-\frac{\theta}{\gamma})}{\Gamma(\beta)\gamma^\beta}, \quad \text{where} \quad \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,
\]

with parameters

\[
\beta = \sum_{i=1}^n y_i + 1 \quad \text{and} \quad \gamma = \frac{1}{n}.
\]

For other choices of frequency distribution, it may not be possible to write the posterior distribution in such a form. In these cases one can use Gibbs/ARMS sampling methods.\(^{30}\)

Fig. 11 shows a comparison for the percentile graphs of the parameters simulated using the two methods, bootstrap and Bayesian. The Bayesian approach leads to a greater volatility in the parameters.

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\(^{28}\) See Borowicz/Norman (2006a).

\(^{29}\) See Borowicz/Norman (2006a).

\(^{30}\) See Gilks/Richardson/Spiegelhalter (1995) and Gilks/Best/Tan (1994).
In the next step we model the predictive distribution for the number of large claims for quantifying prediction risk. The predictive distribution is simulated by first simulating the Poisson parameter $\lambda^s$ from the gamma distribution in (13) per simulation $s$, and then for each simulation $s$ we draw a single number $z^s$ from a Poisson distribution with parameter $\lambda^s$. So we obtain the empirical claim frequency distribution including parameter and process uncertainty. In this case the underlying Poisson model allows prediction distribution representation as negative binomial distribution with the following parameters:

$$m = \sum_{i=1}^{n} y_i + 1 \quad \text{and} \quad q = \frac{1}{n + 1}.$$  \[31\]

Fig. 12 shows some of the percentiles from the simulated large claim frequencies including and excluding parameter uncertainty (bootstrap and Bayes). Greater volatility in large claim frequencies emerges where parameter uncertainty is included.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Bootstrap</th>
<th>Bayes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>5.2</td>
<td>5.3</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.61</td>
<td>0.73</td>
</tr>
<tr>
<td>Median</td>
<td>5.2</td>
<td>5.3</td>
</tr>
<tr>
<td>Min</td>
<td>3.0</td>
<td>2.7</td>
</tr>
<tr>
<td>Max</td>
<td>7.3</td>
<td>8.6</td>
</tr>
<tr>
<td>50th percentile</td>
<td>5.2</td>
<td>5.3</td>
</tr>
<tr>
<td>60th percentile</td>
<td>5.3</td>
<td>5.4</td>
</tr>
<tr>
<td>70th percentile</td>
<td>5.5</td>
<td>5.6</td>
</tr>
<tr>
<td>80th percentile</td>
<td>5.7</td>
<td>5.9</td>
</tr>
<tr>
<td>90th percentile</td>
<td>6.0</td>
<td>6.2</td>
</tr>
<tr>
<td>99th percentile</td>
<td>6.6</td>
<td>7.1</td>
</tr>
<tr>
<td>99.9th percentile</td>
<td>7.1</td>
<td>7.9</td>
</tr>
<tr>
<td>99,99th percentile</td>
<td>7.3</td>
<td>8.8</td>
</tr>
</tbody>
</table>

Fig. 12: Percentiles for process risk and prediction risk (bootstrap and Bayesian approach)

\[31\] The convention is such that the expected value of $\text{NegBin}(m,q)$ is $mq/(1-q)$ and the variance is $mq/(1-q)^2$. 
Fig. 13: Simulated frequency distributions for the two parameters, beta and gamma

After including parameter uncertainty in modeling large claim frequencies, the next step is to include this in modeling large claim severity. We have sixty-five large claims for the last ten years as observations (ultimates, suitably indexed). Assume that a translated gamma distribution $\text{Gamma}(\beta, \gamma)$ has been fitted to large claims observed using the usual statistical methods, with parameters $\beta = 1.18$ and $\gamma = 263,998.6$ estimated using the maximum likelihood method (threshold €250,000).

