Dynamic Portfolio Choice with Deferred Annuities

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Track E – Financial Risk (AFIR)

Abstract
We derive the optimal portfolio choice and consumption pattern over the life-cycle for households facing uncertain labor income, risky capital market, and mortality risk. In addition to stocks and bonds, the household have access to deferred annuities. Deferred payout annuities are financial contracts providing – similar to social security or defined benefit pension plans – lifelong income to the annuitant after a specified period of time conditional on survival. We find that deferred annuities play a significant role in household portfolios and generate significant welfare gains. Our base case investor with high benefits from state pensions and moderate risk aversion and moderate labor income risk starts purchasing deferred annuities at age 40 and gradually increases their portfolio share until retirement at age 65. Then, deferred annuities already account for 78 percent of total financial wealth, with the rest being invested in stocks. Facing low replacement rates from state pensions and a high exposure to labor income risk, the household will purchase more annuities and earlier. Introducing uncertainty with respect to future mortality rates has the same effect, i.e. investors actively use deferred annuities as a hedge against longevity shocks.

JEL Codes: D91 G11 J26
Keywords: Portfolio Choice, Deferred Annuities, Stochastic Mortality

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1 Introduction and Motivation

Previous research on dynamic portfolio choice over the life cycle argues that purchasing annuities with immediate life contingent payouts is important to finance consumption of risk-averse households with uncertain lifetime and no bequest motive (see originally Yaari 1965 and more recently Davidoff et al. 2005). Life annuities are financial products that allow investors to protect against outliving retirement assets while enhancing expected returns through the ‘mortality credit’. Despite this theoretical dominance, investors around the world are reluctant to voluntary annuitize their wealth. This discrepancy between theoretically predicted and empirically observed behaviour is known as the annuity puzzle. Arguments explaining this divergence include incomplete annuity markets, bequest motives, high costs charged by annuity providers, and behavioural factors (Hu and Scott 2007, Yagi and Nishigaki 1993). Against this background, the number of advocates of embedding annuitization as a default option in tax-sheltered pension plans is increasing. In this context, deferred annuities more and more attract the attention of policymakers, regulators and financial intermediaries.

Similar to an immediate annuity, the provider of a deferred annuity promises lifelong periodic payouts to the annuitant in exchange for a non-refundable premium. While payments from immediate annuities start at the date purchase, those from deferred life annuities only commence after a certain number of years (the deferring period) has passed, subject to the individual’s survival. Due to the discounting effect as well as the possibility that the annuitant perishes before payouts start, a deferred annuity is much cheaper than an immediate annuity with identical payouts. According to Milevsky (2005a, p. 110), the comparably low price of deferred annuities may help to overcome psychological barriers to voluntary annuitization, as he points out that “engaging in irreversible financial transactions - that is annuitization - involving large lump sums will never be appealing to individuals regardless of (whether they grasp) the importance of longevity insurance”.

Cash flow patterns similar to those of deferred life annuities are provided by state-organized social security systems and occupational pension plans of the defined benefit variety. In both pension schemes, workers (repeatedly) contribute a certain fraction of their labor income during their working life. In retirement, they receive pension benefits determined by their contributions and working years for as long as they live. Consequently, in a world with declining relevance of both state-organized social security systems and occupational defined benefit pensions, the relevance of deferred life annuities offered in the private market might increase in future.

While immediate life annuities have recently been studied to a large extent, the role of deferred annuities for private households has not been considered in the literature on dynamic portfolio choice over the life cycle. Prior work on the role of life annuities in realistically calibrated dynamic portfolio choice models within an incomplete market setting includes Milevsky and Young (2007) and Horneff et al. (2008 a, b), who analyze the possibility of gradual annuitization for the case of fixed immediate payout annuities. We extend this work in two directions: First, we incorporate life annuities with deferred payouts in line with those proposed and analyzed in Milevsky (2005a) and derive the optimal consumption, saving and portfolio choice pattern for a CRRA utility maximizing household facing uncertain life time, un-spanned labor income, and risky capital market returns. Second, we extend this model by integrating time-varying mortality rates that can exhibit both expected improvements in future life expectancy as well as stochastic variations around this trend. This results in uncertainty with
respect to future prices of deferred annuities and individual survival probabilities, which a rational investor has to anticipate when making decisions on consumption, saving and portfolio allocation.

We find that deferred annuities play a significant role in household portfolios and generate significant welfare gains. Our base case investor with high benefits from state pensions and moderate risk aversion and moderate labor income risk starts purchasing deferred annuities at age 40 and gradually increases their portfolio share until retirement at age 65. Then, deferred annuities already account for 78 percent of total financial wealth, with the rest being invested in stocks. Facing low replacement rates from state pensions and a high exposure to labor income risk, the household will purchase more annuities and earlier. Introducing uncertainty with respect to future mortality rates has the same effect, i.e. investors actively use deferred annuities as a hedge against longevity shocks.

In what follows, we introduce the life-cycle model applied to find the optimal consumption and portfolio allocation into stocks, bonds, and deferred annuities. Here, we discuss preferences, risky labor income patterns, annuity as well as capital market specifications, wealth accumulation, and stochastic mortality dynamics. Out findings are presented in section three. We discuss the optimal life-cycle asset allocation for our base case and in a scenario analysis we show expected life-cycle profiles for various calibrations of key parameters. We continue by analyzing the impact of time-varying mortality and the role of annuities with very high deferring age, before we study the welfare implications of adding deferred annuities to the investment menu. A final chapter concludes.

2 The Model

2.1 Preferences

In our study, we employ a time discrete model with \( t \in \{0, \ldots, T+1\} \), where \( t \) constitutes the investor’s adult age (calculated as the actual age minus 19) and \( T \) is the maximal age. We denote \( p_t^s \) as the investor’s subjective probability to survive from \( t \) until \( t + 1 \). Furthermore, we assume that the investor’s preferences are given by a time-separable CRRA utility function defined over a single non-durable consumption good. Let \( C_t \) be the consumption level at time \( t \) the recursive definition of the value function is given by

\[
V_t = \frac{C_t^{1-\rho}}{1-\rho} + \beta p_t^s E_t [V_{t+1}]
\]

(1)

Here \( \rho \) is the level of relative risk aversion and \( \beta \) is the personal time preference discount factor. We assume that the household does not derive utility from bequeathing potential heirs.iii Today’s utility is given as the utility from consumption and tomorrow’s discounted utility from future consumption. We have \( p_T^s = 0 \) for the final period, i.e. in \( T \) equation (1) boils down to: \( V_T = \frac{C_T^{1-\rho}}{1-\rho} \), which gives us the terminal condition for \( V_T \). From the final value, we can work backwards to find the optimal strategies how to consume, how to invest in bonds, stocks, and how to purchase deferred annuities over the life cycle.
2.2 Deferred Annuity and Capital Markets

The household has access to capital markets by investing in riskless bonds and risky stocks. The real bond gross return is constant over time and given by $R_p$, while the real gross risky stock return at time $t$ is denoted by $R_s$. The risky stock returns are assumed to be serially independent and identically log-normally distributed with an expected return $\mu$ and volatility $\sigma$.

