# FORECASTING MORTALITY OF SMALL POPULATIONS BY MIXING MORTALITY DATA


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# 1 INTRODUCTION

Our focus: the robustness of the forecasted mortality when dealing with small populations

In the case of small sample size, data are very volatile and yield unreliable estimates

This problem emerges even if large populations are referred to, when addressing the oldest ages

Our proposal:

- ▷ not a specific new projection methdod
- ▷ "replicate" the mortality of the small population by mixing appropriately mortality data from "neighboring" populations

The mortality forecast (which can be performed by adopting any established projection method) is then build on a larger sample
$\Rightarrow$ estimates should be less volatile and (asymptotically) more robust

# 2 MOTIVATION

Two main needs:

- Assess the basis risk when hedging longevity risk through longevity-linked securities

  Longevity securities are contingent on the mortality of a large population (reference population), while used to hedge the risk in a small population (i.e. a life annuity portfolio, or a pension fund)

- Improve projection accuracy

  A broader topic, regarding national populations, pension funds, life annuity portfolios, etc.

# 3 LITERATURE

The problem of projecting mortality for small populations is receiving increasing attention in recent demographic and actuarial literature

***Possible target***

Projection of the mortality of:

- ▷ an insurance portfolio, or a pension fund in respect of the general population

  See, for example:

  Li and Hardy [2009], Plat [2009]

  as regards hedging and basis risk

- ▷ the population of a specific geographical area in respect of the general population of a country
- ▷ the population of a small country, either with a well established tradition in mortality reporting or not

### *Approaches*

One dominant population from which estimating the long-term trend and a small population experiencing short-term deviations from such a trend

See:

Jarner and Kryger [2011]

Gravity model in which two populations are attracted to each other

See:

Dowd et al. [2011]

Two-population mortality model in a Bayesian framework, with a single-step procedure for the parameter estimate

See:

Cairns et al. [2011]

Multi-population projection model, assuming that the various populations share a common trend

See:

Börger and Aleksic [2011]

Mortality correlations among countries $\Rightarrow$ analysis of mortality-linked securities

See:

Lin et al. [2012]

Extensions to Lee-Carter

See:

Li and Lee [2005], Russolillo et al. [2010], Debón et al. [2011]

# 4 THE BASIC IDEA

Mortality data from $n + 1$ countries:

- country $0$, which has a small population $\Rightarrow$ possible severe random fluctuations
- country $k$, $k = 1, 2, \ldots, n$ with large / medium / small population

We propose (and investigate) three possible approaches to the replication of mortality data, namely via:

(1) probability distribution of the lifetime - method ($f$)

(2) mortality profile - method ($m$)

(3) mortality improvements (in absolute terms) - method ($\Delta m$)

(2) and (3) are “distribution-free” methods

# 5 THE REPLICATION OF MORTALITY DATA

***Replicating the probability distribution of the lifetime: method ($f$)***

Let $f_{x,t}^{[k]}(u)$ = probability density function (pdf) of the lifetime $T_x$ of an individual age $x$, in country $k$, calendar year $t$

Conditional on the mortality trend, lifetimes are assumed to be independent and identically distributed

Let

$$f_{x,t}^{[\mathrm{AVE}]}(u) = \sum_{k=1}^{n} w_k \, f_{x,t}^{[k]}(u)$$

be the average pdf of countries $1, \dots, n$

Weights $w_1, \dots, w_n$, with $w_k \geq 0$ for $k = 1, \dots, n$, $\sum_{k=1}^{n} w_k = 1$

Aim: to select the weights $w_k$'s so that the distance between $f_{x,t}^{[\mathrm{AVE}]}(u)$ and $f_{x,t}^{[0]}(u)$ is minimum $\Rightarrow$ $f_{x,t}^{[\mathrm{AVE}]}(u)$ can effectively replace $f_{x,t}^{[0]}(u)$

Note: in $f_{x,t}^{[\mathrm{AVE}]}(u)$ we do not include the mortality of country $0$, to avoid the trivial solution $w_0 = 1$, $w_1, \ldots, w_n = 0$

