Dynamic asset allocation techniques
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Abstract
Investment strategy is often static, punctuated by infrequent reviews. For most long-term investors, this practice results in large risks being taken that could otherwise be managed with a more dynamic investment policy. The bulk of this paper is aimed at analysing and describing two multi-period investment strategy problems – in order to derive potential dynamic strategies. Along the way, we show how static investment strategies can fail to deliver an investor's long-term objectives and describe the relationship of our work to other areas of the finance literature. This paper does not cover trading strategies such as Tactical Asset Allocation.

This paper sets out two main approaches to the multi-period problem. The first approach optimises a utility function. The second approach uses partial differential equation (PDE) technology to optimise a target statistic (in this case, TailVaR) subject to return and long-only constraints.
1: Introduction

1.1. Beyond single-period investment theory
Modern portfolio theory, as introduced by Markowitz and developed by many others, provides practical techniques for analysing – and optimising – the trade off between risk and reward. This body of work is applied in a number of contexts and in a number of ways (for example, Sharpe (1964), Ross (1976), Rom and Ferguson (1994)). A common strand is that these are single-period theories: there is a single time horizon, risk and return estimates are made over that horizon and these enable a choice of investment strategy to be made.

An equally important but less widely understood stream of work within the financial academic literature examines the investment problem over multiple periods. A challenge - and a key attraction - of these papers is that the objectives of the investor come much closer to the surface. For a real-world investor – a long-term institution such as a pension plan or an insurance company, or an individual seeking to manage their savings so as to smooth their lifetime income – the multi-period experience is much more relevant, and the investment strategy can and should evolve through time in sympathy with their objectives.

(To see the importance of this, consider performance measurement as a simple example. Modern portfolio theory provides clear and useful guidance on the construction of risk-adjusted performance: Sharpe ratios, Sortino ratios and the like are heavily used across the capital markets industry. In contrast, for a specific investor in the real world, it is the money-weighted returns that are important: not these ratios that depend upon time-weighted returns generated by a notional investment.)

This paper seeks to describe, and apply, some techniques inspired by this multi-period approach: we are concerned with investment strategies that are dynamic in some way. These are strategies where the investment policy changes through time in some defined manner, in order to better meet an overall investment goal. We spend a fair amount of time expounding approaches to the multi-period problem that deserve to be better known. The problems faced by long-term investors are often multi-period in nature, breaking such a problem down into a series of single-period problems often means that some opportunities for a better solution are missed.

Table 1. Approaches compared

<table>
<thead>
<tr>
<th>Single-period approach</th>
<th>Multi-period approach</th>
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<tbody>
<tr>
<td>Short-term focus predominates</td>
<td>Long-term objectives can play a role alongside short-term risk management</td>
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<tr>
<td>Strategy will only evolve when another investment review takes place</td>
<td>Methodology for strategy evolution is put in place on day one</td>
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<tr>
<td>Symmetric view of risk likely to be chosen for simplicity or computational reasons</td>
<td>Asymmetric, goal-focused, approach naturally leads to asymmetric risk measures</td>
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1.2. Policy evolution over time
The usual output from Modern Portfolio Theory is an efficient frontier of investment strategies: there is a hedge portfolio, a growth portfolio, and the choice of a particular point on this frontier (the amount invested in the growth portfolio) is a matter of investor judgement. This choice might typically be expressed in terms of either a return target, risk target or risk aversion. When it
comes to updating this view at the next review (which might be between a week and three years later), should this judgement change? Describing a framework in which it is sensible for the point on the efficient frontier to move, and the manner of its movement over time, is one of our objectives here.

Dynamic strategies offer the opportunity to manage risk across time, but this also leads to a sudden explosion in the number of possible strategies. Fortunately, this is not a new problem in financial theory: quite the reverse. The dynamic investment problem for individuals has been the subject of a huge volume of financial theory, starting, in its modern form, with Merton (1969) and Merton (1971). While the initial model economies had constant interest rates, more recent work has moved this into more complex (stochastic interest rate) models, where the risks of assets and liabilities can be more fairly treated together.

The actuarial community has also contributed to this body of work. For example, Appendix A in Smith (1996) contains an exposition of various techniques. Kemp (1996) discusses how these techniques could be applied in practical situations, and shows how important it is to work within a model that does not allow anomalously attractive investment strategies – dynamic optimisation is often much more able to find and exploit these opportunities than a static approach. This leads to results that say more about the model than about sensible investment strategy, and may have acted as a disincentive to dynamic optimisation. Cairns (2000) solves a dynamic optimisation problem in a pension fund context, by solving a Bellman equation, and discusses the impact of different preference structures on the appropriate strategy. Neilsen (2006) presents a good overview of the academic literature. The recent EDHEC paper, Martellini et al (2009) takes a similar perspective to ours here. On the numerical side, the Dempster et al paper (2003) describes numerical approaches that form the inspiration for the work described in section 6 below.

1.3. Market evolution over time
We should emphasise that the evolutions in portfolio allocation that we discuss in this paper are triggered by the investor’s objectives. Although the investor’s preferences are assumed to stay constant, they say different things at different values of the portfolio. In contrast, it is very common for investors to change their portfolio allocation in response to changes in markets. The market opportunity set does change over time, and an informed investor should of course be wise to this and adjust accordingly.

This adjustment could take several forms. At its simplest, it would be awareness that market risk is not constant – and therefore risk forecasts should be updated regularly to reflect new information. A much further move would be to actively allocate assets, based on an investor’s views of the returns from the changing opportunity set. This is hard, and usually expensive. This form of tactical asset allocation is outside the scope of this paper.

1.4. Motivations and scope
As discussed above and more fully in section 4, one motivation behind our work was to assess the potential benefit to long-term investors of dynamic strategies. We develop strategies in relatively simple model settings in order better to analyse these benefits. These candidate strategies can then be tested in more complicated models and in more practical market contexts.

A related motivation behind our work was to provide a framework within which option strategies can be considered. There has been an increased interest among (the advisers to) pension schemes in the use of options to tailor the distribution of future funding levels. Although these can be fairly easily presented with plausible arguments in their favour, it is usually very difficult to see why a particular strategy has been chosen above another, unless perhaps the profit to the writer of the options is taken into account. We wanted to support the general thrust behind these option
strategies, while providing a means whereby less expensive and potentially more liquid options can be used to meet the same objectives.

This paper therefore seeks to meet several goals:

- To provide a brief description of the utility-based approach often used to analyse multi-period problems, and to give an overview of some of the results in this significant vein of the finance literature;
- To provide a parallel description of a non-utility based solution;
- To provide examples from the finance literature, from life insurance and from pensions where a dynamic policy is often considered;
- To show how some of the insights from the utility approach can be applied to these problems; and
- To analyse the improvement that could result from the use of more dynamic investment strategies, and in particular the use of options within pension plans.
2: Risk and return measures

2.1. Quantifying the trade off between risk and return

When choosing an investment strategy, investors select assets that are likely to meet their particular return goals while managing the likelihood and/or severity of returns falling short of these goals. Assessing the right trade-off between return and risk therefore requires careful thinking about:

- **Target outcome.** The investor’s goals should be clear. This may be expressed in terms of a benchmark or a return target, for example. Ideally these goals will be embedded in the wider situation facing the investor – the investor’s asset allocation is likely only one of several levers that can be adjusted to meet these goals. This goal then helps determine the average return required of the investor’s assets.

- **Risk tolerance.** A judgement of the extent to which the investor is prepared to tolerate the return falling short of this target return. In the simplest situation, this could be expressed as an aversion to volatility, although more explicitly focusing on the probability or severity of poor returns may be more appropriate.

With these in place, an investor will then seek to carry out a form of optimisation – finding a strategy which gives as good a trade off as possible between the average return and the variability relative to this average. In carrying out this optimisation, further choices have to be made: what time horizon is relevant for the investor? What instruments are available in the investment universe? Is the strategy static or is it able to evolve dynamically in response to events?

2.2. Choosing appropriate measures

The particular choices of return and risk measure will vary according to the investor, or user, of the statistics. Some possible considerations include:

- Regulators are likely to focus on tail risk; the risks may be measured via deterministic stress tests in addition to more probabilistic measures such as value at risk.

- An investor with a longer-term perspective may be happier to focus on the dispersion of returns around the general level, rather than on the tail. This may lead to worse short-term performance, e.g. if market liquidity dries up, but the investor will expect to get paid sufficient premium for bearing this risk.

- Particular portfolio characteristics may be more intuitive in specific circumstances. For example, active duration or PV01 would often be a more natural risk metric in a fixed income setting than a statistical measure such as tracking error.

- A risk metric agreed between an investor and an investment manager will need to be verifiable. Ex-ante risk, used in portfolio construction, and ex-post risk, derived from actual returns achieved, may both need to be tracked to ensure that they are compatible. The latter may be especially hard for some measures of tail risk (we don’t want to wait 200 years to see if a tail event only happened 1 year in 200) or for strategies where the risk budget varies dynamically, so some visibility into the ex-ante risk methodology – and perhaps the use of a third party model - may be required.

- The timescale(s) for risk measurement should be clear, particularly for instruments or strategies whose exposures are not constant over time. For example, the characteristics of an option in a portfolio held for a short period would often be materially different if the option is held until maturity, and risk is measured over this longer timeframe.
2.3. Mean-variance approaches

The classic Markowitz approach involves trading the expected return outcome off against the volatility of returns around this outcome. This can be framed in multiple equivalent ways:

- Maximise expected return for a given level of volatility;
- Minimise volatility for a given level of average return
- Maximise risk-adjusted return = return – aversion × variance

These are equivalent: varying the level of volatility, return or aversion respectively in the 3 versions above sweeps out the same ‘efficient frontier’ of portfolios that are mean-variance efficient.

In a single-period situation, this can work very well. If we use one of the simplest models of asset price uncertainty, namely the assumption asset returns are distributed according to a multivariate normal distribution, we can readily create risk/return trade-offs for a variety of risk measures. For many risk measures, the joint-normality assumption means that these generally give identical frontier portfolios, so the simplicity of mean-variance optimisation makes it an attractive approach.

The situation in a multi-period situation is rather different. Joint-lognormality is the natural simple choice in a multi-period framework. If we work in a continuous time dimension, then this results in multivariate Brownian motion with constant drift and covariance.

Efficient frontiers can be established for different rebalancing frequencies: from static portfolios, through portfolios that rebalance \( n \) times over a fixed period, through to a theoretical limit of continuous rebalancing.

Figure 1 (from Goetzmann et al 2002 – see appendix B for a derivation) shows the distribution of outcomes from a portfolio that is continuously-rebalanced in order to optimise the realised Sharpe ratio.