In addition to bonds and stocks, the household can also purchase at each time $t < K$ for an initial non-refundable premium $A_t$ deferred life annuities with constant lifelong real payouts $L_K$ starting at the end of the deferring period a certain age $K$. In our base case the end of deferring period is equal with the retirement age $K = 65$, i.e. when the individual receive sure social security benefits instead of risky labor income. Later we consider also a situation whereby the annuity would begin to start payouts at the advanced age of 85. The life annuity does neither contain any survival or estate benefits nor a cash value at any age. Using the actuarial principle of equivalence the annuity provider calculates the gross single premium as the present value of expected benefits paid to the annuitant (including expense loadings), or formally

$$A_t = L_K h_t \quad t < K$$

where $h_t$ is the annuity factor for an individual with adult age $t$ which is given by

$$h_t = (1 + \delta) \left( \prod_{u=t}^{K-2} p_u^a \right) R_f^{(K-1-t)} \sum_{s=1}^{T-K} \left( \prod_{u=s}^{K-1} p_u^a \right) R_f^{-s} \quad t < K$$

Here $\delta$ is the expense or loading factor charged by the annuity provider to organize the risk-pool of annuitants and the discount factor $R_f$ is equal to the riskless bond return. The year-to-year survival probabilities $p_u^a$ used to price the annuity are specified by a mortality table and may be higher than the individual’s subjective survival probabilities ($p_u^c > p_u^s$). This allows to model asymmetric mortality beliefs whereby the additional price increment is thought of as a compensation for both the adverse selection problem in the private annuity markets (Brugiavini, 1993, Finkelstein and Poterba, 2004) and the macro longevity risk (Cairns, Blake, and Dowd, 2006a). Adverse selection in annuity markets arises because individuals who believe themselves to be healthier than average buy annuities more likely. Macro longevity risk refers to the risk of unexpected changes of survival probabilities.

We consider a highly incomplete annuity market. The only annuities available are deferred annuities with life-long constant payouts, which can only be purchased during the deferral period, i.e. before payments commence. Thus, the investor neither has access to deferred annuities after reaching the deferring age nor to immediate payout annuities. We do not account for annuities which payout at only one specific age and state (as in the complete markets case in Davidoff, Brown, and Diamond, 2005) or on annuities with payouts specified according to the performance of an underlying portfolio of risky assets (see Horneff, Maurer, Mitchell, and Stamos 2010).

The illiquidity related to deferred annuities adversely affects the investor’s ability to react to either adverse developements of labor income or sudden declines in the stock market. In return for the illiquidity and the forgiven bequest opportunity, the household gains conditional on survival a spread over the payout from an investment in riskless bonds, also
referred in the literature as the mortality credit. The spread comes about because the funds of those who die in the annuity pool are distributed among the living members of a cohort. Therefore, a deferred annuity simply constitutes a separate asset class with distinctive risk and return characteristics. We treat the purchase of deferred annuities as a portfolio choice problem by putting them on an equal footing with equity and bond investments. In the remainder, we model the annuitization decision essentially in a dynamic portfolio choice framework akin to Horneff, Maurer, and Stamos (2008).

To illustrate the return-enhancing effects of the mortality credit in deferred annuities, we follow Milevsky (2005a, b) and present the Implied Longevity Yield (ILY) at various ages for annuities with deferring age 65 or 85. The ILY is the excess return over the discount rate that has to be generated on the deferred annuity premium in each year until deferring age to be able to then purchase an immediate annuity that pays the same benefit. Only in case excess returns over the discount rate generated in the capital markets are equal or in excess of the ILY an individual would benefit from postponing annuitization until the deferring age and then purchasing an immediate annuity. If the excess capital market return falls short of the ILY, an investor interested in annuity payments from deferring age on is better off by immediately purchasing a deferred annuity. Table 1 shows the ILY in basis points (BP) based on our 2000 US Annuitant Basic mortality table ($q_{100} := 1$) and a real annual discount rate of 2%. When purchasing a deferred annuity due from age 65 (85), a woman aged 20 already locks in an ILY of 12 (84) BP. The mortality credit increases with age and so does the ILY. Buying the same annuities at age 60 will provide her with ILYs of 48 (207) BP. Particularly when deferring until age 85, it will be increasingly difficult to generate this excess return, making it profitable to purchase these annuities well before deferring age. As men on average are subject to higher mortality rates, their ILY significantly exceed those of women: for a deferring age of 65 (85) by about 100 (50) percent.

Table 1 here

The long deferral periods combined with the increases in return due to the aforementioned mortality credit have a significant positive impact on the individual’s budget and liquidity. At ages 20, 40, and 60 prices for a deferred annuity paying 1 from age 65 on are 6.94, 10.33, and 15.78. Purchasing an immediate annuity offering the same benefit at age 65 would cost 17.84. Naturally, this effect is even more pronounced when annuity payments are deferred until age 85. Then, at ages 20, 40, 60, and 80 prices for a life-long income of 1 are 1.20, 1.78, 2.72, and 5.31, while an immediate annuity purchased at age 85 would cost 7.40. This emphasizes the insurance character of deferred annuities against real longevity risk.

2.3 Labor Income Process
In order to understand how the illiquidity of deferred annuities affects the overall asset allocation, we model transitory and permanent income shocks. Previous literature on strategic asset allocation such as Bodie, Merton, and Samuelson (1992), Cocco, Gomes and Maenhout (2005), Heaton and Lucas (1997), and Viceira (2001) highlighted the relevance of considering unspanned labor income when analyzing the portfolio choice decisions of
households over the life cycle. The labor income \( Y_t \) is given by:

\[
Y_t = \exp(f(t))P_tU_t, \tag{4}
\]

\[
P_t = P_{r_t}N_t, \tag{5}
\]

where \( f(t) \) is used to recover the hump shape of the empirically typically observed income profile over time \( t \). Here, \( P_t \) is a permanent component with innovation \( N_t \), \( U_t \) is a transitory shock. The logarithms of both \( N_t \) and \( U_t \) are i.i.d normally distributed with means zero and with volatilities \( \sigma_N, \sigma_U \), respectively. The shocks are assumed to be uncorrelated. In retirement \((t \geq \tau)\), we assume for the sake of simplicity that the individual receives constant pension payments \( Y_t = \zeta \exp f(\tau)P_t \), where \( \zeta \) is the constant replacement ratio. The variable \( \phi_a(\phi_b) \) describes the correlation between the risky stock returns and the permanent (transitory) labor income shocks.

### 2.4 Wealth Accumulation

At the beginning of every period, the utility maximizing household under consideration can decide on how to spread wealth on hand \( W_t \) across bonds \( B_t \), stocks \( S_t \), new annuities purchases \( A_t \) (as long as \( t < K \)), and consumption \( C_t \). Therefore, the budget constraint is

\[
W_t = \begin{cases} 
B_t + S_t + A_t + C_t & t < K \\
B_t + S_t + C_t & t \geq K
\end{cases}, \tag{6}
\]

where we refer to \( B_t + S_t \) as the value of financial wealth. The individual’s disposable wealth on hand in \( t + 1 \) is given by

\[
W_{t+1} = \begin{cases} 
B_t R_t + S_t R_t + Y_{t+1} & t < K \\
B_t R_t + S_t R_t + L_{t+1} + Y_{t+1} & t \geq K
\end{cases}. \tag{7}
\]

where \( B_t R_t + S_t R_{t+1} \) denotes the next period’s value of financial wealth, \( L_{t+1} \) is the sum of annuity income which the investor gets from all previously purchased annuities and \( Y_{t+1} \) is the labor income or, after retirement, the exogenously provided pension income. At any time \( t (\leq K) \), the state variable \( L^{(i)}_t \) represents the previously accumulated claims on annuity payouts due at adult age \( K \), whereas from age \( K \) on, the state variable \( L_t \) denotes the payouts from previously purchased deferred annuities. These state variables follow the processes:

\[
L^{(i)}_K = L^{(i)}_K + A_t/h_t & \quad t < K \\
L_{t+1} = L_t & \quad t \geq K
\]

where \( A_t/h_t \) is the additional annuity payment purchased in \( t \). We prevent households from borrowing against human capital and from selling annuities. Both restrictions are binding because otherwise households would engage in highly leveraged stock positions financed by short positions in bonds and/or annuities in order to compensate the over-investment in human capital at young ages. Thus, in every year the optimal policy has to
satisfy:

\[ B_i, S_i, A_i \geq 0. \]  