Distance between $f_{x,t}^{[0]}(u)$ and $f_{x,t}^{[\mathrm{AVE}]}(u)$ assessed through the Kullback-Leibler divergence measure (KLD)

See:

Edwards and Tuljapurkar [2005]

In general terms, given two distributions with (strictly positive) pdf $f_1(u)$ and $f_2(u)$, the KLD measure of divergence of $f_2(u)$ from the baseline $f_1(u)$ is defined as follows:

$$KLD\big(f_1(u), f_2(u)\big) = \int_{-\infty}^{+\infty} f_1(u)\, \ln\left(\frac{f_1(u)}{f_2(u)}\right) \mathrm{d}u$$

If $f_1(u)$ and $f_2(u)$ are normal densities, the KLD measure can be expressed as follows:

$$KLD\big(f_1(u), f_2(u)\big) = \ln\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{\sigma_1^2}{2\sigma_2^2} + \frac{(\mu_1 - \mu_2)^2}{2\sigma_2^2}$$

Larger $KLD \;\Leftrightarrow\;$ stronger dissimilarity

Minimizing $KLD \;\Leftrightarrow\;$ optimizing fit of a model to real data

Normality assumption $\;\Rightarrow\;$ disregarding infant mortality and young-mortality hump $\;\Rightarrow\; x'$ truncation age (say, $x' = 30$)

Let

$$KLD_t = KLD\big(f_{x',t}^{[0]}(u), f_{x',t}^{[\mathrm{AVE}]}(u)\big)$$

Then, accepting normality:

$$KLD_t = \ln\left(\frac{\sigma^{[\mathrm{AVE}]}}{\sigma^{[0]}}\right) + \frac{\sigma^{2[0]}}{2\sigma^{2[\mathrm{AVE}]}} + \frac{(\mu^{[0]} - \mu^{[\mathrm{AVE}]})^2}{2\sigma^{2[\mathrm{AVE}]}}$$

where:

$$\mu^{[k]} = \mathbb{E}^{[k]}[T_{x'}], \quad \sigma^{2[k]} = \mathbb{V}\mathrm{ar}^{[k]}[T_{x'}], \quad k = 0, 1, \ldots, n$$

$$\mu^{[\mathrm{AVE}]} = \sum_{k=1}^{n} w_k \, \mu^{[k]}, \quad \sigma^{2[\mathrm{AVE}]} = \sum_{k=1}^{n} w_k \, \sigma^{2[k]}$$

To determine the weights:

$$\begin{array}{ll} \min_{\{w_k\}} & \sum_{t=t_0}^{t_1} (KLD_t)^2 \\ \text{s.t.} & w_k \geq 0, \quad k = 1, \ldots, n \\ & \sum_{k=1}^{n} w_k = 1 \end{array}$$

Let $\{w_k^{(f)}\}$ = optimal set of weights

Replacing $f_{x',t}^{[0]}(u)$ with $f_{x',t}^{[\mathrm{AVE}]}(u)$ $\Rightarrow$ we would disregard completely the information about country $0$

Therefore 2nd step $\Rightarrow$ to define the replicating mortality $f_{x',t}^{[\mathrm{R0}]}(u)$

$$f_{x',t}^{[\mathrm{R0}]}(u) = f_{x',t}^{[0]}(u)\, z_{x',t}^{(f)} + f_{x',t}^{[\mathrm{AVE}]}(u)\,(1 - z_{x',t}^{(f)})$$

where $z_{x',t}^{(f)}$ = "credibility" of the mortality dataset of country $0$

We define:

$$z_{x',t}^{(f)} = \frac{E_{x',t}^{[0]}}{E_{x',t}^{[0]} + \sum_{k=1}^{n} w_k^{(f)}\, E_{x',t}^{[k]}} \qquad (*)$$

where $E_{x,t}^{[k]}$ = number of exposed to risk in country $k$, age $x$, calendar year $t$

Final step: replace $f_{x',t}^{[0]}(u)$ with $f_{x',t}^{[\mathrm{R0}]}(u)$, and project mortality through a standard projection model (e.g. the Lee-Carter model)