Using the log-likelihood function of the gamma distribution and the sampling methods described in Gilks/Richardson/Spiegelhalter (1995), the parameter distribution (Fig. 13) can be simulated as shown in Borowicz/Norman (2006a), where the negative dependencies between parameters $\beta$ and $\gamma$ remain included as shown in Fig. 14.\footnote{See Borowicz/Norman (2006a).}

Fig. 14: Dependency structure between the parameters simulated (beta and gamma) and parameter estimated using the maximum-likelihood method
After separately modeling large claim frequency and large claim severity (both including parameter uncertainty), the next step is to simulate large claim losses (Fig. 15), assuming that the claim severities are independent and identically distributed random variables and that they are independent of claim number, thus fulfilling the condition of a collective model.\textsuperscript{33}

The final step is to quantify the risk-capital requirement for large claim loss in the example segment including and excluding the parameter risk. Assume a TVaR as risk measure at confidence level at 99.8%.\textsuperscript{34} While risk-capital requirement amounts to €5.8 million for process risk, including parameter risk we need a risk capital of €6.7 million.

The extent of parameter uncertainty depends strongly on the data volume observed. Few observations will usually lead to substantially higher parameter uncertainty compared to observed data based on a long history with lots of observations.

To quantify these effects the following example is based on a segment with only five large claims as observed data instead of sixty-five in the above example. We have also applied gamma distribution in this example for modeling claim severities. Fig. 10 shows the percentile graphs for large claim severity including and excluding parameter uncertainty compared to historical data.

The percentile graph for large claim severity including parameter uncertainty (prediction risk) shows substantially higher values in the tail due to the low observation number compared to where parameter uncertainty is ignored (Fig. 16). This means that ignoring the parameter risk can lead to a substantial underestimation of the risk situation, especially in segments with only few observed data.

\textsuperscript{33} See Diers (2007a) for modeling large claim losses following the collective model.\textsuperscript{34} We define risk capital requirement on the random variable $E(L) - L$, where $L$ represents the random variable of large claim loss with expectation value $E(L)$. 
These examples also show that ignoring parameter risk – even in a history of sixty-five claims – can lead to a sizable underestimation in risk-capital requirement. However, note that we have modelled large claims from one line of business, and that these effects may occur in modeling attritional, large and catastrophe claims from all lines of business of the company. In addition a parameter uncertainty in dependency structures should be considered.\textsuperscript{35} In this Section we considered parameter uncertainty in premium risk. Parameter uncertainty can be found in a similar form in other risk categories.

5. Conclusion

The aim of modern management techniques such as value and risk-based management is to evaluate and manage risks and returns in a multi-year period. In the actual literature several questions concerning the use of internal models in a multi-year management context are not answered to date. With this study we want to make a contribution to this emerging field of research. We presented a multi-year model approach and defined a multi-year risk-capital concept that can serve as an internal capital requirement (ORSA) spanning several years.

In the context of multi-year internal models we need techniques to quantify the one-year development result for each period. Moreover Solvency II requires the quantification of one-year reserve risk instead of ultimate reserve risk. In this paper we used the stochastic re-reserving method based on simulations. We gave a quantitative comparison of the re-reserving method to the analytical methods discussed in the actual literature.\textsuperscript{36} In a case study we calculated multi-year risk capital for reserve risk in order to illustrate the potential usefulness of the models for its application within multi-year internal risk models.

\textsuperscript{35} See Borowicz/Norman (2006b).

\textsuperscript{36} See Merz / Wüthrich (2008) and Wüthrich / Merz / Lysenko (2008).
Another part of insurance risk – parameter risk in premium risk – is often ignored in internal models. We presented two modeling approaches and quantified the effects of parameter risk in premium risk using example data in order to increase the awareness of the importance of these risks. We have demonstrated that parameter risk in premium risk may have a substantial effect on the company’s risk situation, and therefore constitutes a risk that should be taken into account to a sufficient degree.

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