(9)

2.5 Mortality Dynamics

There is growing consent among academics and practitioners, that mortality rates over time are neither fixed nor following deterministic trends but rather follow stochastic processes (see Milevsky and Promislow (2001), Milevsky, Promislow, and Young (2009), Cairns, Blake, and Dowd (2006, 2008)). In our context uncertain mortality could be an important factor for the household’s demand-pattern of deferred annuities. By buying deferred annuities early in life the investor is able to insure against the risk of unexpected declining mortality rates occurring late in life. Declining mortality rates could have two effects: First, subjective survival probabilities used by the household to value future consumption and cash flows are increasing. This makes cash flows streams from annuities already purchased earlier in life more attractive. Second, to reflect the new information about unexpected decreases in mortality rates, the insurance company has to adjust the actuarial assumptions underlying the pricing of the annuities. This makes the purchase of additional annuities late in life more expensive for the household.

To examine the implications of stochastic changes of mortality rates on optimal life cycle consumption, portfolio choice and the demand for deferred annuities we use a stochastic mortality model. One of the standard approaches to modeling stochastic mortality, widely spread among demographers and actuaries, is the framework developed by Lee and Carter (1992). In their model, the natural logarithm of age-specific mortality rates is described by a linear function of age-dependent parameters and a latent, unobservable and period-specific index variable:

\[
\ln(m_{x,t}) = a_x + b_x \cdot k_t + \epsilon_{x,t},
\]

(10)

where \( m_{x,t} \) is the central death rate for those aged \( x \) in year \( t \), \( a_x \) and \( b_x \) are age-specific constants, \( k_t \) is the latent, time-varying index and \( \epsilon_{x,t} \) is an age-specific error term with zero mean and a variance \( \sigma_e^2 \). The index variable \( k_t \) itself follows a random walk with drift. Stating (10) differently by taking the exponential, the model can be interpreted as

\[
m_{x,t} = \exp(a_x) \cdot \exp(b_x \cdot k_t + \epsilon_{x,t}) = q_x \cdot \Delta_{x,t},
\]

(11)

where \( q_x \) is the mortality rate in the base year and \( \Delta_{x,t} \) is the cumulative change in this mortality until year \( t \).

While this is already a parsimonious approach to modeling stochastic mortality, the curse of dimensionality renders its application impractical in our dynamic stochastic programming framework, as the level of \( k_t \) represents an additional continuous state variable. To circumvent this problem, we assume that instead of following a geometric random walk with drift, \( \Delta_{x,i} \) is described by a symmetric and bounded trinomial tree with an odd number of \( n \) states \( (i \in \{- (n-1)/2, \ldots, 0, \ldots, (n-1)/2\}) \), where the age-specific mortality reduction factor in mortality state \( i \) is \( \Delta_{x,i} = i \cdot \Delta_x \). Here, \( \Delta_x \) is a time-invariant, age-specific mortality reduction factor adequately calibrated to \( b_x k_t \) from the Lee/Carter model, refraining from additional age-specific shocks. The jump probabilities are defined by
\[ p_u = \begin{cases} p_u & |i < (n-1)/2 \\ 0 & |i = (n-1)/2 \end{cases}, \quad p_d = \begin{cases} 0 & |i = -(n-1)/2 \\ p_d & |i > -(n-1)/2 \end{cases}, \quad p_r = 1 - p_u - p_d \quad (12) \]

where \( p_u \) (\( p_d \) [\( p_r \]) specifies the probability to jump from state \( i \) to \( i+1 \) (\( i \) to \( i-1 \)) \( [i \) to \( i] \) as time passes from \( t \) to \( t+1 \). This approach can be interpreted as a generalization of the binomial tree model used in Milevsky and Promislow (2001).

Assuming that the probabilities for upward and downward jumps are equal, i.e. \( p_u = p_d \), this model generates purely stochastic mortality rates that remain constant in expectation. To recover the empirically observable increase in life expectancy, we superimpose a deterministic trend over the trinomial tree. Hence, the age-specific mortality reduction factor in mortality state \( i \) at time \( t \) is \( \Delta_{x,t,i} = (1 - MT)^{t-1} \cdot \Delta_{x,0,i} \). This set-up enables us to individually study the effects of stochasticity and trends in mortality rates on the individual’s life-cycle behavior.

2.6 Numerical Method and Model Calibration

Optimization problems of this type cannot be solved analytically due to the untradeable labor income, the irreversibility of annuity purchases, and the shortselling restrictions. Therefore we adopt the standard approach of dynamic stochastic programming to solve the household’s optimization problem. The household maximizes (1) under budget and short-selling restrictions (6), (7), (8), and (9), whereby the choice variables in each year the household is alive are the demand for stocks \( S \), bonds \( M \), new life annuities \( A \), and consumption \( C \). The optimal policy depends on five state variables: the permanent income level \( P \), wealth on hand \( W \), annuity payouts from previously purchased annuities \( L \), age \( t \), and mortality state \( i \). First of all, the curse of dimensionality can be partly mitigated by reducing the state space by one state variable as we exploit the scale independence of the optimal policy if we normalize the continuous state variables with the permanent labor earnings component \( P \) (see for example Cocco, Gomes, and Maenhout, 2005). We solve the problem in a four-dimensional state space by backward induction. The continuous state variables normalized wealth \( w \) and normalized annuity payouts \( l \) have to be discretized and the only discrete state variables are age \( t \) and mortality state \( i \). The size of the grid is \( 40(w)x40(l)x81(t)x9(i) \) (in our base case with fixed mortality, the state space is reduced to three dimensions). The grid we use is equally spaced for the logarithms of \( w \) and \( l \) since the policy functions and value function are especially sensitive in the area with low \( w \) or \( l \). For each grid point we calculate the optimal policy and the size of the value function.

To provide numerical insight into our setup, we calibrate the stylized base case as follows: The starting age is set to 20, the maximum age to 100 (\( T = 81 \)), and the retirement age 65 (\( K = 46 \)) is equal the end of the deferring period for annuities purchase during working life. In addition, we also study the case when annuity payouts start only at age 85. The preference parameters are set to standard values found in the life-cycle literature (e.g. Gomes and Michaelides, 2005): coefficient of relative risk aversion \( \rho = 5 \) and the personal discount factor \( \beta = 0.96 \). In our fixed mortality scenarios, we use the 2000 US Annuitant Basic mortality table for pricing the annuities and the 2000 US Population Basic mortality table for evaluating the utility from consumption, both for females. In our stochastic mortality scenarios, we employ these mortality rates as the \( q_t \) from (12). The probability upward or downward jumps in mortality is exogenously set to \( p_u = p_d = 10 \) percent, recovering the empirical observation that in many countries annuitant mortality tables are typically updated every decade. Based on the calibrations in Lee and Carter (1992), the age-
specific mortality improvement factor $\Delta_x$ is set to 5.77 percent to recover the average 1-period volatility of mortality rates and the mortality trend factor is set to $MT = 1.42$ percent to recover the average 1-period mortality improvement. Furthermore, we assume that changes in mortality identically affect both, annuitant as well as population tables. The expense rate for annuities is set to $\delta = 0$ in the base case, and 2.38% and 7.20% in the sensitivity analysis. The deterministic age-dependent labor income function $f(t)$ for individuals with high school education excluding college education and the replacement rate $\zeta = 0.68$ are taken from Cocco, Gomes, and Maenhout (2005). Our volatility parameters $\sigma_u = 0.15$ and $\sigma_n = 0.1$ are in line with the estimates found by Gourinchas and Parker (2002). Furthermore, we select a real interest rate $R_f - 1$ of 2 percent, an equity premium $\mu - R_f$ of 4 percent, and a stock volatility $\sigma$ of 18 percent. We choose correlations between the stock returns and the transitory (permanent) income shocks of $\phi_u = 0, \phi_n = 0$.