It is readily seen that (a) despite the underlying process being normally distributed, the resulting optimal distribution is far from normal (b) the left-hand tail of ‘bad’ outcomes is significantly fatter than the static portfolio comparator. The Sharpe ratio measure, based on standard deviation, may fail to adequately capture an investor’s risk preferences once we move away from normally distributed outcomes. A metric which appears to adequately select between ‘good’ and ‘bad’ strategies in a simple situation may no longer be an adequate description of the investor’s goals in situations that allow more complex outcomes.
2.4. Utility-based approaches

In principle, therefore, we would like to find a way to rank possible sets of outcome – we might express these outcomes as cash flows or values that arise from our strategy over time. A scoring function

\[ u : \{ c_t(\omega) \} \rightarrow \mathbb{R} \]

could be used to do this for a strategy that produces cashflows \( \{ c_t \} \) in scenario \( \omega \). Although this is not crucial, a common approach is to consider expected-utility scoring functions, i.e. of the form

\[ u(\{ c_t(\omega) \}) = \sum_t \mathbb{E}(U_t(c_t)) \]

Where \( U_t(w) \) is the utility score from wealth \( w \). The utility function applies a score to each consumption and wealth outcome; two strategies can then be directly compared by comparing the expected utility, using the distributions of outcomes from following the two strategies.

In this basic set-up, utility is acting as both return and risk measure. Alternatively, utility could be combined with a risk measure such as volatility: maximising utility subject to a cap on the variance of outcomes would then provide a way to focus separately on the desired outcomes and express tolerance for risk.

In principle, this is in fact little different to the expected return / standard deviation trade off, which can be expressed in terms of wealth \( w \) as:

\[ \text{Maximise } \mathbb{E}(w_f) - \lambda \mathbb{E}(w_f^2) \]  

(1)

Maximising expected utility, \( U \), is simply
Clearly a quadratic utility function will suggest the same investment strategies as those for an investor using the Sharpe ratio as their risk measure. It is easy to mock the quadratic utility function (e.g. it makes no sense for utility to fall beyond a certain level of wealth – the so-called bliss point) however if the time period is short and the impact of rebalancing choices small, a quadratic approximation to any utility function will be likely to do a good job.

The utility function formulation is explicitly more general, so the question arises: what choices of utility function make sense / give ‘sensible’ portfolio outcomes? Time-homogeneity is a natural requirement to seek. This means that maximising expected utility over two adjacent time periods separately should give the same answer as maximising over the whole time period, at least if the two sub-periods are independent and distributionally identical (i.i.d.). Such considerations lead to a family of ‘constant relative risk aversion’ (CRRA) utility functions. These are of the form

\[ U(w) = \frac{w^\gamma - 1}{1 - \gamma}. \]

CRRA utility function

Surprisingly, although (1) is amazingly successful in the single period case, this does not generalise neatly to multi-period investment policies, and maximising a function of the form (3) becomes the more natural choice. The results of using (3) in simple models are also much more intuitive, of which more below.

Investment strategy problems using the CRRA utility function are well-studied. Although this leads to mathematically-precise, aesthetically-pleasing results, the challenge in practical situations that interest us is to identify how a CRRA function fails to properly express investor preferences. Utility functions are in practice impossible to calibrate to an investor (indeed a significant strand of the behavioural finance literature argues that this is not possible even in theory) – however we can hope to use utility functions to express key features and then use optimisation techniques to learn what investment strategies make sense as a result. This is just a case of George Box’s oft-quoted statement “All models are wrong, but some models are useful.”

Appendix A provides some additional background on utility functions.
2.5. Asymmetric outcomes and tail risk measures

Variance has the notorious property that it penalises above-average outcomes as much as below-average outcomes. As we have seen, volatility is most appropriate if the underlying distributions to which it is applied are symmetrical. If the distributions have material skewness – for example if options form part of the allowable investment universe or dynamic strategies could give rise to option-like payoffs – then using a symmetric risk measure may give unappealing results. The average outcome plus dispersion around this may give a misleading sense of the real outcome distribution.

This simply re-emphasises the point that the focus should be on the investor’s goals – if these are properly captured by an average return, well and good. If not, an appropriately structured utility function may provide more control. Alternatively, measures of risk have been developed that focus instead on the region of interest, i.e. losses – the ‘left hand tail’ of returns well below the average outcome.

Value at risk (VaR) is such a measure that has seen widely use for many years. More recently, academics have tended to use other risk measures, in a growing literature that was kick started by the famous article of Artzner et al on coherent risk measures, which pointed out the lack of coherence of VaR. Whereas VaR essentially reports quantiles such as the ‘1 in 200 event’ TailVaR, ‘tail value at risk’ or ‘conditional value at risk’ looks at the average of the worst 0.5% of possible outcomes. Due to its coherence, it has attractive theoretical and analytical properties that make it more suitable for modern risk management.

Therefore, trading off the mean outcome against a tail risk measure such as TailVaR or the mean utility against this measure can be an attractive means to compare tail-risk management procedures.

2.6. Value at risk (VaR) and tail value at risk (TailVaR)

Value at risk (VaR) measures the potential loss over a particular time frame. The loss is stated at a particular probability level, e.g. 99.5% of the time the annual loss will be no more than 20%.

Although authors generally agree upon the meaning in practice, there is considerable scope for confusion when writing down a definition: do you think of the probability level as 0.5% or 99.5%? Is the focus on the portfolio return or the portfolio loss? For clarity, the definition we use here is as follows. Suppose $X$ is a random variable representing the portfolio value at the time horizon we are interested in, and $\varepsilon$ (e.g. $\varepsilon = 5\%$ ) is the tail probability we are interested in:

$$VaR_\varepsilon(X) = \inf\{x : P(X \leq x) > \varepsilon\}.$$
This definition is illustrated in the figure 2, which is based on a Student t distribution. In this continuous-density example, the vertical red dotted line represents the point at which the 5% probability level is breached. The value at risk is -4.2 in this example.

The picture also shows the CVaR, the conditional or tail value at risk, also sometimes known as expected shortfall (ES), although this terminology is used elsewhere with a slightly different meaning. This represents the average loss in the blue area shown: first define the quantile function

$$q_x(p) = F^{-1}_x(p) = \inf \{ x : F_x(x) \geq p \} \in \mathbb{R} \cup \{-\infty, \infty\}.$$  

And then

$$\text{TailVaR}_\epsilon(X) = \alpha^{-1} \mathbb{E} \left[ \left( F^{-1}_x(\epsilon) - X \right)^+ \right] - F^{-1}_x(\epsilon).$$

In the intuitive case when there are no jumps in the cumulative distribution function, the quantile function is, up to a sign, the same as the value at risk, and the CVaR is the expected loss in the tail bounded by the VaR. That is, informally, TailVaR is the average over the worst alpha% of outcomes (e.g. alpha = 5%). Where the portfolio can take particular values with positive probability (e.g. this will be the case for typical option payoffs; or when working with simulations) slightly more care is required.

Interest in tail-risk measures has increased in the wake of Artzner et al (1999). A substantial corpus of papers examines measures of risk, with particular focus on those measures that are coherent.

An alternative definition of TailVaR/CVaR is as the solution of an optimisation problem:

$$CVaR_{\alpha}(X) = \inf \left\{ \frac{\mathbb{E} \left( (s - X)^+ \right)}{\alpha} : s \in \mathbb{R} \right\}.$$
This formulation is due to Rockafellar and Uryasev (1999); see also Pflug (2000). Crucially, the functional (of $s$ and $X$) being optimised here is a convex function. This means that very often – for example, when seeking the minimum CVaR for a given expected return – any local minimum is guaranteed to be a global minimum, and convex optimisation techniques can be used to steadily locate the desired optimum.

A similar feature is that the nature of the optimisation is often changed from being one of a ‘global’ nature to one that is ‘local’. By fixing $s$, small changes in one part of the distribution of $X$ can be considered, and the effect on the functional analysed near where these changes are made. In the more usual form (as in the definition of tail mean above), the percentile (the Value at Risk) has to be recalculated for any small change – and this is a global calculation (i.e. it depends on the whole distribution). Features of the optimum can therefore be derived before optimising $s$, and this makes the search for the optimum considerably easier.
3: Techniques for finding an optimal investment strategy

3.1. Introduction
In the previous section, we introduced two possible measures that could be used to help determine good investment strategies: TailVaR and utility functions. In this section, we describe two techniques that can be applied to these problems. The first uses a ‘forward’ technique to solving optimisation problems. The second uses a ‘backward’ one, more familiar from option pricing theory.

3.2. The Kolmogorov equations
Suppose that $X$ is a positive diffusion process satisfying a stochastic differential equation of the form

$$dX = \beta(u, X(u))du + \gamma(u, X(u))dW$$

(4)

The Kolmogorov equations are (partial, rather than stochastic) differential equations for the transition density $p(s, t, x, y)$. This is the probability density of $X$ at time $t$ given its position at time $s$: informally,

$$\mathbb{P}(X_t \in (y, y + dy) | X_s = x) = p(s, t, x, y)dy$$

The Kolmogorov backward and forward equations describe how this density changes according to changes in the starting point and ending point respectively:

$$\frac{\partial}{\partial s} p(s, t, x, y) = \beta(s, x) \frac{\partial}{\partial x} p(s, t, x, y) + \frac{1}{2} \gamma(s, x)^2 \frac{\partial^2}{\partial x^2} p(s, t, x, y)$$

$$\frac{\partial}{\partial t} p(s, t, x, y) = - \frac{\partial}{\partial y} \left[ \beta(t, y) p(s, t, x, y) \right] + \frac{1}{2} \gamma(t, y)^2 \frac{\partial^2}{\partial y^2} p(s, t, x, y)$$

The forward equation (the second one above) is also known as the Fokker-Planck equation. The backward equation (the first one) is well known in option pricing theory via the Feynman-Kac formula. A special case of this states that a solution $f(s, x)$ to

$$- \frac{\partial}{\partial s} f(s, x) = \beta(s, x) \frac{\partial}{\partial x} f(s, x) + \frac{1}{2} \gamma(s, x)^2 \frac{\partial^2}{\partial x^2} f(s, x)$$

with boundary condition $f(t, y) = g(y)$, can be expressed in the form

$$f(s, x) = \mathbb{E}_s \left[ g(X_t) | X_s = x \right] = \int g(y) p(s, t, x, y)dy .$$

Indeed, provided sufficient technical conditions hold for these expectations to exist and for ‘differentiation under the integral sign’, the Feynman-Kac formula follows immediately from the backward equation. Since option prices can be expressed as the expected discounted payoff under a suitable measure, the option price is revealed to satisfy a Kolmogorov backward equation-type PDE, which provides a standard route to its calculation, using classical PDE techniques.

The Kolmogorov forward equation is also used in option pricing – when a model that captures the ‘volatility smile’ is required. Option prices can be used to infer a risk-neutral evolution of the probability density, and the Fokker-Planck equation is then inverted to determine a volatility surface consistent with that evolution.
Below, we use the expectation representation of a value – the martingale representation – to calculate optimal strategies for the expected utility approach. In contrast the Fokker-Planck equation is used to help determine optimal strategies for our numerical solution of the mean-CVaR optimisation problem.

3.3. Optimising utility over multiple periods
Modern portfolio theory, as initially developed by Markowitz in the 1950s, provides a way to determine optimal strategies over a single investment horizon. In a well-known ‘separation theorem’, the efficient frontier can be expressed as linear combinations of two portfolios: the minimum-risk portfolio and an optimal risky asset portfolio. In the presence of linear constraints (e.g. no short positions), the efficient frontier is a finite number of intervals, over each of which the efficient portfolios are once more linear combinations of the boundary portfolios.