### ***Other methods***

Same logical structure, relying on weights $\{w_k\}$, calculated via optimization problem

Simpler definition of "distance"

### ***Replicating mortality profile: method ($m$)***

Mortality profile in terms of central death rates $m_{x,t}^{[k]}$

Then:

$$m_{x,t}^{[\mathrm{AVE}]} = \sum_{k=1}^{n} w_k \, m_{x,t}^{[k]}$$

Optimization:

$$\begin{array}{ll} \min_{\{w_k\}} & \sum_{x=x_0}^{x_1} \sum_{t=t_0}^{t_1} (m_{x,t}^{[0]} - m_{x,t}^{[\mathrm{AVE}]})^2 \\ \text{s.t.} & w_k \geq 0, \quad k = 1, \ldots, n \\ & \sum_{k=1}^{n} w_k = 1 \end{array}$$

Let $\{w_k^{(m)}\}$ = optimal set of weights

Then:

$$m_{x,t}^{[\mathrm{R0}]} = m_{x,t}^{[0]}\, z_{x,t}^{(m)} + m_{x,t}^{[\mathrm{AVE}]}\, (1 - z_{x,t}^{(m)})$$

with $z_{x,t}^{(m)}$ calculated as in Eq. $(^*)$, but now for all relevant ages $x$ and replacing weights $\{w_k^{(f)}\}$ with weights $\{w_k^{(m)}\}$

Finally: replace $m_{x,t}^{[0]}$ with $m_{x,t}^{[\mathrm{R0}]}$

***Replicating the (absolute) mortality improvements: method (*$\Delta m$*)***

Mortality improvements:

$$\Delta m_{x,t}^{[k]} = m_{x,t}^{[k]} - m_{x,t+1}^{[k]}$$

Then:

$$\Delta m_{x,t}^{[\mathrm{AVE}]} = \sum_{k=1}^{n} w_k \, \Delta m_{x,t}^{[k]}$$

Optimization:

$$\begin{array}{ll} \min_{\{w_k\}} & \sum_{x=x_0}^{x_1} \sum_{t=t_0}^{t_1} (\Delta m_{x,t}^{[0]} - \Delta m_{x,t}^{[\mathrm{AVE}]})^2 \\ \text{s.t.} & w_k \geq 0, \quad k = 1, \ldots, n \\ & \sum_{k=1}^{n} w_k = 1 \end{array}$$

Let $\{w_k^{(\Delta m)}\}$ = optimal set of weights. Then:

$$\Delta m_{x,t}^{[\mathrm{R0}]} = \Delta m_{x,t}^{[0]} \, z_{x,t}^{(\Delta m)} + \Delta m_{x,t}^{[\mathrm{AVE}]} \, (1 - z_{x,t}^{(\Delta m)})$$

with $z_{x,t}^{(\Delta m)}$ calculated as in Eq. $(^*)$, but now for all relevant ages $x$ and replacing weights $\{w_k^{(f)}\}$ with weights $\{w_k^{(\Delta m)}\}$

Finally: replace $\Delta m_{x,t}^{[0]}$ with $\Delta m_{x,t}^{[\mathrm{R0}]}$

# 6 SOME RESULTS

Our main ultimate purpose: projecting mortality for pension and life annuity business

Time interval $[1975, 2008]$

- $[t_0, t_1] = [1975, 1995] \;\Rightarrow$ model calibration
- $[1996, 2008] \;\Rightarrow$ backtesting projection

Ages - calibration

- $x' \geq 30$ for method ($f$)
- $[x_0, x_1] = [60, 80]$ for methods ($m$) and ($\Delta m$)