To gain further insights, we then vary selected parameters and conduct comparative static analyses. The additional scenarios include low ($\rho = 2$) and high ($\rho = 8$) risk aversion, medium ($\zeta = 0.60$) and low ($\zeta = 0.50$) replacement rates, no ($\sigma_u = \sigma_n = 0$) and high ($\sigma_u = 0.30, \sigma_n = 0.20$) labor income risk, as well as low ($\delta = 0.0238$) and high ($\delta = 0.072$) loading factors (Maurer, Mitchell, and Rogalla 2009).

In what follows, we present the results of our optimization model assuming constant as well as stochastic mortality dynamics in the case annuity payouts starting at the end of the working life. Next, we turn to a situation where the individual can purchase only annuity with payouts starting late in life at age 85. Finally we present the welfare gains for households having access to private annuity markets with deferred payouts.

3 Results

3.1 Deferred Annuities under Constant Mortality

3.1.1 Optimal Asset Allocation

In this section, we turn to the optimal asset allocations we obtained for our fixed-mortality base case from solving the Bellman equation under the short-selling restrictions. For each age and level of cash on hand, i.e. remaining wealth after consumption, Figure 1 displays how the individual will optimally spread its financial means over the three asset classes available, risky stocks (Panel B), risk-free bonds (Panel B), and deferred life annuities (Panel C), assuming that no deferred annuities were purchased before.

Figure 1 here

At young ages and with low cash on hand, the individual will be fully invested in stocks seeking the opportunity to cash in on the equity premium. With little wealth at stake and a long investment horizon remaining, there is little need for diversifying into less-risky assets. With rising cash on hand, however, the investor will hold an increasing fraction of wealth in risk-free assets, even at lower ages. Here, in an attempt to insure against risky labor income, young individuals will predominantly choose liquid bonds rather than deferred annuities. It is noteworthy, however, investors with very high levels of cash on hand will devote a measurable
amount of their means to purchasing deferred annuities, accepting to wait for another 45 years before receiving any payoffs from this investment.

With rising age and cash on hand, we see the typical life-cycle pattern of decreasing fractions of stock investments. As human capital depreciates, the investor is more willing to purchase bond-like financial assets, with deferred annuities becoming the predominant form of risk-free investment as the individual ages due to an increasing mortality credit. By the age of 60 deferred annuity purchases have completely crowded out bonds for any level of cash on hand. Except for very low levels of cash on hand, purchases of deferred annuities skyrocket at age 64, resulting in a sharp decline of stock holdings. This can be attributed to two facts. First, with deferred annuities beginning to pay benefits at age 65, the deferral period has shrunk to its minimum. More important, however, is the fact that age 64 provides the last chance for the individual to purchase any annuities. As can be seen in Panel C, annuity purchases from age 65 on are zero by definition and cash on hand has to be distributed among stocks and bonds only.

Hence, it comes as no surprise, that individuals, who haven’t purchased annuities until age 65 and thus won’t hold any in the future, will be significantly invested in bonds, resulting in the sharp increase in optimal bond holdings at age 65. Bond holdings increase with both, cash on hand as well as age, reflecting the decreasing investment horizon. Nevertheless, even individuals with very high wealth levels will until the end of the life-cycle invest about 30 percent of their cash on hand in stocks. For those with very low remaining financial wealth it is even optimal to be fully invested in equities.

3.1.2 Expected Life-cycle Profile

We now analyze the simulated distributions of the relevant choice and state variables by conducting an extensive Monte Carlo analysis based on the optimal feedback controls derived above. Drawing 50,000 independent stochastic scenarios, we compute the expected life-cycle profile for our stylized base case with risky labor income. Results are summarized in Figure 2, where Panel A shows the development of labor income, consumption, liquid savings, annuity purchases and annuity income and Panel B shows the expected allocation of wealth to equities, bonds and annuities over the individual’s life-span.

*Figure 2 here*

We first attend to Panel A of Figure 2. The expected trajectory of labor income exhibits the empirically observed hump-shape until retirement at age 65 when it is replaced by a risk-less pension of 68 percent of final labor income. Early in the life-cycle consumption falls short of labor income, helping to build up a significant level of liquid savings. This nest egg peaks at 55 when savings are about 6 times the average labor income. From age 40 on, the individual starts purchasing deferred annuities in expectation and continues to do so until retirement. Thus, the individual is initially willing to wait for 25 years before receiving the first pay-offs from the annuity investment. With annuity purchases gradually rising over the individual’s forties and fifties and continuously increasing consumption exceeding labor income from age 49, liquid savings gradually decrease until age 64, when the individual takes the final chance and shifts a significant fraction of liquid wealth into deferred annuities, which then commence to provide a second stable income from retirement age 65 on. Consumption continues to gradually increase during retirement, peaking at age 83 and marginally
decreasing thereafter, yet not falling short of consumption in the mid-fifties until the end of life. With no interest in bequeathing any financial means, this high level of retirement consumption is financed by continuously depleting liquid savings over the remaining life-cycle.

Turning to Panel B of Figure 2, we obtain the well-known result that the individual is fully invested in equities early in life, due to the low level of available savings, the mainly bond-like characteristics of human capital and expected excess returns on equity investments. With depreciating human capital, the individual’s appetite for bond-like investments increases. Thus, from the mid-thirties, the allocation to equities decreases and a small but increasing fraction of wealth is invested in liquid bonds. Although deferred annuities are calculated at the same rate of return, liquid bonds are preferred over annuities as the individual needs to insure against today’s risky labor income, while the annuities would only provide an additional stable income from retirement age onward. Only over time, the excess return on annuities resulting from an increasing mortality credit is sufficiently high to overcompensate the annuities’ illiquidity and deferral. With annuity purchases starting at age 40, the allocation to both annuities and bonds increases over the forties while the equity weight is slowly reduced. At age 45, equities still make up 92 percent of the portfolio, the allocation to bonds has increased to five percent and three percent of the wealth is invested in deferred annuities.

With about 6 percent, the allocation to bonds peaks at age 49. From that age on, the growing mortality credit, decreasing exposure to labor income risk, and the shrinking deferral period increase the appeal of annuities, which completely crowd out bonds until the late fifties. Until age 55, the annuity share has already increased to 26 percent, while the allocation to equities (bonds) had dropped to 71 (3) percent. Massive purchases of deferred annuities at the end of the working life turn this relation upside down and entering retirement at age 65, 78 percent of the individual’s wealth is invested in annuities with the remaining 22 percent remaining in equities. With liquid savings remaining virtually constant over the following decade, the fraction of overall wealth invested in equities slightly increases due to depreciating annuity wealth. Until age 75, the equity share rises to 26 percent, and at age 85 equities still account for 24 percent of overall assets. Only then does the allocation to equities begin to continuously decline until the end of the life-cycle.