In a discrete setting, the dynamic problem can be solved sequentially starting with the final period: for given starting levels of wealth, the expected utility can be maximised given optimal expected utilities already calculated for levels of wealth at the end of the period. That is, the function:

\[ V(w, t) = \max \mathbb{E}_t \left( U(w_t) \middle| w_t = w \right) \]

can be calculated iteratively starting at \( T \), since \( V(w, T) = U(w) \):

\[ V(w, t) = \max \mathbb{E}_t \left( V(w_{t+1}, t+1) \middle| w_t = w \right) . \]

This recursive equation is an example of a Bellman principle. The multi-period model is broken down, in a straightforward way, into single-period problems. In simple models, this ‘dynamic programming’ problem can often be solved explicitly; and more generally a numerical approach can be used.

At each stage (for each time interval), the problem is to find the asset strategy that will maximise expected utility. Mathematically, this strategy is known as a (stochastic) ‘control’ and the full problem as a stochastic control problem.

3.4. Continuous-time modelling
The continuous-time setting is used very commonly in the academic literature. This is of course less realistic than the discrete-time setting, and requires much stronger model assumptions and mathematical sophistication, but allows some impressive results to be derived. For example, a separation result very similar to the single-period result mentioned above can be proved.

One particular gain that the continuous-time framework can provide is due to Cox & Huang (1987). The dynamic programming problem is reinterpreted as two steps, each of which is considerably easier than the original problem, as follows. The utility-maximisation problem (2) can be interpreted as a problem to find a final wealth \( w_T \) for each possible asset price path (‘state’). Convex optimisation methods (notably, the formation of a Lagrangian) can be used to determine this wealth function. Then as a second step, a search can be undertaken for an investment strategy that has this wealth function as payoff. The second step is familiar from option pricing theory, and in a complete model the existence of such a strategy is often assured by the martingale representation theorem.

3.5. Analytical tractability
In the following utility approach we have restricted ourselves to models that are analytically tractable. This enables clean results to be derived, and intuition to be gained, but ultimately more realistic models are needed to make decisions in practical situations. We seek to identify general principles (e.g. on the risk budgeting process over time) that are expected to hold when more
realistic asset pricing, discrete rebalancing, a reduced investment set and transaction costs are introduced. A more complex addition is to include non-investment strategy (e.g. contribution rate strategy; benefit improvement strategy) into the optimisation problem. Consumption was an integral part of Merton’s original work in this area (see below).

3.6. A special case of Merton’s portfolio problem
Merton proposed the following problem. A CRRA utility function (2) and an initial level of wealth $w_0$ are given. The problem is to find investment strategy and consumption (or drawdown) $c_t$ to be taken from the portfolio at each time $t$ in order to maximise

$$\int_0^T U(c_t) \, dt + U(w_T).$$

Merton showed that in the case of a Black-Scholes model set up (constant interest rates, lognormal Wiener process for the single risky asset) the optimal policy is to invest a constant fraction of wealth in the risky asset and to consume wealth at a rate defined by a simple formula.

Similar results can be derived in more complicated models. In a model driven by Brownian motion where asset volatilities are deterministic functions of time, and ignoring consumption, expected CRRA utility is maximised by a portfolio $X(\gamma, T)$ that invests $1/\gamma$ of wealth in the growth-optimal portfolio $h_t$ and $(1 - 1/\gamma)$ in a bond maturing at time $T$.

The growth-optimal portfolio process $h_t$ can be defined as the portfolio that has the highest expected rate of growth $\ln (h_T / h_t)$; alternatively, up to a multiple, it is just the inverse of the pricing kernel (state price deflator) for the model. (In the Black-Scholes model this growth-optimal portfolio is just a leveraged holding in the risky asset, and all bonds have the same deterministic price process.)

3.7. Approach taken
In the next section we focus on the choice of utility function. This function is supposed to express the investor’s preferences and is therefore worthy of serious consideration. We show what plausible functions might look like, and identify the resulting investment strategies. As far as we know, this focus on the utility function is essentially new, and appears to us to offer interesting insights.

3.8. Portfolio constraints
In many analytically-tractable situations, constraints are few. The classic example of this is a ‘long only’ investor, who is unable to create short, or leveraged exposures to a basic set of asset classes. These constraints may well be binding for problems which are otherwise rather benign. There are at least three approaches to these kinds of problem:

- Bellman equation. This approach is most generic, and thus requires most creativity, to solve. This typically works in special situations.
- Duality approach. As described in Karatzas & Shreve (1998) and Rogers (2003), Lagrange multipliers (in fact, a Lagrange multiplier process) can be introduced to produce a modified optimisation problem. A solution to the unconstrained modified problem is often a solution to the unmodified constrained problem, in the usual way, given appropriate Lagrange multipliers. Again, in some situations, the dual problem (effectively, finding the appropriate Lagrange multipliers) can be more readily solved and a solution then found for the original (‘primal’) problem.
- Numerical approach. This is a brute force approach, using a discretisation of the problem on a suitable tree. Care and attention is required to set up the algorithm to ensure that a solution can be found in reasonable time. Analytical work can sometimes be undertaken to simplify the
problem before resorting to the computer. For example, at each point the chosen portfolio will usually be mean-variance optimal and a parameterisation of the mean-variance efficient frontier means that only a single parameter rather than $n$ portfolio holdings need be found at each point of the tree. The parameter used can be thought of as a Lagrange multiplier, and thus this approach can be linked into the duality approach above.

In section 6 below, we show how a numerical approach can be used to examine a problem with portfolio constraints.
4: Optimal utility approach

4.1. Motivation
One of the problems that motivates our interest in utility functions is the question of how a pension plan’s investment strategy should evolve over time. The standard approach in the finance literature has been to express an investor’s goals in terms of a utility function, and use that to derive the dynamic investment policy.

In the pensions context, this is difficult to apply. The main difficulty to answer is: who is the investor? Who is taking risks & benefiting from returns in a pension plan? Plan members, acting through the trustees as their agents, have a defined benefit, so that, provided the sponsoring employer does not default, they have no exposure to the funding level of the plan. Of course, the proviso that the employer remains solvent is a strong one: companies regularly go insolvent. On the other hand, the sponsoring employer - or rather its shareholders - is exposed to the funding level of the plan: as the funding level goes up or down, less or more future contributions will be required. But shareholders will measure utility at the level of their entire portfolio; the question for them is one of maximising the value of the company rather than trading off risk and return.

This theory is of course at variance with common practice: trustees are very concerned to improve the funding level of their pension plan, so that if insolvency occurs members will receive a higher level of benefits than those promised by the PPF. A lot of time is therefore spent discussing, formulating and documenting investment objectives, investment principles and the like. Finance directors are typically also keen to improve funding / reduce risk – they will probably not be willing to place any weight on the possibility of company insolvency. We therefore take the point of view of these agents, albeit with an awareness that this perspective may not be appropriate.

For a fuller discussion of the perspectives of different stakeholders, we refer the reader to the draft report of the Dynamic Investment Strategies working party, Barnes et al (2008).

4.2. Overview
The technical problem we examine in this approach is to determine what happens to the optimal investment strategy as we alter the utility function away from the ubiquitous CRRA family described above. Much of the finance literature is involved with applying this family of utility functions to steadily more complicated financial models. The appeal of following an orthogonal route is that we want to understand how the optimal strategies change as the investor’s goal varies. In particular, we examine the challenge of a target asset level and/or a minimum asset level relative to a (stochastic) liability target.

Derivations of the various optimal strategies are provided in appendix C.

4.3. Minimum solvency level
Starting from a CRRA utility function, we examine two potential ways of introducing a minimum wealth level $M$. The first is simply to shift the utility function to the right:

$$U(w) = \frac{(w - M)^{1-\gamma} - 1}{1-\gamma}.$$
The second way is to truncate the utility function at the desired level (and have utility of $-\infty$ below this level):

$$U(w_T) = \begin{cases} 
\frac{w_T^{1-\gamma} - 1}{1-\gamma}, & w_T \geq M \\
-\infty, & w_T < M 
\end{cases}.$$  

Maximising expected utility in either of these examples will lead to a final wealth that is always above the guarantee level $M$. However, the distribution of outcomes is quite different. In the first case, we see significant additional benefit for small increases in wealth above the guarantee level (the derivative becomes infinite as we approach the minimum level $M$). In the second case, there is only a small difference in utility from small increases above the minimum $M$ (the derivative is finite).
Denote by $X_t$ the value at time $t$ of an investment strategy that maximises the (unmodified) expected utility given by:

$$U(w_t) = \frac{w_t^{1-\gamma} - 1}{1 - \gamma},$$

As noted above, in typical cases, $X$ consists of a continuously-rebalanced mixture of the growth-optimal portfolio (with weight $1/\gamma$) and a bond maturing at time $T$ (with weight $1 - 1/\gamma$). It is easy to calculate the result of maximising the modified expected utility functions in terms of this portfolio.

Case 1: Modifying the utility function (ie a shift right), the optimal strategy is of the form:

$$M.P(t,T) + \xi X(\gamma,T),$$

for some $\xi$ determined by the initial wealth. $P(t,T)$ has the usual meaning of the value at time $t$ of a payment of 1 at time $T$. Note that if the initial wealth is insufficient to purchase the value of the guarantee $M.P(t,T)$, then the problem is insoluble, so $\xi > 0$). Splitting $X$ into its components, this then amounts to investing $\xi / \gamma$ of $(w_t - M.P(t,T))$ in the growth-optimal portfolio, and the rest in the bond expiring at time $T$. In other words,

Proportion in risky asset $\propto w_t - M.P(t,T)$.

The constant proportion is called the multiplier, and the simple investment strategy that we have derived is known as Constant Proportion Portfolio Insurance (CPPI).

Case 2: modifying the utility function (ie a truncated utility function), the optimal portfolio consists of an investment $\xi X_t$ in the portfolio $X$ and $\xi$ put options on $X$, with strike price $M / \xi$. Again, $\xi$ is determined by the initial level of wealth.

So the investment strategy that is optimum for a truncated utility function includes put options.

**4.4. Comparison**

In the CPPI case, the final portfolio value is always at least $M$, with the distribution of possible values being close to $M$. In the put option case, the final portfolio typically has a high chance of being exactly $M$, with the residual distribution being more widely spread than the CPPI case.
At the risk of stating the obvious, each of these strategies is optimal for the particular utility function that was used to derive it, and sub-optimal for the other.

4.5. Target liability payment
Rather than focusing on a minimum wealth constraint, a more relevant example involves satiation: more wealth is good up until a target is reached, at which point additional wealth generates no additional utility.

The [above] picture shows a simple function with these properties, of the form:

$$u(w_T) = \frac{\min(w_T, L)^{1-\gamma} - 1}{1 - \gamma}.$$  

Here $L$ need not be certain: for example, it could be the uncertain value (at time $T$) of payments due after time $T$. Indeed, mathematically, this makes any analysis slightly clearer, as the separation
of the optimal strategies into three pieces (a liability hedge, an optimal risky portfolio and a bond maturing at the target horizon) becomes more evident.

This utility function gives no credit for having a funding level of more than 100% relative to the target wealth. Low funding levels are penalised without rewarding high funding levels: an asymmetric view of risk is the effective result. In a way, this is similar to the minimum-solvency examples, but crucially differs in that the target level is now above the current asset level, whereas in the minimum solvency cases the guarantee level was below the current asset level. A more flexible boundary was therefore required in this case than in the minimum case, where a hard constraint was imposed.