Ages - projection

- $[60, 100]$

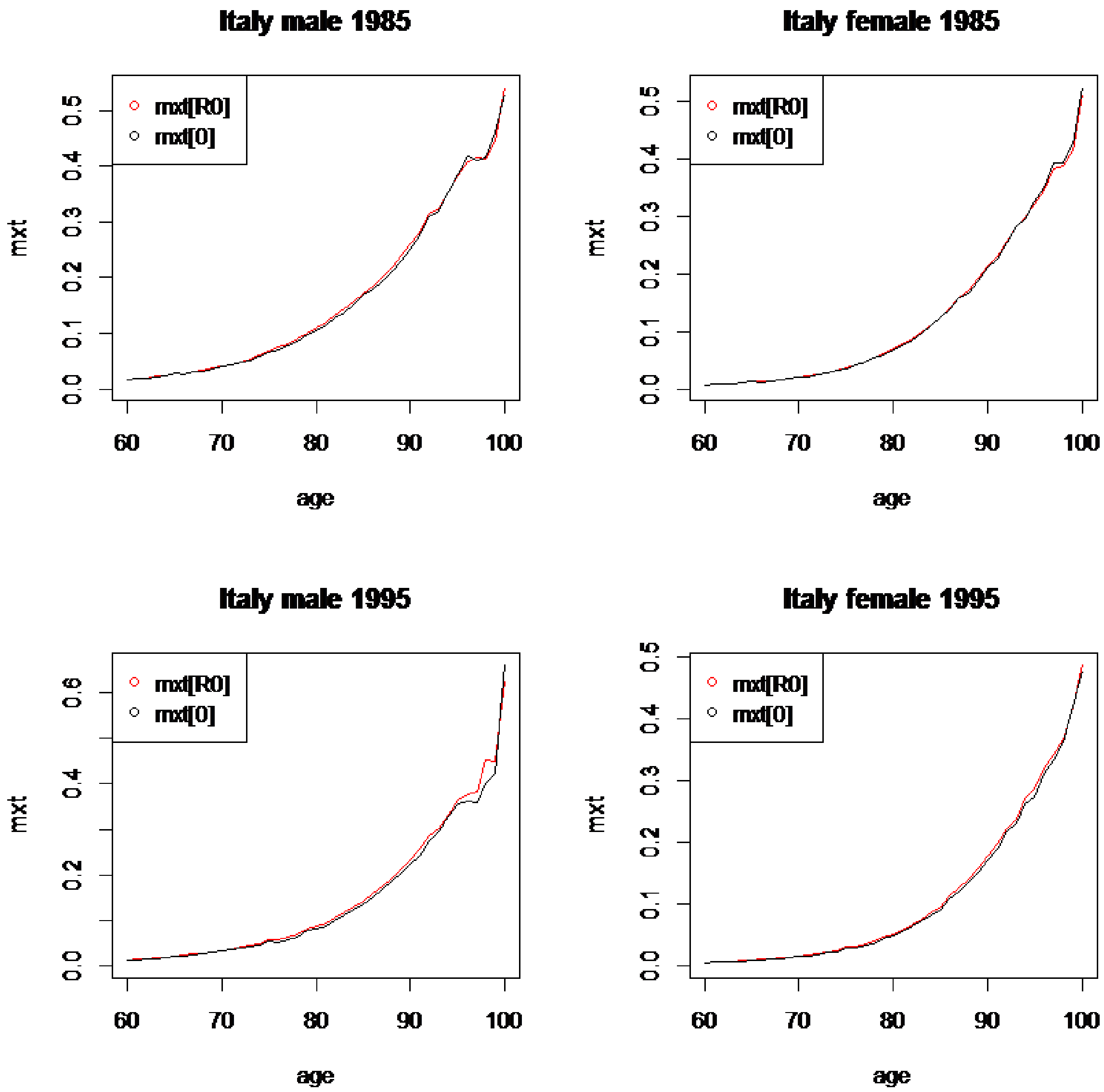
Mortality data taken from HMD, except for Slovenia (data from the Statistical Office of the Republic of Slovenia)

# Some results *(cont'd)*

| Code | Country | Code | Country | Code | Country | Code | Country |
|---|---|---|---|---|---|---|---|
| AT | Austria | FI | Finland | LV | Latvia | PT | Portugal |
| BE | Belgium | FR | France | LT | Lithuania | SCO | Scotland |
| BG | Bulgaria | DE | Germany | LU | Luxembourg | SK | Slovakia |
| CZ | Czech Republic | HU | Hungary | NL | Netherlands | SI | Slovenia |
| DK | Denmark | IS | Iceland | NIE | Northern Ireland | ES | Spain |
| ENG | England & Wales | IE | Ireland | NO | Norway | SE | Sweden |
| EE | Estonia | IT | Italy | PL | Poland | CH | Switzerland |