3.1.3 Impact of Central Risk Parameters

Having discussed the results for our base case in the previous section, we now direct our focus to analyzing the impact of varying central risk parameters on expected life-cycle asset allocation. In particular, we will focus on alternative parameterizations of the individual’s risk aversion, the exogenous pension replacement rate, the level of labor income risk and the loadings charged by the annuity provider, varying only the respective parameter and otherwise maintaining the base case calibration. Results are presented in Table 2, which shows the age of initial purchases of deferred annuities and the allocation to equities, bonds, and deferred annuities as well as the level of liquid savings for various ages from 45 to 85. In addition to that, Figure 3 provides more detailed insights into the allocation to deferred annuities over time, comparing for each parameter dimension our base case and two alternative calibrations.

Table 2 here

Figure 3 here
In our base case, we assumed our investor to have CRRA utility with a risk aversion coefficient of $\rho = 5$. We now analyze investment decisions for an individual with lower ($\rho = 2$) and for one with higher ($\rho = 8$) risk aversion. As one would expect, the individual with low risk aversion has little appetite for either saving or risk-less investments, remaining fully invested in equities virtually until retirement. Deferred annuities are purchased only from age 63 onward and even then they only make up for an insignificant fraction of wealth; 0.8 percent at age 65, while our base case investor would already hold 78.2 percent of total wealth in annuities. With ongoing depletion of liquid savings, the allocation to annuities steadily increases, but even at age 85 a mere of 4.6 percent of total wealth is held in annuities. Only then does this fraction begin to dramatically increase until liquid savings are completely exhausted in the early nineties, resulting in an annuity fraction of 100 percent. By contrast, our highly risk averse individual will save significantly more, start purchasing deferred annuities already from age 28 on and invest liquid assets much more conservatively. At age 45, deferred annuities account for 39.7 percent of total wealth, while 48.2 (12.1) percent are being invested in stocks (bonds). At this time, the investor holds 9.3 times the initial labor income as liquid savings, only marginally less than the 10.6 times the (identical) initial labor income of our base case investor, who in turn only holds 3 percent of overall wealth in deferred annuities. At retirement age 65, our highly risk averse individual holds 93.4 percent of total wealth in annuities and the remaining liquid savings (1.7 times the initial labor income) are fully invested in equities (6.6 percent of total wealth). Even in retirement, this individual will initially continue saving, resulting in again increasing liquid assets – 2.8 (2.6) times the initial labor income at age 75 (85). Saving combined with depreciating annuity wealth leads to again increasing allocations to stocks and bonds – 21.1 percent and 0.2 percent at age 85 – before the allocation to annuities increases again due to depletion of liquid assets.

We now turn our attention to variations of the (exogenous) replacement rate, i.e. the fraction of pension payments from an external pension system with respect to final year labor income. In our base case, this replacement rate was $\zeta = 68$ percent. As it is widely believed that this figure will decrease in future, we analyze two alternative settings with lower replacement rates of 60 and 50 percent. Reducing the replacement rate (without compensating the individual, e.g. through higher labor income) results in marginally increased initial savings, earlier annuity purchases and more conservative investment behavior. While our base case investor will start purchasing deferred annuities from age 40, individuals with a replacement rate of 60 (50) percent will already invest in annuities at age 39 (38). At age 45, the allocation to annuities increases from 3 percent in the base case to 4.3 (6.4) percent with the medium (low) replacement rate. At the same time, the equity fraction drops from 92.1 percent to 90.5 (88.3) percent, while bonds make up for 5.1 (5.3) percent of total wealth compared to 4.9 percent in the base case. These patterns are maintained over the whole life-cycle. Annuity holdings increase from 78.2 percent to 80.6 (83.0) percent at retirement, while the allocation to equities drops from 21.8 to 19.4 (17.0) percent. At age 85, 78.5 (80.9) percent of total wealth is held in annuities compared to 76.2 percent in the base case, while the equity exposure is reduced from 23.6 to 21.3 (18.9) percent.

Next, we analyze the implications of different levels of labor income risk. In our base case, the volatility of the permanent labor income shock was $\sigma_n = 0.1$ and the transitory shocks had a volatility of $\sigma_u = 0.15$. Now, we look at an individual that does not face any labor income risk, i.e. $\sigma_u = \sigma_n = 0$, and an investor who faces very risky labor income with double the volatilities of the base case, i.e. $\sigma_u = 0.3$ and $\sigma_n = 0.2$. In case labor income has no volatility, the individual does not only have a predictable income during working life but also a foreseeable pension, as this is based on final salary. Thus, it comes as no surprise that the
need to insure sufficient pension income by early and more heavily investing in deferred
annuities is less developed than in a situation with labor income risk. As a result, the
individual will only purchase annuities from age 53, when the rising mortality credit of the
annuity investment is increasingly able to over-compensate its illiquidity. As in our base
case, the allocation to annuities continuously increases through the fifties and early sixties
and just before retirement, the investor will shift a significant fraction of liquid savings into
annuities, which reach a maximum share of 68.7 percent of total wealth at retirement age
65. Over the first decade of retirement, depreciating annuity wealth results in decreasing
portfolio shares (63.9 percent at age 75), as savings remain almost constant. With liquid
assets being depleted over the next decades, the allocation to annuities again increases
toward the end of the life-cycle. Having a secure labor income also influences the
allocation of liquid assets. While in our base case bonds play a certain role from the mid-
thirties to the mid-fifties, the individual will now be virtually fully invested in equities, as
there is no need to hedge risky labor income. The opposite is true for our individual that
faces high labor income risk. Already early in the life-cycle, the investor has a strong
appetite for risk-free investments, both liquid an illiquid, as human capital shows many
characteristics of equities rather than those of bonds. Annuities are being purchased from
age 24 and reach a share of 22.7 percent of total assets at age 45. At the same time, the
individual holds another 31.3 percent in bonds, while equities only account for 46 percent
of overall assets. As in all other cases, annuities crowd out bonds until retirement, when the
allocation to annuities reaches a temporary maximum of 82 percent, with the remaining 18
percent being invested in equities. With retirement, risky labor income is replaced by risk-
less pension income and investment patterns mimic those in the cases with less-risky labor
income. Equity shares first rise with depreciating annuity wealth (22.0 percent at age 75)
and later drop as liquid assets are continuously depleted.

Finally, we are interested in how the picture changes when we give up the (unrealistic)
assumption of cost-free deferred annuities. Instead we introduce expense loadings of 2.38%
and 7.2 percent, the former being representative for comparably cheap deferred annuities
within company pension plans, the latter representing market loadings of private annuity
providers. As in our base case, the individual will start purchasing deferred annuities at age 40
for both our alternative expense rates, and also at age 45, there are only negligible differences
in saving and investment patterns over the three alternative settings. While savings are
virtually unchanged, the allocation to deferred annuities marginally drops from 3.0 percent in
our base case to 2.9 (2.7) percent for low (high) loadings. Consequently, equity holdings are
slightly higher under high expense rates (92.3 percent) compared to the base case (92.1
percent). Bond holdings remain unchanged. The marginal impact of introducing loadings at
young ages results from the fact that deferred annuities, which only become due 25 years in
the future, are then comparably cheap. Adding a 7.2 percent levy on the annuity price thus has
a low absolute impact. As the individual ages, however, deferred annuities with loadings
become increasingly more expensive in absolute terms compared to the case where no
expenses are levied on them. Consequently, the individual is less inclined to purchase these
products. Thus, at retirement, individuals that face low (high) expense ratios only hold 74.6
(67.2) percent of their total wealth in annuities compared to 78.2 percent in the base case. In
turn, these individuals keep more of their liquid savings (5.2 (6.7) compared to 4.5 times the
initial labor income) and invest more heavily in equities (25.4 (32.8) compared to 21.8 percent
of total assets). This pattern is maintained over the rest of the life-cycle.