It is straightforward to show that the final result from an optimal strategy can again be expressed in terms of the solution $X$ to the unmodified utility function (see discussion above in connection with minimum solvency). The portfolio consists of an investment $\xi X_1$ in the portfolio $X$ and selling (writing) $\xi$ call options on $X$, with strike price $L/\xi$. As before, $\xi$ is determined by the initial level of wealth.

The intuition here is simple: below the target level, returns should be generated in an optimal way, but any returns that bring the funding level over 100% are not required, so these can be sold (by writing call options) – and the proceeds used to generate slightly higher returns.

**4.6. Combined utility functions**

In practice, there may be both a target income level and a minimum required income for the investor. This would suggest a combination of the two approaches, as shown in the graph [below]. Appendix C shows that combinations of this type introduce no new complexity to the problem: the optimal strategy is a combination of those already discussed.

![Utility]  

**4.7. Dynamic strategies**

By expressing the optimal investment strategy as a combination of stocks, bonds and options, the dynamic nature of the solution is hidden. Yet within the theoretical framework (lognormal asset prices, continuous trading, no transaction costs) the optimal strategy can be expressed (uniquely) in terms of a dynamic portfolio of stocks and bonds. The amount of risk relative to the target (or minimum) liability depends on (a) the time to the horizon $T$ and the (b) the ‘funding level’ relative
to the target. This dynamic strategy is illustrated in the chart [below] which corresponds to a finite liability target and a floor of zero: for a given time and funding level, the optimal strategy can be read off.

![Target allocation in stock increases as time to horizon or funding level falls](image)

In broad terms, the strategy is as we would expect: at a fixed time, the target amount of risk is higher if the deficit is higher; and for a fixed deficit, the target amount of risk is higher if the time remaining to the target date is longer.

### 4.8. Discrete rebalancing strategies

Of course, we do not live in the frictionless continuously-trading world of this model. It is therefore interesting to relax these assumptions to see whether the strategies remain attractive. As an example, we now explore how well discretely-rebalanced strategies perform relative to this optimum. Again suppose that there is a target income being sought and no minimum level.

We can illustrate the efficiency loss from discrete rebalancing by calculating the expected utility level from the optimal annual-rebalance strategy compared to the theoretical (continuous-rebalancing) case. The following graph compares the expected utility with 5 years until the target date:

![Expected utility comparison](image)
The utility scale is deliberately not shown here: changing the scale or the origin of utility has no economic effect. But we can nonetheless, as illustrated, calculate the wealth-equivalent of the utility loss. At many funding levels (and reducing as we approach 100%), we need a 2% higher funding level in order to achieve the same expected utility as we would have been able to achieve with continuous rebalancing.

One simple way to close this gap faced by the annual rebalancer is to use call options. If options expiring at the terminal horizon are available, then a static option strategy replicates the (continuous-time) dynamic strategy. More realistically, suppose that each year, options with a one-year maturity are used, i.e. expiring on the next rebalancing date. (This is in a more liquid part of the market, where transparent pricing is more likely to be available.)

The utility increases accordingly, to the point that this has allowed the investor to come very close to the theoretical optimum of continuous rebalancing.
Similar results hold when a minimum income level is sought: in the continuous model, the investor’s risk budget needs to fall either as the assets approach the minimum or the higher target level. If rebalancing is only allowed at discrete intervals, this tension leads to a more significant utility loss than in the zero-minimum case. But this loss can be significantly reduced by introducing put and call options to protect the investor between rebalancing dates.
5: Risk-constrained utility optimisation

5.1. Introduction
As mentioned above, tail risk constraints can be used in addition to, or instead of, utility functions as a way to control the losses in a portfolio. In the next section, we look at a mean-tail-risk optimisation problem, and a numerical approach to its solution. As an intermediate example, we first look at a utility-tail-risk optimisation problem. This turns out to be a straightforward addition to the utility approach of the previous section.

Further details on the results outlined in this section can be found in Gandy (2005) and Gabih et al (2004).

5.2. Statement of the problem
The objective is to find a strategy that maximises utility subject to the CVaR risk being below a limit:

\[ \text{Max } \mathbb{E}(U(X_T)) \]
\[ CVaR_{\epsilon}(X_T) \leq -c \]  

We assume a geometric Brownian motion model for returns on assets, i.e. the total return of the \( i \)'th asset evolves according to a stochastic differential equation of the form

\[ S_i(t) = S_i(0) \exp \left( \left( \mu_i - \frac{1}{2} \Sigma_{ij} \right) t + A_i^t W(t) \right). \]

Here \( W(t) \) is a multivariate Brownian motion; \( A_i \) is a vector containing the (constant) exposures of asset \( i \) to each driver; and \( \Sigma_{ij} = A_i^t A_j \) is the covariance of assets \( i \) and \( j \). To avoid arbitrage opportunities, the expected returns \( \mu_i \) must be of the form

\[ \mu_i = r + A_i^t \lambda \]  

Here, \( \lambda \geq 0 \) is a vector each of whose components is the market price of risk for the underlying drivers; and \( r \) is the (again, assumed constant) rate of interest on cash. The pricing kernel \( m(t) = \exp(-rt - \lambda^t \lambda t/2 - \lambda^t W) \) is a key object of interest: the produce of this kernel and any total return index will be a martingale (under the real-world measure). The existence of this pricing kernel, i.e. the condition (7), is thus a familiar one on the existence of an equivalent martingale measure.

5.3. Optimal payoff
The model as described is dynamically complete, i.e. any payoff at time \( T \) can be replicated using a suitable dynamic hedging strategy. This therefore allows us again to use the martingale approach (rather than the more complicated Bellman approach) to solving the optimisation problem (6). We can first derive the optimal payoff \( X_T \) and subsequently determine the investment strategy that supports it.

This is straightforward using standard variational calculus: the optimal payoff is of the form

\[ X_T = \begin{cases} \left( \lambda_i m_T \right)^{-1/\gamma} & \text{if } h < \left( \lambda_i m_T \right)^{-1/\gamma} \\ h & \text{if } \left( \lambda_i m_T \right)^{-1/\gamma} < h < \left( \lambda_i m_T - \lambda_2 \right)^{-1/\gamma} \\ \left( \lambda_i m_T - \lambda_2 \right)^{-1/\gamma} & \text{if } h > \left( \lambda_i m_T - \lambda_2 \right)^{-1/\gamma} \end{cases} \]
For clarity, in the absence of the risk constraint, the optimal payoff profile would simply be 
\[ X_T = \left( \xi m_T \right)^{1/\gamma} \] for some constant \( \xi \). These payoffs are all functions of the so-called ‘growth-optimal’ portfolio (GOP) \( G_t = m_t \gamma \) at time \( T \). The typical structure of the payoff from the constrained problem (6) is illustrated in figure 2. (Special cases can arise from some constraints: if the constraint does not bite then \( \lambda_2 = 0 \) and the unconstrained-optimum applies; in other cases we can have \( X_T \geq h \) everywhere or even \( X_T \in \{0, h\} \) everywhere. The optimal \( X_T \) is always non-decreasing with the value of the GOP.)

**Figure 3. Impact of conditional value at risk constraint on optimal payoff at time horizon, one-asset case**

![Figure showing impact of CVaR constraint on optimal payoff](image)

The constants \( \lambda_1, \lambda_2, h \) must satisfy the budget constraint \( \mathbb{E}(m_T X_T) = X_0 \) and the shortfall constraint \( \mathbb{E}\left( (h - X_T)^+ \right) = \varepsilon(h - c) \). The remaining degree of freedom can be found by maximising the expected utility. Each of these constraints and operations are straightforward to carry out numerically.

### 5.4. Investment strategy example

The weights in each risky asset are always in the same relative proportions: these can for example be identified with the weights in the growth-optimal portfolio, or equivalently the weights in the unconstrained optimal portfolio. The allocations thus vary over time along a single dimension, namely the total exposure to risky assets can vary (with the balance made up with cash or bonds). The evolution of this exposure could equally be described as the variation of the risk budget, or return target of the portfolio over time. In the pictures below, we choose to show the total risky asset allocation as a percentage of the unconstrained asset allocation: where this percentage is above (below) one we are therefore taking more (less) risk than we would in the absence of a risk constraint.

There is therefore no additional complexity here with having more than one risky asset; so we keep things simple in the following example, which has one asset, with 
\[ \Sigma_{i1} = (15\%)^2, \hat{\lambda} = 1/3, r = 5\% \]. The CVaR constraint we impose is \( CVaR_{0.05} \leq 70\% \), i.e.
The average loss for the worst 5% of scenarios can be at most 30% over a 1-year horizon. This is a fairly mild restriction, which allows the graphs below to be easily readable. (The tail VaR for the unconstrained optimal portfolio is $CVaR_{0.05} = -57\%$.)

The evolution of the strategic allocation over time is illustrated in the following charts. **Figure 4. Optimal portfolio risk for problem (6), as function of portfolio value**

![Variation of portfolio risk over time and portfolio value](image)

Figure 3 shows how the risk level, as a proportion of that which would apply in the absence of a risk constraint, varies over time. Initially, when the portfolio value is 1, the optimal level of risk is below the unconstrained version. If the portfolio value stays around or above a threshold value, the risk remains below 1; but if the value falls too far below then it is optimal to take more risk. The highest possible level of risk is obtained if we are just below the threshold level very close to the time horizon.

Figure 4 illustrates this from a different perspective: the time zero and time 1 slice is given at the bottom, to make the structure of the surface in figure 3 clearer; and on the right we also give the equivalent surface if we plot against the unconstrained portfolio value rather than the constrained portfolio value. As the optimal constrained portfolio attains the threshold value with a strictly positive probability (see figure 1), this gives a slightly smoother picture.
5.5. Comments

Clearly a TailVaR constraint is very different from a hard constraint that such as led to CPPI or put option strategies in the previous section. Controlling, but not eliminating, tail risk, is clearly cheaper, enabling better average outcomes, and may be attractive for most investors. Using a VaR constraint also exerts some control, but although this has an impact on the size of the tail, it is much less effective within the tail itself.

On the other hand, it is questionable whether the TailVaR constraint, as formulated here, is appropriate. It focuses explicitly on the outcomes as seen at the outset of the strategy. In contrast to the expected utility approach, which has a consistent interpretation through time, that is not the case here: if very poor returns are experienced, the strategy will tend to raise risk, in contrast to the strategies considered above. While this is sensible when viewed at the outset (this is likely to happen rarely), it may not be once the returns have occurred. A risk constraint which moves appropriately through time is required - remaining solvent with a 199 in 200 confidence level at all times is rather different from the requirement we have analysed here. We leave this to further research.
6: Dynamic Stochastic Programming Approach

6.1. Introduction
The preceding discussion has focused on the use of methods that rely on the use of a utility function. As individuals, we do not always know what our utility function is and in a collective investment such as a pension scheme or a company it is not always clear how the utility functions of individual members or shareholders should be aggregated to form the utility preferences of the institutional investor. Therefore it is often useful to formulate an investment approach that is not dependent on the existence of a utility function.