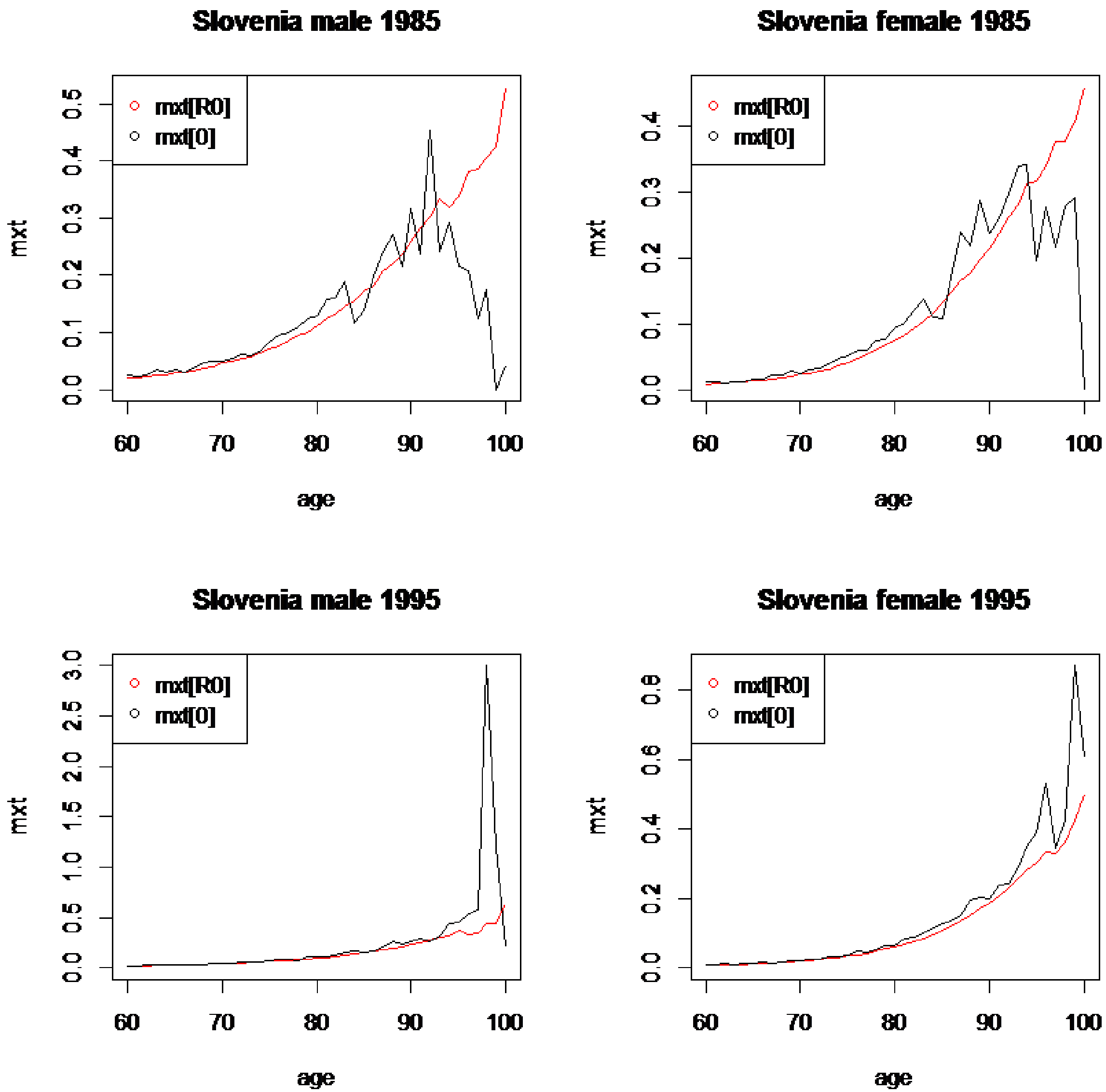
*Basket of countries*

## Some results *(cont’d)*




*Original and recalibrated data for Italy; method* ($\Delta m$)