3.2 Deferred Annuities under Time-Varying Mortality
As discussed above, the assumption that mortality rates remain constant over the individual’s whole life-cycle is somewhat unrealistic. Not only does empirical evidence show clear trends in mortality over time, it is also apparent that mortality rates exhibit some extent of stochasticity. These factors may well affect the individual’s annuitization decisions, as both annuity prices over time as well as individual life expectancy will vary over time. Thus, in this section, we revisit our base case from section 3.1 and analyze the impact of time-varying mortality modeled as described in section 2.5. Again, our representative investor has CRRA utility with a risk aversion coefficient of $\rho = 5$, faces medium labor income risk, will retire at age 65 with an exogenously provided pension of 68 percent of the final salary, and is – until retirement – offered deferred annuities without loadings. Our findings are presented in Table 3 and Figure 4, both of which, for ease of comparison, also restate our base case results from section 3.1 where mortality is constant over time.

Table 3 here

Figure 4 here

First, looking at the case with stochastic but not trending mortality, we find that the individual will begin purchasing deferred annuities from age 37, three years earlier than in the constant mortality scenario. While earlier in the life-cycle the individual prefers liquid assets as an insurance against labor income fluctuations, it faces an increasing dispersion of possible mortality levels as time goes by. The investor will therefore prepone annuity purchases compared to the constant mortality case in an effort to avoid possibly higher annuity prices later in life. At age 45, annuities already make up for 6.3 percent of the investor’s portfolio, more than double the fraction our base case investor holds in annuities. At the same time, under stochastic mortality, liquid savings are higher by 0.9 times the initial labor income, indicating that the individual measurably reduces consumption in the late thirties and early forties to finance annuity purchases. As in our base case with constant mortality, liquid assets are primarily invested in equities (88.7 percent) while bonds only play a small role (5 percent). The annuity fraction steadily increases to about 50 percent in the early sixties, and just before retirement, the individual will shift a substantial fraction of liquid savings into annuities. Hence, at age 65, annuities make up for 82 percent of the investor’s portfolio, 4 percent more than in the fixed mortality scenario, while the investor still holds liquid savings of 4.2 times the initial labor income, about as much as in our base case. As in our constant mortality scenario, the remaining liquid assets are held in equities. Buying more annuities and earlier, results in expected annuity benefits that exceed those in the constant mortality scenario by about 17 percent.

We next turn to our world with deterministically trending mortality. Here mortality rates for all ages continuously drop by 1.42 percent per year. While in our pure stochastic set-up there is a significant chance of mortality rates remaining constant over some time, the investor in this scenario anticipates that mortality rates will decline even from the beginning of the life-cycle, driving up both prices for deferred annuities as well as the investor’s life expectancy. As annuities in any period are priced at the prevailing mortality rates, the investor has a strong incentive to annuitize early in life. At the same time, the individual again faces risky labor income, which cannot be hedged with annuities, as payments only commence at retirement. Thus, the investor has to postpone annuitization in an effort to build up a buffer stock of liquid assets that can assist in bolstering labor income shocks. Trading-off in expectation increasing annuity prices and over time declining labor income risk, the investor will annuitize earlier than in our constant mortality case as well as our pure stochasticity case. Annuity purchases
commence at age 35 and by age 45, deferred annuities make up for 10.4 percent of the individual’s portfolio. The annual increase in the annuity fraction, however, is lower than in our two previous cases, despite the fact that the value of previously purchased deferred annuities will increase with decreasing mortality, indicating that the individual is less inclined to purchase deferred annuities later in the working life as their price is increasing continuously. This becomes most apparent just before retirement, when last-order annuity purchases significantly fall short of those in both previous cases. At age 65, annuities only account for 64.1 percent of the individual’s portfolio, with the rest being held in equities. Liquid savings in turn amount to 8 times the initial labor income, almost twice the sum we found in our two earlier scenarios. The individual is aware of the fact that declining mortality rates require a bigger retirement nest-egg. Annuities, however, appear to be too expensive at that time. Hence, while the individual begins to purchase deferred annuities early, benefits fall short of those in the constant mortality case by about 24 percent and by about 35 percent of those in the purely stochastic scenario.

In our final scenario, we combine the two previously analyzed patterns of time-variability of mortality and assume that mortality rates are stochastic and additionally follow the deterministic, downward sloping trend. As we saw earlier in this section, superimposing stochasticity on deterministic (previously, deterministic meant constant) mortality behavior led to the individual being interested in buying more annuities and earlier. Not surprisingly, we derive a similar result now. When combining the deterministic trend with stochastic mortality development, we find that purchases of deferred annuities commence at age 34, one year earlier than in the pure deterministic trend scenario and asset allocation throughout the life-cycle is also tilted more toward annuities. At age 45, 13.4 percent of the individual’s wealth is invested in annuities, three percentage points more than in the pure deterministic trend scenario. Liquid savings at that time are up by one times the initial labor income, indicating that the individual reduced consumption early in the life-cycle even more than in the previous case. At retirement, the allocation to annuities exceeds that in the previous case by 4 percentage points, while liquid savings are virtually the same. Purchasing more annuities and earlier, results in annuity benefits exceeding those in the deterministic mortality trend case by 21 percent. Compared to the constant mortality scenario, however, annuity income still falls short by about 9 percent.

In conclusion, we find that negative mortality trends result in earlier but in total fewer annuity purchases, as annuity prices continuously increase. At the same time, introducing stochasticity with respect to future mortality rates will induce risk averse individuals to purchase more annuities and earlier, hence, insuring against the possibility of adverse annuity price developments as well as the risk of not having purchased enough annuities to maintain a high and smooth consumption stream should mortality rates decrease in retirement.

3.3 Deferred Annuities as Pure Longevity Insurance

So far, we assumed that deferred annuities provided benefits from retirement age 65 on. Alternatively, deferred annuities might be purchased to insure against the risk of real longevity, as for example put forward by Milevsky (2005). It is argued, that in a world were DB pension plans are increasingly being replaced by DC-type schemes, the reluctance to protect against longevity by annuitizing might be overcome by offering comparably cheap products that only offer longevity protection at very high ages, i.e. when individuals need it the
most. Along this line of thought, German Riester plans, tax-qualified supplementary private pension plans available since 2002, require mandatory annuitization from age 85 onward. Against this background, we in this section analyze the optimal life-cycle investment patterns in a world that provides deferred annuities that start paying benefits from age 85. In particular, we will again attend to the four scenarios from the previous section, and analyze the constant mortality scenario (i.e. the base case from section 3.1) as well as the scenarios with purely stochastic, deterministically trending, and stochastically trending mortality. Results are summarized in Table 4 and Figure 5.