Institutional investors are increasingly becoming familiar with the use of various statistical measures of risk such as tracking error, VaR and TailVaR so we have set out the following method to optimise the trade off between portfolio growth and those risk measures rather than rely on utility. For the sake of exposition we set out what follows in terms of TailVaR.

Next we need to consider whether to proceed down an analytical or numerical route. We note that the original Merton work was analytical but that the nature of many real-world problems including constraints such as long only investment usually force one to use a numerical algorithm.

The discussion so far is likely to lead us to follow an approach similar to Dempster et al (2003). He used a tree based approach which we significantly simplify as follows.
1. Writes down stochastic differential equations for the various asset classes.
2. Divides the time interval up into a number of decision points when the portfolio would be rebalanced.
3. For each asset class/economic variable used, discretise the possible outcomes. So far we are following a similar approach to the familiar Cox Rubinstein approach to valuing American options using a tree with the differences that the tree is non-recombining and there are multiple asset/economic variables being modelled so the problem is multidimensional.
4. One can then assume that various portfolio weights are applied to each asset at each decision point.
5. Combining the portfolio weights and the asset class behaviour allows us to calculate the portfolio value at any point on the tree. We also have the probability of each node being passed through. Together we can therefore calculate the portfolio probability distribution at any point in time.
6. We can then optimise the weights to produce an optimum portfolio probability distribution.

We initially attempted to build a model with similar characteristics and found that the number of variables tends to increase exponentially and make the method intractable for the types of problem we wished to solve. More detail of the reasons for this are to be found in Appendix D.

We were therefore in a position of wishing to use Michael Dempster’s tree approach while avoiding the exponential proliferation of variables. There were two observations that allowed us to simplify the problem being:
1. A recombining tree will have fewer nodes than a non-recombining tree.
2. There are certain sets of economic events that will have the equivalent effects on the portfolio and so may be grouped together. For example, if the portfolio is equally weighted between US and European equities then we will be indifferent between US equities rising 10% while European equities stay flat or visa versa.

These two observations allow us to significantly reduce the dimensionality of the problem. We reduce the problem by replacing a non-recombining tree in all assets classes by a recombining tree in the whole portfolio value. This means that the number of nodes at a given point in time increases linearly rather than exponentially and hence makes the problem tractable.
This leaves us with something similar to a Cox Rubinstein American option valuation recombining tree. The difference is that at each node we have a set of portfolio weights. The weights at that node enable us to calculate the volatility and growth rate at that node. This is very similar to the local volatility models that have been used to provide option prices consistent with a volatility smile.

Finally we needed to solve some technical problems with the approach. Firstly as with local volatility models it is possible for a model such as this to produce negative probabilities of various nodes. Clearly this aspect is aesthetically unappealing because we are trying to model the real world where probabilities are positive. There is also a more serious concern here in that negative node probabilities can destabilise the tree.

We were therefore in a position where we sought a method that had the following properties:

- No reliance on utility
- Optimises tailVar
- Discrete numerical
- Not exponential numerically
- Stability

We thus decided to remove the negative probabilities problem by recharacterising our recombining tree as a lattice style solution to a partial differential equation (pde). Lattice PDE methods are well known from option pricing theory as they enable a more controlled approach than trees. Characterising the problem as a lattice PDE problem allowed us to alter the stability properties at will by switching between explicit and implicit PDE solution methods.

So to summarise the work so far we now have a pde for the portfolio value. The coefficients of the PDE vary for each point on the lattice because we vary the portfolio weights at each point.

We now define a weight vector to be (for any given node) the asset allocation at that node expressed as a vector (so all the components will add to 1). Further we define a strategy to be a matrix of weight vectors such that each node has a weight vector. Therefore a strategy can be thought of as an object with three indices being time, portfolio value and asset.

Therefore if we have any given strategy we can combine this with the lattice PDE to calculate the terminal probability distribution of the portfolio and hence its risk statistics such as TailVAR. If we then combine this with an optimisation algorithm we are able to find the optimum strategy. Those who are familiar with the theory of PDE Constrained Optimisation may then see this algorithm as an application of that theory.

A final aspect of this is to note that the discussion so far has been in terms of assets that follow some kind of Brownian motion. Such models are often criticised because they produce insufficiently fat tails. One suggestion for further work is to note that although fat tails do not fit well into the methodology of partial differential equations they can be expressed using Patial Integro-Differential Equations (PIDEs). Jump processes can be designed that are consistent with fat tails and finite difference methods have been developed that enable the lattice solution to PIDEs thus we could envisage that our methodology could be modified to allow for a fat tail aware dynamic portfolio optimisation strategy.

Having given an overview of our rationale we now specify the method.

6.2. Method

This method seeks to solve the following problem: given a universe of possible assets that evolve under Geometric Brownian Motion with a vector of known expected growth rates, a covariance matrix, a measure of risk and a given tolerance for that risk what is the optimum way to (re)allocate the portfolio over time. The method does not require use of a utility function but
instead requires use of well known statistical measures of risk (variance, Tail VaR etc.). Importantly we require that the volatility of each asset be constant or at least deterministic with a given term structure.

In outline, the method works as follows:
1. Discretize the portfolio value between a minimum and maximum level and the time between inception and maturity by dividing both into segments so that taken together they form a grid.
2. Enumerate the universe of possible assets.
3. Define a (3 dimensional) weight matrix as the percentage holding of the portfolio in a given asset at a given node.
4. Make an initial guess at what the weight matrix might be (e.g. by doing a single period Markowitz optimization then assuming that the weight matrix holds those weights at every node).
5. Use the weight matrix to calculate using matrix multiplication the portfolio mean growth rate and volatility separately at every node.
6. The portfolio is considered as a probability density function over the space of future time and portfolio values. It therefore follows the Fokker Planck equations with time and space varying volatility and drift.
7. The initial condition for the FP equation is that the portfolio has infinite point density at its initial value (as the Dirac delta function).
8. Discretising the FP equations gives a set of difference equations which can then be solved.
9. Solving the discretized equations tells us the terminal probability distribution.
10. From this we calculate the risk measure.
11. We then use an optimization routine at step 4 together with the risk calculation in step 10 to iteratively estimate the optimal weight matrix.

Filling in the detail, we assume that the economics are that each asset follows Geometric Brownian Motion, possibly with term dependence in the drift and volatility. Our strategy is to define the SDE then deduce the PDE and hence the difference equations.

Thus, we assume that the price of a share in asset i follows the SDE
\[
dS^i = \mu^i (t) S^i dt + S^i \sigma^i (t) dZ^i
\]
Suppose that we hold \(n^i (P, t)\) shares in asset i then the value of the portfolio \(P\) at time \(t\) is:

\[
P = \sum n^i S^i
\]
We then use the multidimensional Ito formula to give:
\[
dP = \sum \mu^i (P, t) S^i dt + \sum n^i S^i \sigma^i (P, t) dZ^i
\]
Or
\[
dP = \mu^P (P, t) P dt + \sigma^P (P, t) P dZ \quad \text{Equation A}
\]
Where we define the \(w, \mu^P (t), \sigma^P (t), Z\) by:
\[
n^i S^i = w^i P
\]
\[
\mu^P (t) P = \sum \mu^i (t) S^i
\]
\[ \sigma^P(t)^2 = w^T C w \]

with \( C \) being the variance covariance matrix and \( Z \) being a standard Brownian motion.

We now wish to transform the SDE equation \( A \) as a partial differential equation (PDE). This is common in derivative pricing although usually in that case one is working backwards from known values. However we are not trying to value a derivative but to find the paths of the portfolio total value. Therefore we use the Kolmogorov forward equation or Fokker-Planck equation to give us a PDE for the probability density of \( P \). Let the probability density be \( p(x,t) \) then

\[
\frac{\partial}{\partial t} p(x,t) = - \frac{\partial}{\partial x} [\mu(x,t) p(x,t)] + (1/2) \frac{\partial^2}{\partial x^2} [\sigma^2(x,t) p(x,t)]
\]

Equation B

The above problem has stochastic volatility. It is well known that such problems can have stability problems induced by negative probabilities. We avoid this by using PDE finite difference methods. We use an implicit method because of the extra stability that it provides.

First then we need to set up a grid. We have time go from 0 to \( T \) in units of \( \Delta t \) (chosen to generate an integer number of time steps) denoted by index \( i \) so we consider times:

\[ 0, \Delta t, 2\Delta t, \ldots, i\Delta t, \ldots, T \]

We also let the portfolio range in value between \( P_{\text{min}} \) and \( P_{\text{max}} \) in steps of \( \Delta P \) with grid points denoted by index \( j \) so the portfolio can take values:

\[ P_{\text{min}}, P_{\text{min}} + \Delta P, P_{\text{min}} + 2\Delta P, \ldots, P_{\text{min}} + j\Delta P, \ldots, P_{\text{max}} \]

Therefore we denote positions on the grid by \((i, j)\). We will be setting boundary conditions that \( P_{\text{min}} \) and \( P_{\text{max}} \) have probability zero so they need to be sufficiently far away from the starting point for this. We also need to choose them so that \( P_{\text{start}} \) lies on a grid line.

We will also need initial conditions and boundary conditions. Now we know the portfolio value at time zero. Thus the probability density function at time zero is the Dirac delta function, i.e. infinitely dense at \( P_{\text{start}} \) and 0 density elsewhere. On a grid this means that

\[ p(0, j) = \begin{cases} 
1 & \text{if } P_{\text{min}} + j\Delta P = P_{\text{start}} , \\
0 & \text{otherwise}.
\end{cases} \]

We also need boundary conditions and these are:

\[ P(i, 0) = 0 \]

\[ P(i, (P_{\text{max}} - P_{\text{min}})/\Delta P) = 0 \]

Finally we need to express the PDE as a recurrence relationship, noting that we have boundary conditions at initial rather than terminal time as is normally the case in a derivatives pricing algorithm and therefore what would normally be an implicit method is an explicit method and visa versa:

\[
\frac{\partial}{\partial t} p = [p(i+1, j) - p(i, j)]/\Delta t
\]

\[
\frac{\partial}{\partial x} [\mu p] = [\mu(i, j+1) p(i, j+1) - \mu(i, j) p(i, j)]/\Delta x
\]

\[
\frac{\partial^2}{\partial x^2} [\sigma^2 p] = [\mu(i, j+1) p(i, j+1) - 2 * \mu(i, j) p(i, j) + \mu(i, j-1) p(i, j-1)]/\Delta x
\]

Having set up the PDE grid we now have a mapping from strategies to our objective function. We are therefore able to impose any constraints (e.g. long only weights) and optimise the objective function. The objective function would be
6.3. Results
We now show the results of this method which are typically displayed graphically. The types of output that we have developed are as follows.

1. A Markowitz style efficient frontier. As in traditional mean variance portfolio optimisation we show for each level of risk the maximum return that can be obtained. However we show 2 differences from the traditional model. Firstly we show the x axis as Tail VAR rather than portfolio volatility. Secondly we show, in addition to the single period optimisation, the optimisation that is produced by breaking the period down into several sub-periods and allowing rebalancing at each sub-period. As might be expected, the dynamically rebalanced portfolio always shows at least as great return for a given level of risk. Also the two curves meet at the right hand side because that corresponds to both strategies choosing maximum risk maximum return assets throughout.