# Some results *(cont'd)*




*Original and recalibrated data for Slovenia; method* ($\Delta m$)

To check the goodness of mortality replication: compare the mean square error ($\mathrm{MSE}$) between projected and observed mortality data in the case the projection is based on the original data ($\mathrm{MSE}^{[0]}$) or on the replicated data ($\mathrm{MSE}^{[\mathrm{R}0]}$) respectively

Lower values of $\dfrac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}}$ $\Rightarrow$ more satisfactory results

| | Method $(f)$ | | | Method $(m)$ | | | Method $(\Delta m)$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | Males | Females | All | Males | Females | All | Males | Females | All |
| Average value of $\frac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}}$ | $124.24\%$ | $108.83\%$ | $116.53\%$ | $67.90\%$ | $78.13\%$ | $73.02\%$ | $60.89\%$ | $68.50\%$ | $64.69\%$ |
| # of cases | 28 | 28 | 56 | 28 | 28 | 56 | 28 | 28 | 56 |
| Average value of $\frac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}}$, for the cases with $\frac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}} < 1$ | $63.41\%$ | $66.46\%$ | $64.90\%$ | $58.91\%$ | $60.62\%$ | $59.67\%$ | $52.28\%$ | $56.12\%$ | $54.12\%$ |
| # of cases with $\frac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}} < 1$ | 21 | 20 | 41 | 25 | 20 | 45 | 25 | 23 | 48 |
| Average value of $\frac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}}$, for the cases with $\frac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}} > 1$ | $306.73\%$ | $214.75\%$ | $257.67\%$ | $142.85\%$ | $121.93\%$ | $127.64\%$ | $132.65\%$ | $125.43\%$ | $128.14\%$ |
| # of cases with $\frac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}} > 1$ | 7 | 8 | 15 | 3 | 8 | 11 | 3 | 5 | 8 |

*Average values of the ratio* $\dfrac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}}$

# Some results *(cont'd)*

| | Country $0$: Slovenia | | | | | | Country $0$: Spain | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $w_k^{(f)}$ | | $w_k^{(m)}$ | | $w_k^{(\Delta m)}$ | | $w_k^{(f)}$ | | $w_k^{(m)}$ | | $w_k^{(\Delta m)}$ | |
| | Males | Females | Males | Females | Males | Females | Males | Females | Males | Females | Males | Females |
| $\mathrm{MSE}^{[0]}$ | $0.0543$ | $0.0240$ | $0.0543$ | $0.0240$ | $0.0543$ | $0.0240$ | $0.0002$ | $0.0001$ | $0.0002$ | $0.0001$ | $0.0002$ | $0.0001$ |
| $\mathrm{MSE}^{[\mathrm{R}0]}$ | $0.0266$ | $0.0011$ | $0.0092$ | $0.0012$ | $0.0085$ | $0.0012$ | $0.0002$ | $0.0001$ | $0.0001$ | $0.0001$ | $0.0001$ | $0.0001$ |
| $\frac{\mathrm{MSE}^{[\mathrm{R}0]}}{\mathrm{MSE}^{[0]}}$ | $49.00\%$ | $4.50\%$ | $16.94\%$ | $5.04\%$ | $15.70\%$ | $5.03\%$ | $89.12\%$ | $97.18\%$ | $78.75\%$ | $113.51\%$ | $74.11\%$ | $66.00\%$ |

*Backtesting for Slovenia and Spain*

# 7 CONCLUDING REMARKS

Current results are encouraging, but further investigations are appropriate; in particular:

- ▷ sensitivity analyses to check dependence on the reference range of ages and years
- ▷ selection of the basket of countries (possibly through some constraints in the minimization problem, based on similarity measures)

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