**Table 4 here**

**Figure 5 here**

The individual facing constant mortality starts purchasing deferred annuities from age 42, accepting to wait for 43 years before receiving any payoffs from this investment. The allocation to deferred annuities slowly rises from 0.8 percent at age 45 to 12.3 percent at retirement age 65. During retirement, the fraction of total wealth invested in deferred annuities increases rapidly, reaching 30 percent at age 75, and by the age of 85, when the deferred annuities begin to pay benefits, the individual is fully invested in them. The exponential growth of the annuity fraction results, on one side, from continuously increasing annuity purchases, on the other side, with a shrinking deferral period, pension claims increase in value due to decreasing discount factors and increasing (conditional) survival probabilities. As deferred annuities will only provide payments from age 85 on, the individual faces a significant reduction of periodic income at retirement. To bolster this income shock and to bridge the gap until annuity payments will commence, the individual has to rely on its liquid savings. At retirement, liquid savings amount to 20 times the initial labor income at age 65, continue to rise to 20.5 times at age 70, and are then depleted until age 85. The allocation of liquid savings to equities and bonds exhibits the well-known life-cycle pattern. Equity holdings continuously drop from 92.9 percent at age 45 to 46.8 percent at age 75. While in section 3.1 the role of bonds was diminishingly small, they are of much more appeal in this set-up. The decreasing investment horizon, depreciating human capital, and the need to compensate the income shock at retirement with other sources of stable income drives the bond fraction up from 6.3 percent at age 45 to 26.4 percent at age 65. While the allocation to bonds remains quite stable over the subsequent decade (23.4 percent at age 75), bond holdings are then also depleted until age 85.

As we derived in the previous section, in a world with purely stochastic mortality, the individual will again buy more deferred annuities and earlier. Purchases commence at age 37 and when the deferred annuities begin paying benefits at age 85, the individual is again virtually fully invested in annuities. Prior to that, allocations to deferred annuities and bonds as well as liquid savings are slightly higher than in the constant mortality case. While the difference appears to be small in terms of overall portfolio weights, purchasing more annuities and earlier results in expected benefits that exceed those in the constant mortality case by about 21 percent.

In the deterministic mortality trend scenario, the individual will already begin purchasing deferred annuities at age 33, anticipating increasing future annuity prices and higher individual benefits due to trending mortality improvements. As these deferred annuities are significantly cheaper than those in section 3.2, the individual can annuitize earlier at lower cost and still build up sufficient liquid assets to insure against labor income risk. By the age of 45, deferred
annuities already account for 9.5 percent of total wealth. As in section 3.2, we find that while the fraction of wealth invested in deferred annuities in the stochastic mortality scenario is always above its constant mortality scenario counterpart before payments begin, the rate of increase in the annuity fraction is lower. From age 45 to 65, the annuity fraction only doubles to 20.3 percent, while it increased by factor 15 in the fixed mortality case. Allocation to annuities continues to rise until age 85 when annuities start to pay benefits and no further annuities may be purchased. With liquid savings comparable to those in the constant mortality case, increased annuity weights result from reductions in both equity and bond holdings. As in the previous section, increasing annuity prices due to decreasing mortality rates have a significant impact on the annuity benefits. While the allocation to annuities significantly exceeds that in the constant mortality case, annuity income at age 85 falls short by about 20 percent.

Finally, we again turn our attention toward the scenario that combines stochastic mortality with mortality trends and derive familiar results. Annuity purchases commence even earlier, at age 31. Liquid savings as well as allocations to annuities and bonds are up compared to the purely deterministic case, while allocations to equities are lower before the deferring age. Again, differences in the allocation to deferred annuities appear to be of minor importance. As we have seen before, however, this small deviation results in significant differences in annuity benefits, which are up by 19 percent compared to the pure deterministic trend case.

3.4 Welfare Implications

Our analyses so far have shown that in many of the scenarios discussed above the individual’s optimal portfolio includes a significant fraction of deferred annuities already early in life. This indicates that including those annuities in the investment menu will have welfare enhancing effects. To test this hypothesis, we in this section analyze the utility gains attributable to granting access to markets for deferred annuities for the various scenarios scrutinized above. For individuals aged 20 and 65, we determine the increase in utility attributable to the availability of deferred annuities by calculating the lump-sum payment required by individuals without access to annuities in order to have the same utility level in terms of the value function as in a world with annuities. Table 5 presents these lump-sum payments in percent of the respective average labor income.

Table 5 here

Our calculations suggest that access to deferred annuities generates measurable utility gains already for those aged 20 in most of our scenarios, with the low risk aversion and the no labor income risk cases being the exceptions. In a world without deferred annuities, our base case investor would require a lump-sum of 16.2 percent of the average labor income to have the same utility as in a world with annuities. As could be expected, this lump-sum rises with increasing risk aversion and labor income risk and with decreasing expense loadings and replacement rates from exogenous pensions. Utility gains significantly increase as the individual ages. At retirement, our base case investor already requires almost two times the average income to be indifferent between the worlds with and without annuities. The highly risk averse individual would require a lump-sum of 4.5 times the average income, while even at retirement an investor with low risk aversion would only ask for a negligible 1.4 percent.

Looking at our scenarios with time-varying mortality, we find that deferred annuities are
welfare enhancing particularly in those cases where mortality is stochastic, while their positive impact is sharply reduced by mortality trends. An investor aged 20 facing non-trending stochastic mortality will require a lump-sum of 27 percent of the average income, almost twice the amount required by his counterpart in a world with constant mortality. By contrast, if mortality is decreasing deterministically, our investor would abandon the option to invest in annuities for a mere 10 percent of the average labor income, which is only about 60 percent the compensation needed under constant mortality. In a world where mortality is stochastic and at the same time trending downward, these effects offset one another to some extent but welfare gains through annuitization still exceed those in the constant mortality scenario by about 30 percent. As before, utility gains significantly increase with the age of the investor. This especially holds for purely stochastic mortality, where an individual would require almost nine times the average income to be indifferent.

We finally turn to our cases where the deferring age is 85. Again, we find some welfare gains even for individuals aged 20. These are slightly lower than before when mortality is either constant or purely stochastic, while they are even marginally higher in case mortality exhibits a trend.

4 Conclusion
In a world that sees the relevance of predictable retirement income streams from state-organized social security systems and occupational defined benefit pension plans continuously diminishing, deferred life annuities offered in the private market may provide households with an adequate means to independently replicate these cash flow patterns and the inherent insurance against longevity risk. In this study, we scrutinize the role of deferred annuities in optimal portfolios of households facing un-insurable labor income risk, uncertain capital market returns, and stochastic mortality.

We set off by analyzing the relevance of annuities with a deferring age of 65, which is equal to the retirement age. Hence, they serve as a direct supplement to retirement income from the social security system. We show that investors with moderate risk aversion and some exposure to labor income risk will start purchasing deferred annuities from their early forties and will then gradually shift funds from liquid savings into annuities until their portfolio share reaches about 80 percent at retirement. With increasing risk aversion, higher labor income risk, lower replacement rates from state pensions, and stochastic mortality rates, annuity purchases commence even earlier and the fraction of total financial wealth invested in annuities increases. Consistent with findings in other studies, early in the life-cycle liquid savings are fully invested in equities and then gradually shifted into annuities. Liquid bonds in most cases only play a minor role and are quickly crowded-out by annuities.

We also find that households use deferred annuities to actively hedge against longevity risk. Even when payouts only commence at the very advanced age of 85, the investor slowly begins purchasing deferred annuities at age 42. In case the household faces to stochastic and trending life expectancy, investments in deferred annuities already begin at age 31 and the allocation to annuities is significantly higher compared to a constant mortality scenario.