2. A terminal probability distribution showing at the end of the portfolio projection period the probability distribution of the portfolio. This allows one to check that the TailVaR is optimised.

3. A contour map showing the evolution of the portfolios probability distribution. Thus the last time period of this is equivalent to the distribution in 2. At time zero the probability density will be infinite at one point (the initial portfolio value). This initial probability distribution can be thought of as the Dirac Delta Function.

4. A separate contour map for each asset showing its weight within the portfolio for each future state. This can be used to check that the portfolio is reallocating into lower risk assets following falls in portfolio value.

![Static and Dynamic Efficient Frontiers for Tail VaR optimized portfolio](image_url)
The efficient frontier shows the same kind of shape that we would expect given what we have discussed. The terminal probability distribution shows an interesting skewness to the right. We interpret this as follows. The algorithm is punished heavily for any probability that the portfolio falls below a certain level. Therefore as that level is approached it reallocates assets into low risk assets. This produces a peak in probability close to but above the level that we use to define TailVaR. However on paths where financial markets perform well, the portfolio can afford to reallocate towards riskier higher returning assets, hence the skew to the right.
7: Conclusion

How could these theories be applied in practice? We have outlined two separate methods, a utility-based and a statistics-based method. The utility method allowed us to construct optimal strategies analytically and produced strategies that make intuitive sense.

Choosing a utility function is fraught with difficulty. Multiple stakeholders can have different views on what risks are appropriate to take or to hedge. Even in the case of a single investor or agent, the calibration of a utility function is extremely difficult. We have sought to suggest that progress can nonetheless be made by seeking to identify the broad shape of the utility curve (specifically, any target or guarantee level). The resulting strategies are then internally coherent: a framework is built within which these and other strategies can be assessed.

If a statistical method is preferred over the utility method then we need to use an algorithm to choose a dynamic strategy that optimises the combination of our favourite risk measure and growth ambitions. The advantages of this are that it makes sense for organisations that have a regulatory requirement to use a specific statistical risk measure such as VaR or TailVAR. It also makes sense in situations where the utility function is unknown. In this situation one could use the PDE method outlined here.
8: Bibliography


Appendix A: Common utility functions

We provide some more detail on the utility functions that are regularly used in the finance literature.

A.1. Definition
The basic idea of a utility function is to express mathematically the personal value that will arise from different levels of wealth. Generally, utility functions have two reasonable properties:

– **Increasing.** More wealth is always better.
– **Convexity.** The increase in utility for each additional dollar of wealth diminishes as the initial level of wealth increase.

A.2. Measures of risk aversion
The Arrow-Pratt measure of absolute risk aversion (ARA) at a particular level of wealth \( w \) is:

\[
ARA(w) = -\frac{U''(w)}{U'(w)}.
\] (8)

This effectively expresses the trade off required between additional return and variance when starting from a given level of wealth.

The relative risk aversion is:

\[
RRA(w) = w \cdot ARA(w) = -\frac{wU''(w)}{U'(w)}.
\] (9)

This measures the aversion to risks that are proportional to the initial level of wealth.

Arrow argued that investors’ aversion should decrease with wealth (i.e. \( ARA \) is a decreasing function of \( w \)), i.e. that willingness to face a risk of a given size increases with wealth. But if the risk rises in the same proportion as the wealth then the willingness falls (i.e. \( RRA \) is an increasing function of \( w \)).

A.3. Constant relative risk aversion (CRRA) utility functions
As noted in the main text, the CRRA family of utility functions can be defined via:

\[
U(w) = \frac{w^{1-\gamma} - 1}{1 - \gamma}
\] (10)

It is easy to check that this family does indeed have constant relative risk aversion \( \gamma \).

Since it is the expected value of a utility function that is maximised, changing a utility function by the addition of, or multiplication with, a constant has no effect on the investment strategy that is followed. A utility function \( U(w) = w^{1-\gamma} \) would therefore be equivalent. The form of (6) is used so that the \( \gamma = 1 \) limit exists: it is simply \( U(w) = \ln w \).

Note that:

– The \( \gamma = 0 \) case is linear
– The cases \( 0 < \gamma < 1 \) are bounded below
– The \( \gamma = 1 \) case is logarithmic
The cases $\gamma > 1$ are bounded above. This family is sketched below for different levels of the relative risk aversion $\gamma$.

A.4. Hyperbolic absolute risk aversion (HARA) utility functions

The absolute risk aversion of the CRRA family is $\gamma / w$, a function whose graph is a hyperbola. Other hyperbolic absolute risk aversion functions, of the form $ARA(w) = a + (bw + c)^{-\gamma}$, define the HARA family of utility functions. These utility functions are also commonly used in academic papers.

A.5. Further reading

For a fuller introduction to the use of utility functions within the context of dynamic investment strategy problems, we recommend Nielsen (2006).
Appendix B: Optimal Sharpe ratio portfolios

B.1. The model
This appendix summarises results in Goetzmann et al (2002). They investigate models which allow rebalancing or which contain assets such as options whose returns are not normally distributed. They show how to construct portfolios that have superior Sharpe ratios. This appendix looks at just one of their examples (continuous-time rebalancing) to illustrate their point.

The model consists of the usual ‘Black Scholes’ set up of lognormally distributed asset prices:

- Deterministic bond prices $P_{bt} = P(t,T) = \exp(-r(T-t))$;
- Stock price $S_t = \exp\left( (\mu - \sigma^2/2) t + \sigma W_t \right)$ driven by a Brownian motion (Wiener) process $W_t$;
- Stochastic discount process $m_t = \exp\left( -(r + \lambda^2/2)t - \lambda W_t \right)$, where $\lambda = (\mu - r)/\sigma$ is the instantaneous Sharpe ratio for the stock.

The Sharpe ratio of investing a static stock investment to time $T$ is

$$\frac{\exp(\mu T) - \exp(r T)}{\exp(\mu T) (\exp(\sigma^2 T) - 1)} = \frac{\mu - r}{\sigma} \sqrt{T} = \lambda \sqrt{T} \text{ for small } T.$$ 

B.2. Maximum Sharpe ratio portfolio
Suppose a portfolio has value (payoff) $X_T$ at time $T$. This has value $E_0(m_TX_T)/m_0$ at time zero, and so the expected Sharpe ratio is

$$\frac{E_0(X_T) - E_0(m_TX_T)/E_0(m_0P(0,T))}{\text{Var}(X_T)}$$

This can be optimised by maximising a Lagrangian

$$E_0(X_T) - \lambda E_0(X_T^2) - \mu E_0(m_TX_T).$$

This then gives $X_T = \alpha - \beta m_T$ which has value $X_T = \alpha P(t,T) - \beta m_t \exp\left( (\lambda^2 - 2r)(T-t) \right)$ and expected return and variance $E_0(X_T) = \alpha - \beta P(0,T), \text{Var}(X_T) = \beta^2 \exp(-2rT)\left( \exp(\lambda^2 T) - 1 \right)$. Therefore the expected Sharpe ratio is

$$\text{sign}(\beta) \sqrt{\exp(\lambda^2 T) - 1}.$$ 

B.3. Return distribution
The probability density function for the stock and the maximum Sharpe ratio portfolio follow directly from the pdf for the normal distribution (set $\beta = 1$ for simplicity here):

$$f_{S_t}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( -\frac{(\ln(x) - (\mu - \sigma^2/2)T)^2}{2\sigma^2T} \right)$$

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi\lambda}(\alpha - x)} \exp\left( -\frac{(\ln(\alpha - x) + (r - \lambda^2/2)T)^2}{2\lambda^2T} \right)$$

These are plotted in the graph shown in section [1.1] above and repeated below.
Return from max Sharpe portfolio
Appendix C: Derivation of expected utility results

C.1. Definition of a family of modified utility functions

The most common utility function in the finance literature is of the form

\[ U(w_T) = \frac{w_T^{1-\gamma} - 1}{1-\gamma} \]

In order to model various floor and cap structures, this section analyses the following family of modified utility functions:

\[ U(w_T) = \left( \frac{[w_T - FP(T,T')]_{\max(L,F,M)}}{[M-F,P(T,T')]_{\min(L,F,M)}} \right)^{-\gamma} - 1 \]

Here,
- \( [x]_+ = \max(A,\min(B,x)) \) denotes a value \( x \) limited to lie in the interval \([A,B]\);
- \( P(s,t) \) is the price at time \( s \) of a zero-coupon bond paying 1 unit at time \( t > s \); and
- \( \gamma \) is a relative risk aversion parameter.

The idea behind the three modifications are:
- \( F \) is a floor relative to which risk is measured;
- \( M \) is a minimum payment level, below which some kind of guarantee may apply; and
- \( L \) is a target liability payment. All these payments are due at time \( T' > T \).

In addition, we may seek to impose a hard constraint \( w_T \geq GP(T,T') \). Here \( G \leq M \).

A graph may help to illustrate the idea here (\( G = 25, F = 50, M = 150, L = 350 \)):

The above graph shows the typical situation in the case that \( F \leq M \). When \( F \geq M \), then \( F \) becomes a hard floor to the wealth level, and \( M \) becomes irrelevant.
C.2. Optimal terminal wealth

The optimal terminal wealth profile can be calculated fairly simply. The optimisation problem is

$$\text{Max } U(w_T) \text{ subject to } w_T = \mathbb{E}_t \left( \frac{m_T w_T}{m_t} \right), \ w_T \geq G.P(T,T').$$

To solve this optimisation problem, it can be helpful to define a convex version of the utility function: this amounts to establishing the point $M'$ at which a line drawn from $(G,U(M))$ touches the utility function, as illustrated below.

It may be that $M' = L$, and certainly $M' \leq L$. The solution to the original problem is also a solution to the problem with this convex utility function, and in particular the optimal wealth does not take values on the (open) interval $(G, M')$. When $M' = L$, the optimal payoff is thus the payoff from a binary option (taking one of the two values $G, L$) and more generally the solution is

$$w'_T = \begin{cases} 
G.P(T,T') & \text{if } F.P(T,T') + \xi X(\gamma, T) < M'.P(T,T') \\
F.P(T,T') + \xi X(\gamma, T) & \text{if } M'.P(T,T') \leq F.P(T,T') + \xi X(\gamma, T) \leq L.P(T,T') \\
L.P(T,T') & \text{if } L.P(T,T') < F.P(T,T') + \xi X(\gamma, T)
\end{cases}$$

Here $X(\gamma, T)$ is the price of the $(\gamma, T)$-optimal portfolio at time $t$, i.e. the solution to the unmodified optimisation problem $(F = 0, M = 0, L = \infty)$. The value of $\xi$ is set by equating the value of this terminal wealth (for which a formula is derived below) with the current wealth $w_t$.

This terminal wealth function is illustrated in the graph below – it is a piecewise linear function of $X(\gamma, T)$. 

![Graph of Utility vs Funding Level](image-url)
Although this looks complex, this is simply a combination of the bond \( P(t, T') \) or the T-optimal portfolio \( X(\gamma, T) \) and familiar option payoffs: a binary option (giving the jump from \( G \) to \( M' \) shown above), and two call options (or, due to put-call parity, this can be expressed in terms of put options).

