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Pension Options Valuation and Hedging Bounds

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Key Questions

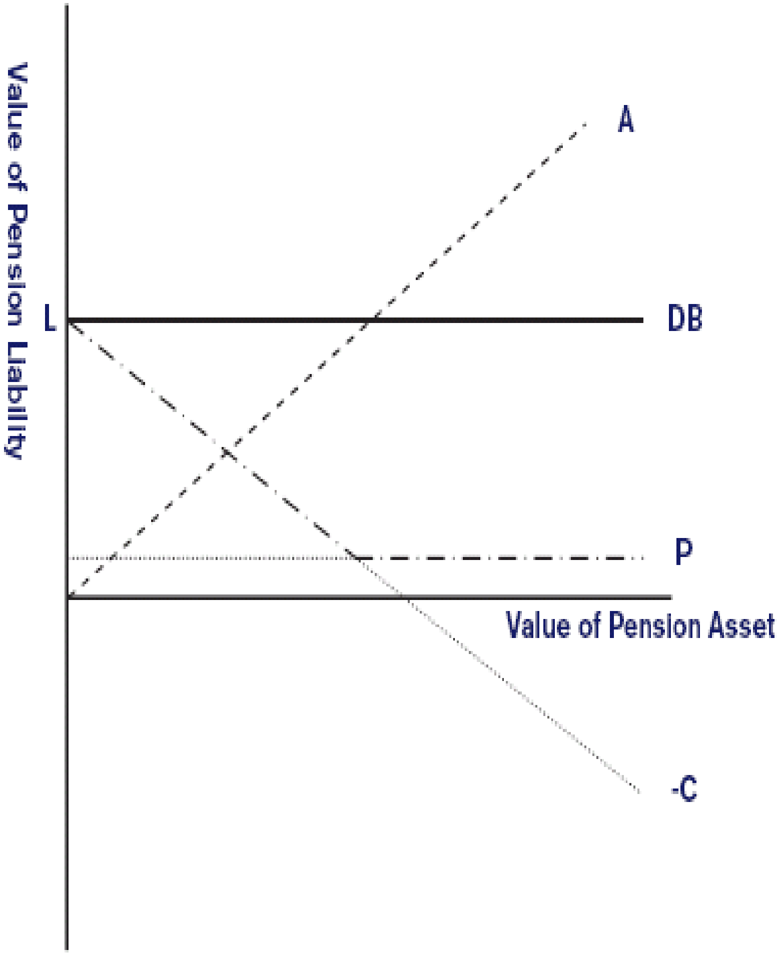
- Why unfunded pension liability is option-like debt?
- How to value "pension options" in incomplete markets?
  - Black Scholes model
  - Margrabe exchange model
  - Stulz rainbow model
- What are the financial implications of pricing bounds?
- What are the asset allocation implications of incomplete hedging bounds?

Pension liability valuation

- Sharpe (1976) proposed modelling pension liabilities as a long put (P) and a short call (C) on pension assets (PA) with the same exercise price (PL):

PV (PL) = PV (PA) + PV (P) – PV (C)

where
PV(..): present value
P: put options written by the companies and held by the scheme members
C: call options written by the members and held by the companies



Source: Sharpe W (1976), "Corporate Pension Funding Policy"

Further Research

- Subsequent papers such as Willinger (1985,1992), Steekamp (1999), Wilkie (1989) extend the analysis on pension options:
  - If the future liabilities are known and fixed, use Black-Scholes model
  - If the future liabilities are unknown and risky, use Margrabe exchange option model
  - If the pension and corporation are closely integrated, use Stulz rainbow option model,
- However, all models assume the market condition is complete.

Why market isn't complete?