Consistent with prior studies on immediate annuities, we find substantial welfare gains from including deferred annuities in the investment menu. In a world with stochastic and trending mortality, even an investor aged 20 is willing to give up about one quarter of average labor
income to have access to deferred life annuities with payouts starting at age 85. Consequently, due to their comparably low price, deferred annuities might be a good instrument to overcome the reluctance to annuitize.
References


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Notes: Authors' calculations. Implied longevity yields in BP: excess return over discount rate generated by buying deferred annuities at various ages compared to postponing annuitization until deferring age. Real discount rate: 2%. Mortality assumption: 2000 US Annuitant Basic with $q_{100} := 1.$
Table 2: Impact of Central Risk Parameters

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<td>4.7</td>
<td>2.8</td>
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<td>3.9</td>
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<td>8.6</td>
<td>4.8</td>
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<td>2.2</td>
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### Table 3: Impact of Time-Varying Mortality

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<th>Asset Allocation</th>
<th>Age 1st Ann. Purchase</th>
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#### Constant Mortality
- Equities (%) 92.1 71.2 21.8 26.2 23.6
- Bonds (%) 4.9 3.3 0.0 0.0 0.2
- Annuities (%) 3.0 25.5 78.2 73.8 76.2
- Savings 10.6 12.8 4.5 4.2 2.3

#### Stochastic Mortality w/o Trend
- Equities (%) 88.7 66.3 18.0 24.6 24.4
- Bonds (%) 5.0 3.9 0.0 0.0 0.3
- Annuities (%) 6.3 29.8 82.0 75.4 75.2
- Savings 11.5 13.7 4.2 4.6 3.1

#### Deterministic Mortality Trend
- Equities (%) 88.8 67.2 35.9 39.2 41.0
- Bonds (%) 0.8 1.3 0.0 0.8 2.6
- Annuities (%) 10.4 31.6 64.1 60.0 56.3
- Savings 9.9 12.2 8.0 8.0 6.6

#### Stochastic Mortality with Trend
- Equities (%) 85.4 63.8 32.0 37.1 40.2
- Bonds (%) 1.2 1.7 0.0 1.1 3.2
- Annuities (%) 13.4 34.5 68.0 61.8 56.6
- Savings 10.8 13.1 7.9 8.6 7.7

Notes: Authors’ calculation. Constant Mortality (equal to Base Case in section 3.1): Retirement Age and Deferring Age: 65. Replacement Rate: 68%. Risk Aversion: \( \rho = 5 \). Med. Labor Income Risk: Vola of Transitory Income Shock = 15%. Vola of Permanent Income Shock = 10%. Stochastic Mortality (for both. annuitant’s & individual mortality) follows a bounded, symmetric 9-level trinomial tree with \( p_u = p_d = 0.1 \) and jump size equal to 5.77 percent of the original morality rate (other parameters equal to Constant Mortality case). Trend: mortality rates decline geometrically by 1.42 percent per year (other parameters equal to stochastic mortality case). Asset weights in percent of total wealth (liquid savings + present value of annuity claims); Savings as a multiple of initial labor income.
Table 4: Asset Allocation with Deferring Age 85

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<td>Bonds (%)</td>
<td>6.3</td>
<td>18.7 26.4 23.4 0.0</td>
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<tr>
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<td>Equities (%)</td>
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<tr>
<td>Bonds (%)</td>
<td>7.6</td>
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</tr>
<tr>
<td>Annuities (%)</td>
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<td>7.8 13.6 30.7 99.8</td>
</tr>
<tr>
<td>Savings</td>
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<td>18.5 22.2 22.7 0.1</td>
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<td><strong>Deterministic Mortality Trend</strong></td>
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<tr>
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<tr>
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<tr>
<td>Savings</td>
<td>11.4</td>
<td>17.5 21.6 22.1 1.7</td>
</tr>
</tbody>
</table>

Notes: Authors’ calculation. Constant Mortality (equal to Base Case in section 3.1): Retirement Age: 65. Deferring Age: 85. Replacement Rate: 68%. Risk Aversion: $\rho = 5$. Med. Labor Income Risk: Vola of Transitory Income Shock = 15%. Vola of Permanent Income Shock = 10%. Stochastic Mortality (for both. annuitant’s & individual mortality) follows a bounded, symmetric 9-level trinomial tree with $p_u=p_d=0.1$ and jump size equal to 5.77 percent of the original morality rate (other parameters equal to Constant Mortality case). Trend: mortality rates decline geometrically by 1.42 percent per year (other parameters equal to stochastic mortality case). Asset weights in percent of total wealth (liquid savings + present value of annuity claims); Savings as a multiple of initial labor income.
Table 5: Welfare Analysis

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<th>Welfare Gains</th>
<th>(in % of Avg. Labor Income)</th>
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<tr>
<td>Base Case</td>
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<tr>
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<td><strong>Time-Varying Mortality</strong></td>
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<tr>
<td>Stochastic Mortality w/o Trend</td>
<td>27.0</td>
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<td>Deterministic Mortality Trend</td>
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<td>Stochastic Mortality with Trend</td>
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<td><strong>Deferring Age 85</strong></td>
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<td>Constant Mortality</td>
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<td>Stochastic Mortality w/o Trend</td>
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<td>Deterministic Mortality Trend</td>
<td>11.5</td>
</tr>
<tr>
<td>Stochastic Mortality with Trend</td>
<td>23.0</td>
</tr>
</tbody>
</table>

Figure 2: Expected Optimal Life-Cycle Profile and Asset Allocation: Base Case

A) Expected Life-cycle Profile

B) Expected Asset Allocation

Figure 3: Impacts on Allocation to Deferred Annuities over the Life-Cycle

Relative Risk Aversion

Replacement Rate

Labor Income Risk

Annuity Loadings

Figure 4: Impact of Time-Varying Mortality on Annuity Allocation

Notes: Authors’ calculation. Constant Mortality (equal to Base Case in section 3.1): Retirement Age and Deferring Age: 65. Replacement Rate: 68%. Risk Aversion: \( \rho = 5 \). Med. Labor Income Risk: Vola of Transitory Income Shock = 15%. Vola of Permanent Income Shock = 10%. Stochastic Mortality (for both annuitant’s & individual mortality) follows a bounded, symmetric 9-level trinomial tree with \( p_u = p_d = 0.1 \) and jump size equal to 5.77 percent of the original morality rate (other parameters equal to Constant Mortality case). Trend: mortality rates decline geometrically by 1.42 percent per year (other parameters equal to stochastic mortality case). Asset weights in percent of total wealth (liquid savings + present value of annuity claims).
Figure 5: Annuity Allocation with Deferring Age 85

Notes: Authors' calculation. Constant Mortality (equal to Base Case in section 3.1): Retirement Age: 65. Deferring Age: 85. Replacement Rate: 68%. Risk Aversion: \( \rho = 5 \). Med. Labor Income Risk: Vola of Transitory Income Shock = 15%. Vola of Permanent Income Shock = 10%. Stochastic Mortality (for both. annuitant’s & individual mortality) follows a bounded, symmetric 9-level trinomial tree with \( p_u = p_d = 0.1 \) and jump size equal to 5.77 percent of the original mortality rate (other parameters equal to Constant Mortality case). Trend: mortality rates decline geometrically by 1.42 percent per year (other parameters equal to stochastic mortality case). Asset weights in percent of total wealth (liquid savings + present value of annuity claims).
Notes

i See for instance Blake, Cairns, and Dowd (2003), Horneff, Maurer, and Stamos (2008a, b), Horneff, Maurer, Mitchell, and Dus (2008), Kojien, Nijman, and Werker (2006), Milevsky and Young (2007), Milevsky, Moore, and Young (2006), and Yogo (2008).

ii Other work about deferred annuities includes Scott, Watson, and Hu (2008), who recommend purchasing deferred annuities in case the extent to which annuities can be bought is limited. Gong and Webb (2009) evaluate deferred annuities using the “annuity equivalent wealth concept” introduced by Mitchell et al. (1999).

iii This can be justified that the household put only that part of financial wealth into the annuity which is not intended for a bequest, i.e. we only consider that part of wealth which is required for consumption purchases. Also see Stamos (2008) on this point.