**C.3. T-optimal portfolio process wealth**

In the deterministic-volatility case (the only one treated here), write \( \lambda(t) \) for the market price of risk (vector) process, and \( \sigma(t, T) \) for the instantaneous log-volatility of bond prices; that is:

\[
\begin{align*}
\frac{dm_i}{m_t} &= -r_t dt - \lambda(t) dW_i - r_t dt - \sum_{i=1}^{n} \lambda_i(t) dW_i^i \\
\frac{dm_i PT(t, T)}{m_P(t, T)} &= (\sigma(t, T) - \lambda(t)) dW_i - \sum_{i=1}^{n} (\sigma_i(t, T) - \lambda_i(t)) dW_i^i
\end{align*}
\]

The \((\gamma, T)\)-optimal portfolio can be expressed in terms of the growth-optimal portfolio, i.e. the portfolio that maximises expected growth, which is

\[ h_t = m_t^{-1} = \exp \left( \frac{1}{2} \int \lambda(u) \gamma(u) du + \int \lambda(u) dW_u \right) \]

The \((\gamma, T)\)-optimal portfolio is then

\[
\frac{dX(\gamma, T)}{X(\gamma, T)} = \gamma^{-1} \frac{dh_t}{h_t} + \left( 1 - \gamma^{-1} \right) \frac{dP_{\gamma T}}{P_{\gamma T}}
\]

\[
X(\gamma, T) = \left( h_t \right)^{1/\gamma} \left( P_{\gamma T} \right)^{1-1/\gamma} \exp \left( \int c_x(u) du \right)
\]

\[ c_x(u) = \frac{\gamma - 1}{2\gamma} \left| \sigma(u, T) - \lambda(u) \right|^2 \]  

**C.4. Value of optimal wealth profile**

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(As the optimal wealth is a combination of well-studied options, the value of the optimal profile is easy to write down. This section is therefore only included as a refresher of techniques that are used to calculate the expected utility of this profile in the following section.)

The value \( w^*_t = \mathbb{E}_t (m_t \cdot w^*_t)/m_t \) of the optimal wealth can be calculated using standard option pricing techniques, in particular change-of-numeraire. A numeraire \( N \) is a non-dividend-paying asset whose price is always positive. The measure \( \mathbb{P}^N \) associated to \( N \) is the measure with respect to which all ratios \( R_t/N_t \) of asset prices \( R_t \) to the chosen numeraire \( N_t \) become martingales. This measure is related to the natural measure by the Radon-Nikodym derivative

\[
\frac{d\mathbb{P}^N}{d\mathbb{P}} = m_t N(t)
\]

Denote by \( A, B, C \) the three events

\[
\begin{align*}
A &= \left\{ \xi X(\gamma, T)_T < (M' - F).P(T, T') \right\} \\
B &= \left\{ (M' - F).P(T, T') \leq \xi X(\gamma, T)_T \leq (L - F).P(T, T') \right\} \\
C &= \left\{ (L - F).P(T, T') < \xi X(\gamma, T)_T \right\}
\end{align*}
\]

The value of the optimal wealth profile is then:

\[
w^*_t = m^{-1}_t \mathbb{E}_t (m_t G.P(T, T')I_A) + m^{-1}_t \mathbb{E}_t (m_t F.P(T, T')I_B) + m^{-1}_t \mathbb{E}_t (m_t L.P(T, T')I_C)
\]

\[
= G.P(t, T')\mathbb{P}^P(T, T') \mathbb{P}^P(T, T') (A) + F.P(t, T')\mathbb{P}^P(T, T') (B) + \xi X(\gamma, T)_T \mathbb{P}^P(T, T') (C)
\]

The four probabilities are straightforward to calculate, as follows.

Under \( \mathbb{P}^P(T, T') \), \( X(\gamma, T)_T / P(t, T') \) is a martingale, so writing

\[
X(\gamma, T)_T = \frac{X(\gamma, T)_T}{P(t, T')} \exp \left( -\frac{1}{2} \sigma^2 + \sigma z \right)
\]

(where \( \sigma \) is the log-volatility of the left hand side given information up to time \( t \)), it follows that \( z | \mathcal{F}_t \) is normally distributed under \( \mathbb{P}^P(T, T') \).

Specifically,

\[
\sigma^2 = \int_t^T \left[ \frac{1}{\gamma} \lambda(u) + \left( 1 - \frac{1}{\gamma} \right) \sigma(u, T) - \sigma(u, T') \right]^2 du \tag{12}
\]

Therefore

\[
\mathbb{P}^P(T, T') (A) = \mathbb{P} \left( \frac{\xi X(\gamma, T)_T}{(M' - F)P(t, T')} \exp \left( -\frac{1}{2} \sigma^2 + \sigma z \right) < 1 \right)
\]

\[
= \Phi \left( \frac{1}{\sigma} \left[ -\ln \left( \frac{\xi X(\gamma, T)_T}{(M' - F)P(t, T')} \right) + \frac{1}{2} \sigma^2 \right] \right)
\]
Similarly under \( P^{X(\gamma,T)} \), \( P(t,T')/X(\gamma,T) \) is a martingale so identical arguments give

\[
\begin{align*}
\frac{w_i^*}{G.P(t,T') \Phi(-d_2) + F.P(t,T') \left[ \Phi(-d_1) - \Phi(-d_2) \right]} + & \xi.X(\gamma,T) \left[ \Phi(-d_3) - \Phi(-d_1) \right] + L.P(t,T') \Phi(-d_4) \\
= & F.P(t,T') + (G - F).P(t,T') \Phi(-d_2) + \\
& + \xi.X(\gamma,T) \left[ \Phi(-d_3) - \Phi(-d_1) \right] + (L - F).P(t,T') \Phi(d_4)
\end{align*}
\]

(13)

\[
\begin{align*}
d_i = & \frac{1}{\sigma} \left[ \ln \left( \frac{\xi X(\gamma,T)}{(M' - F)P(t,T')} \right) + \frac{1}{2} \sigma^2 \right], \quad
d_j = \frac{1}{\sigma} \left[ \ln \left( \frac{\xi X(\gamma,T)}{(L - F)P(t,T')} \right) + \frac{1}{2} \sigma^2 \right] \\
d_2 = d_1 - \sigma, \quad d_4 = d_3 - \sigma
\end{align*}
\]

We check a few special cases. Firstly, when \( F = G = M = 0 \), this simplifies to

\[
\begin{align*}
\frac{w_i^*}{\xi.X(\gamma,T) \Phi(-d_3) + L.P(t,T') \Phi(d_4)} \\
d_j = \frac{1}{\sigma} \left[ \ln \left( \frac{\xi X(\gamma,T)}{(L - F)P(t,T')} \right) + \frac{1}{2} \sigma^2 \right], d_4 = d_3 - \sigma
\end{align*}
\]

When \( M = F = G \) and \( L = \infty \) (the CPPI case), this becomes

\[
\begin{align*}
\frac{w_i^*}{F.P(t,T')} + \xi.X(\gamma,T)
\end{align*}
\]

B5 Value of expected utility

It is useful for comparing the efficiency of potential strategies to know the expected utility of this optimal strategy. The details are rather more complicated, but this involves essentially the same technique as the preceding wealth process calculation.

\[
(1 - \gamma) \mathbb{E}_i \left( U(w_i^*) \right) - 1 = \mathbb{E}_i \left( \left[ (M - F)P(T,T') \right]^{\gamma-1} I_\gamma \right) \\
+ \mathbb{E}_i \left( \left[ \xi X(\gamma,T) \right]^{\gamma-1} I_\gamma \right) + \mathbb{E}_i \left( \left[ (L - F)P(T,T') \right]^{\gamma-1} I_\gamma \right)
\]

The second expectation on the right is the easiest. Using (11) above:

\[
\begin{align*}
X(\gamma,T)_r^{\gamma-1} & = X(\gamma,T)^{\gamma-1} \left( \frac{h_r}{h_T} \right)^{\gamma-1} \left( \frac{P(T,T)}{P(t,T)} \right)^{\gamma-1} \exp \left( (1 - \gamma) \int_t^T c_x(u) du \right) \\
& = X(\gamma,T)^{\gamma-1} \left( \frac{m_T}{m_r} \right)^{\gamma-1} \left( \frac{P(T,T)}{P(t,T)} \right)^{\gamma-1} \exp \left( (1 - \gamma) \int_t^T c_x(u) du \right) \\
& = X(\gamma,T)^{\gamma-1} \left( \frac{m_T}{m_r} \right) \left( \frac{X(\gamma,T)_r}{X(\gamma,T)_t} \right) \left( \frac{P(T,T)}{P(t,T)} \right)^{\gamma-1} \exp \left( -\gamma \int_t^T c_x(u) du \right)
\end{align*}
\]

Therefore
\[ E \left( \left[ \xi X(\gamma, T) \right]^{T-r} I_g \right) = \left( \frac{\xi X(\gamma, T)}{P(t, T)} \right)^{T-r} \exp \left( -\gamma \int_t^T c_x(u)du \right) E \left( \frac{m_r X(\gamma, T)}{m_r X(\gamma, T)} I_g \right) \]

\[ = \left( \frac{\xi X(\gamma, T)}{P(t, T)} \right)^{T-r} \exp \left( -\gamma \int_t^T c_x(u)du \right) \mathbb{E}^{x(\gamma, T)} (B) \]

\[ = \left( \frac{\xi X(\gamma, T)}{P(t, T)} \right)^{T-r} \exp \left( -\gamma \int_t^T c_x(u)du \right) \left( \Phi(-d_1) - \Phi(-d_1) \right) \]

where the calculation in the final step was made in (3) above.

To calculate \( E \left( [(M - F)P(T, T')]^{T-r} I_A \right) \), a similar approach can be taken: \( P(T, T')^{T-r} \) needs to be written as (up to a deterministic factor) the discounted payoff from a numeraire, which can then be used as the basis for a change of measure. This is readily achieved:

\[ P(T, T')^{T-r} = \left( \frac{P(t, T')}{P(t, T)} \right)^{T-r} \frac{m_r h_r P(T, T')^{T-r} P(T, T')^{T-r}}{m_r h_r P(T, T')^{T-r} P(T, T')^{T-r}} \]

The term \( h_r P(T, T')^{T-r} P(t, T')^{T-r} \) is almost the price of a numeraire: it just needs a deterministic adjustment factor. Specifically, define a numeraire price process \( Y(\gamma, T, T') \) via the SDE

\[ \frac{dY(\gamma, T, T')}{Y(\gamma, T, T')} = \frac{dh_r}{h_r} + (\gamma - 1) \frac{dP(t, T)}{P(t, T)} + (1 - \gamma) \frac{dP(t, T')}{P(t, T')} \]

Then

\[ Y(\gamma, T, T') = h_r P(T, T')^{T-r} P(t, T')^{T-r} \exp \left( \int_0^{T} c_r(u)du \right), \]

with

\[ c_r(u) = \frac{\gamma - 1}{2} \left[ \left( \lambda(u) - \sigma(u, T) \right)^2 - \left( \lambda(u) - \sigma(u, T') \right)^2 \right] \frac{1}{\gamma} \left( \sigma(u, T) - \sigma(u, T') \right)^2. \]

Note that an alternative expression for \( Y(\gamma, T, T') \) can be derived by noting that

\[ \frac{dY(\gamma, T, T')}{Y(\gamma, T, T')} = \gamma \frac{dX(\gamma, T)}{X(\gamma, T)} + (1 - \gamma) \frac{dP(t, T')}{P(t, T')} \]

So that

\[ Y(\gamma, T, T') = X(\gamma, T) P(t, T')^{T-r} \exp \left( \int_0^{T} c_r(u)du \right), \quad (14) \]

with

\[ c_r(u) = \frac{\gamma(1 - \gamma)}{2} \left( \lambda(u) + (1 - \gamma) \frac{1}{\gamma} \sigma(u, T) - \sigma(u, T') \right)^2. \]

(15)

It can be checked, either by comparing the two expressions for \( Y(\gamma, T, T') \), or by direct calculation, that \( c_r - \gamma c_x = c_r \). Furthermore, comparing (5) with (2) above,

\[ \int_0^{T} c_r(u)du = \frac{\gamma(1 - \gamma)}{2} \sigma^2. \]

With these observations in place, the expectation can now be simplified:

\[ E \left( [(M - F)P(T, T')]^{T-r} I_A \right) = \left( \frac{(M - F)P(T, T')}{P(t, T)} \right)^{T-r} \exp \left( -\int_0^{T} c_r(u)du \right) \mathbb{E}^{x(\gamma, T)} (A). \]

To calculate the final probability, (4) implies that:
\[
A = \begin{cases}
P(T,T')X(\gamma,T)_{T'_{r}} \\
(P(T,T')X(\gamma,T)_{T'_{r}}, (M' - F)P(t,T'))
\end{cases}
\]

By definition, \(P(t,T')/Y(\gamma,T,T')\) is a martingale under \(P^{t,T',Y}(\cdot)\), and indeed from (4), the log-volatility is \(\gamma\) times the volatility of \(P(t,T')/X(\gamma,T)\).

So

\[
\frac{P(T,T')Y(\gamma,T,T')_{T'_{r}}}{P(t,T')Y(\gamma,T,T')_{T'_{r}}} = \exp \left( -\frac{\gamma^2 \sigma^2}{2} + \gamma \sigma z \right)
\]

with \(zF_{t}\) is a unit normal random variable under \(P^{t,T',Y}(\cdot)\). Using (5) then gives

\[
P^{t,T',Y}(A) = \mathbb{P} \left\{ z \geq \frac{1}{\sigma} \left[ \ln \left( \frac{\xi X(\gamma,T)_{T'_{r}}}{(M' - F)P(t,T')} \right) + \frac{2\gamma - 1}{2} \sigma^2 \right] \right\}
\]

The calculation of \(\mathbb{E}(\cdot)\) is almost identical, so in summary

\[
(1 - \gamma)\mathbb{E}(U(w_{t})) = 1 + \left( \frac{(M' - F)P(t,T')}{P(t,T)} \right)^{1-\gamma} \exp \left( -\int_{t}^{T} c_{y}(u)du \right) \mathbb{P} \left\{ \frac{1}{\sigma} \left[ -\ln \left( \frac{\xi X(\gamma,T)_{T'_{r}}}{(M' - F)P(t,T')} \right) - \frac{2\gamma - 1}{2} \sigma^2 \right] \right\}
\]

As a preliminary lemma, the \(d\Phi(x)\) terms can be calculated. Set \(\sigma_{S}^{P(T',T)} = \sigma_{S} - \sigma_{P(T',T)}\) to be the log volatility of \(S_{t}/K.P(t,T)\). Then for \(\varepsilon, \varepsilon' = \pm 1\), and putting

\[
x = \left[ \int_{t}^{T} \sigma^{2}(u)du \right]^{-1/2} \left( \varepsilon \ln (S_{t}/K.P(t,T)) + \frac{\varepsilon'}{2} \int_{t}^{T} \sigma^{2}(u)du \right),
\]

Ito’s lemma can be used to deduce that

C.5. Optimal strategies

Although the value of the optimal payoff has been calculated, this does not immediately identify the dynamic strategy required to obtain this payoff in practice. As usual, this can be calculated by calculating the (stochastic) derivative of the value of the optimal wealth strategy (13) and identifying terms. Again, writing down this delta hedge down is straightforward due to standard calculations in the option pricing literature, but the details are checked here, as the model is slightly more general than the Black-Scholes model that is usually quoted.

As a preliminary lemma, the \(d\Phi(x)\) terms can be calculated. Set \(\sigma_{S}^{P(T',T)} = \sigma_{S} - \sigma_{P(T',T)}\) to be the log volatility of \(S_{t}/K.P(t,T)\). Then for \(\varepsilon, \varepsilon' = \pm 1\), and putting

\[
x = \left[ \int_{t}^{T} \sigma^{2}(u)du \right]^{-1/2} \left( \varepsilon \ln (S_{t}/K.P(t,T)) + \frac{\varepsilon'}{2} \int_{t}^{T} \sigma^{2}(u)du \right),
\]
\[ dx = \left[ \int_{t}^{T} \sigma_{i} \, du \right]^{-1/2} \left( \epsilon \sigma_{i} dW_{i}^{P(t,T)} - (\epsilon + \epsilon') \sigma_{i}^{2} \, dt / 2 \right) + \left[ \int_{t}^{T} \sigma_{i}^{2} \, du \right]^{-1} \sigma_{i}^{*} \, x \, dt / 2 \]

and thus
\[ d\Phi(x) = \phi(x) \left[ \int_{t}^{T} \sigma_{i}^{2} \, du \right]^{-1/2} \left( \epsilon \sigma_{i} dW_{i}^{P(t,T)} - (\epsilon + \epsilon') \sigma_{i}^{2} \, dt / 2 \right). \]

Here, \( dW_{i}^{P(t,T)} = dW_{i} + (\lambda(t) - \sigma(t,T)) \, dt \) is Brownian motion under \( P^{P(t,T)} \).

From the definitions
\[ d_{1} = \frac{1}{\sigma} \left[ \ln \left( \frac{\xi X(\gamma, T)}{(L - F) P(t, T')} \right) + \frac{\sigma^{2}}{2} \right], \quad d_{4} = \frac{1}{\sigma} \left[ \ln \left( \frac{\xi X(\gamma, T)}{(L - F) P(t, T')} \right) - \frac{\sigma^{2}}{2} \right] = d_{1} - \sigma, \]

it follows easily that
\[ \phi(d_{4}) = \phi(-d_{1}) \frac{\xi X(\gamma, T)}{(L - F) P(t, T')} \]

So
\[
\begin{align*}
&d \left( \frac{\xi X(\gamma, T)}{(L - F) P(t, T')} \Phi(-d_{1}) \right) = \xi \Phi(-d_{1}) dX(\gamma, T) + (L - F) \Phi(d_{1}) \, dP(t, T') \\
&\quad + \frac{\phi(-d_{1}) \xi X(\gamma, T)}{\left[ \int_{t}^{T} \sigma_{i}^{2} \, du \right]^{1/2}} \left( -\sigma_{i} dW_{i}^{P(t,T)} + \sigma_{i}^{2} \, dt \right) - \frac{\sigma_{i}^{2} \phi(-d_{1}) \xi X(\gamma, T)}{\left[ \int_{t}^{T} \sigma_{i}^{2} \, du \right]^{1/2}} \\
&\quad + \frac{\phi(d_{1})(L - F) P(t, T')}{\left[ \int_{t}^{T} \sigma_{i}^{2} \, du \right]^{1/2}} \left( \sigma_{i} dW_{i}^{P(t,T)} \right) + \frac{\sigma_{i}^{2} \phi(d_{1})(L - F) P(t, T')}{\left[ \int_{t}^{T} \sigma_{i}^{2} \, du \right]^{1/2}} \\
&\quad = \xi \Phi(-d_{1}) dX(\gamma, T) + (L - F) \Phi(d_{1}) \, dP(t, T')
\end{align*}
\]

This hedge for a covered call, and in particular the ‘magic’ cancellations of terms, is of course well known. A similar approach can be used for terms that relate to a call option struck at \( M \):

Similarly, the expression for the put option at \( M \) is straightforward
\[
\begin{align*}
&d \left( \frac{(M - F) P(t, T') \Phi(-d_{1})}{-\xi X(\gamma, T)} \right) = (M - F) \Phi(-d_{1}) \, dP(t, T') - \xi \Phi(-d_{1}) dX(\gamma, T),
\end{align*}
\]

The remaining term is the value of the binary option \( (G - M) P(t, T') \Phi(-d_{2}) \). Here, the hedge is notoriously more unstable and there is much less cancellation of terms.

\[
\begin{align*}
&d \left( (G - M) P(t, T') \Phi(-d_{2}) \right) = (G - M) \Phi(-d_{2}) \, dP(t, T') \\
&\quad - \frac{(G - M) P(t, T') \phi(-d_{2})}{\left[ \int_{t}^{T} \sigma_{i}^{2} \, du \right]^{1/2}} \sigma_{i} \left( dW_{i}^{P(t,T)} + \sigma_{i}^{2} \, dt \right) \\
&\quad = (G - M) \Phi(-d_{2}) \, dP(t, T') \\
&\quad - \frac{(G - M) P(t, T') \phi(-d_{2})}{\left[ \int_{t}^{T} \sigma_{i}^{2} \, du \right]^{1/2}} \left( dX(\gamma, T) - \frac{dP(t, T')}{P(t, T')} \right)
\end{align*}
\]

This hedge for a covered put, and in particular the ‘magic’ cancellations of terms, is of course well known. A similar approach can be used for terms that relate to a put option struck at \( M \):
In particular, we note that only two portfolios are required over time to achieve this payoff: the liability hedge, $P(t,T')$ and the $T$-optimal portfolio $X(\gamma, T)$.
Appendix D:  Why were non-recombining trees intractible

Previous approaches, as in the literature, show the following:

- One defines a set of investment factors (stock market indices, FX rates, yield curve parameters).
- One then writes down a set of simultaneous SDEs for these variables.
- One constructs a multidimensional tree for them.
- One assigns an unknown set of portfolio weights to each node.
- One then runs an optimization to find the global set of weights that optimizes the risk return trade off for that portfolio.

We found that when we tried to reproduce this method we very quickly had exponentially increasing weights in our optimisation problem. The reason for this is as follows:

Suppose that we have 10 asset factors, and 10 time steps at which decissions are made. The methods then require construction of a non-recombining tree. For a decission to be made at a node, the node has to split in the dimension of each decision variable. Thus with 10 assets we need to split each node into $2^{10}$ paths (approximately 1,000). We then construct a (non-recombining) tree with 10 time steps. Therefore this has $1,000^{10}$ (or $10^{30}$ final nodes). At each node there are 10 weights therefore we have to solve an optimization problem with $10^{31}$ variables.

By contrast, our method described in the text can also optimizes 10 assets for 10 time steps. We use only 1 SDE being the total size of the portfolio. We construct a finite difference mesh with 10 time steps and 10 space steps giving us 100 nodes. At each node we have a set of 10 portfolio weights to choose giving us a problem with 1,000 variables to solve.

Therefore our method has reduced the scale of the problem by a factor of the order of $10^{28}$, making tractable a problem that would not otherwise be.