- However, hypotheses for complete market are problematic in practices:
  - continuous vs. discrete trading
  - price takers vs. price makers
  - frictionless vs. transaction costs
  - constant vs. stochastic covariance
  - random walk vs. mean reversion process
- And the absence of financial market instruments, that closely replicate indexed pension liabilities, imposes further risks.
- Therefore, in an incomplete market, instead of a single arbitrage-free price, there exists an arbitrage-free interval $[h_{low}, h_{up}]$

Valuation in Incomplete Market

- Method one: super replicating portfolios
  - find portfolio whose payoffs are always "≥" the payoff of the derivative, *but unhedgable wage growth is unbounded*
- Method two: utility indifference valuation
  - the wealth invested in market can provide the same utility as pension options, see De Jong (2007)
- Method three: no arbitrage arguments
  - find arbitrage strategies bounded Sharpe's ratio, see Cochrane & Saa-Requejo (2000)
  - find arbitrage strategies using first/second stochastic dominance, see Rodriguez (2003)

Description of the Problem

- Definitions:
  - *C* and *P* are the current price of a European call and put option respectively on the same underlying asset
  - Same exercise price *X* and time to maturity *T*.
  - Current asset price is *S* and its terminal price when the options expire is S_T, with cumulative distribution function $F(S_T)$.
  - The risk-free rate is r_f , and the required rate of return on the underlying asset, call, and the put are r_S, r_C, and r_P respectively.
- The expected value of European call, C_T is

$$E(C_T) = \int_X^{\infty} S_T \, dF(S_T) - X[1 - F(X)]$$

$$E(C_T) = \int_0^{\infty} S_T \, dF(S_T) - \int_0^{X} S_T \, dF(S_T) - X[1 - F(X)]$$

Description of the Problem

- where $\int_0^\infty S_T dF(S_T)$ is the expected underlying asset price at the option's expiration, so

$$E(C_T) = Se^{r_S T} - X + \int_0^X F(S_T) d(S_T)$$

- To find the current price of a call options *E_t {C(S)},* each component in the right hand side of equation should be discounted at a rate commensurate with its risk.
  - The first term in the right hand side is the expected value of the underlying asset's terminal price, which should be discounted at its required rate, r_S.
  - The second term is the exercise price and because it is known with certainty, it must be discounted at the risk free rate, r_f.
  - And the integral term in equation is the expected value of the put option at maturity which should be discounted at the put's appropriate rate, r_P

Description of the Problem

- Hence, the problem is about to find upper and lower bounds for appropriate discount rate for put options ?

$$\mathrm{R}_P(S) = \frac{C(S) - S + XR_f}{I(S)} = V(S) + \frac{C(S)}{I(S)}$$

where

- $$V(S) = \frac{XR_f - S}{I(S)}$$
- Denote integral part as I(S)
- $R_i \equiv \exp(-r_i T)$ for i = f, S, C, P represent the discount factors

Solution of the Problem

- Merton's (1973) well-known lower bound: $L_M(S) = \max[0, S - XR_f]$
- Perrakis and Ryan (1984) lower bound: $L_{PR}(S) = \max[0, S - XR_f + R_f I(S)]$
- Levy (1985) and Ritchen (1985) lower bound:

LL/R(S) = 0 where 0≤S<Smax

= S- XR_f +V (Smax) I(S) where S≥ Smax

- Rodriguez (2003) lower bound:

$$L_R(S) = \max[0, S - XR_f + \{V(S_i) + V'(S_i)(S - S_i)\} I(S)]$$

where V'(Si) is the slope of inflexion line

- Lo (1987) upper bound:

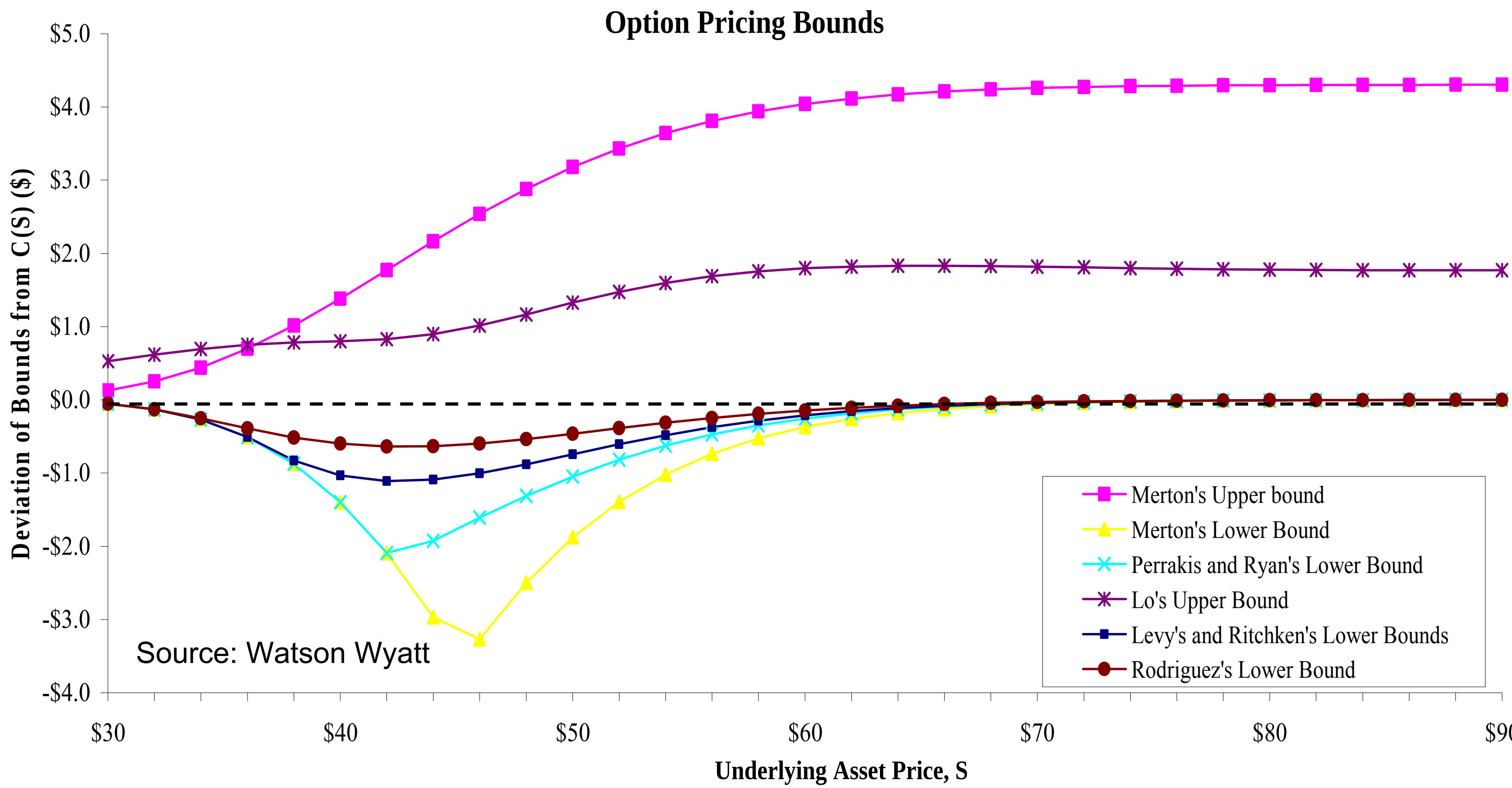
$$U_L = \frac{S - XR_f + SV^* R_f^2}{1 + V^* R_f^2} \quad \text{if} \quad \frac{S}{X} \geq \frac{2R_f}{1 + V^* R_f^2}$$

$$= \frac{1}{2}\left[S - XR_f + \sqrt{(XR_f - S)^2 + S^2 V^* R_f^2}\right] \quad \text{if} \quad \frac{S}{X} < \frac{2R_f}{1 + V^* R_f^2}$$

where $V^* = R_f^2[\exp(\sigma^2 T - 1)]$

Calibration

Deviation of Lower and Upper Pricing Bounds from the Black-Scholes Call Price, \$ (X=\$50, T=1 year, rf=0.1, μ=0.18, σ=0.20)



Arbitrage-free value of at-the-price call is \$6.6 (based on BS model), while it can vary between -\$0.5 to +\$1.2 in incomplete market

Pricing Bounds of Pension Put

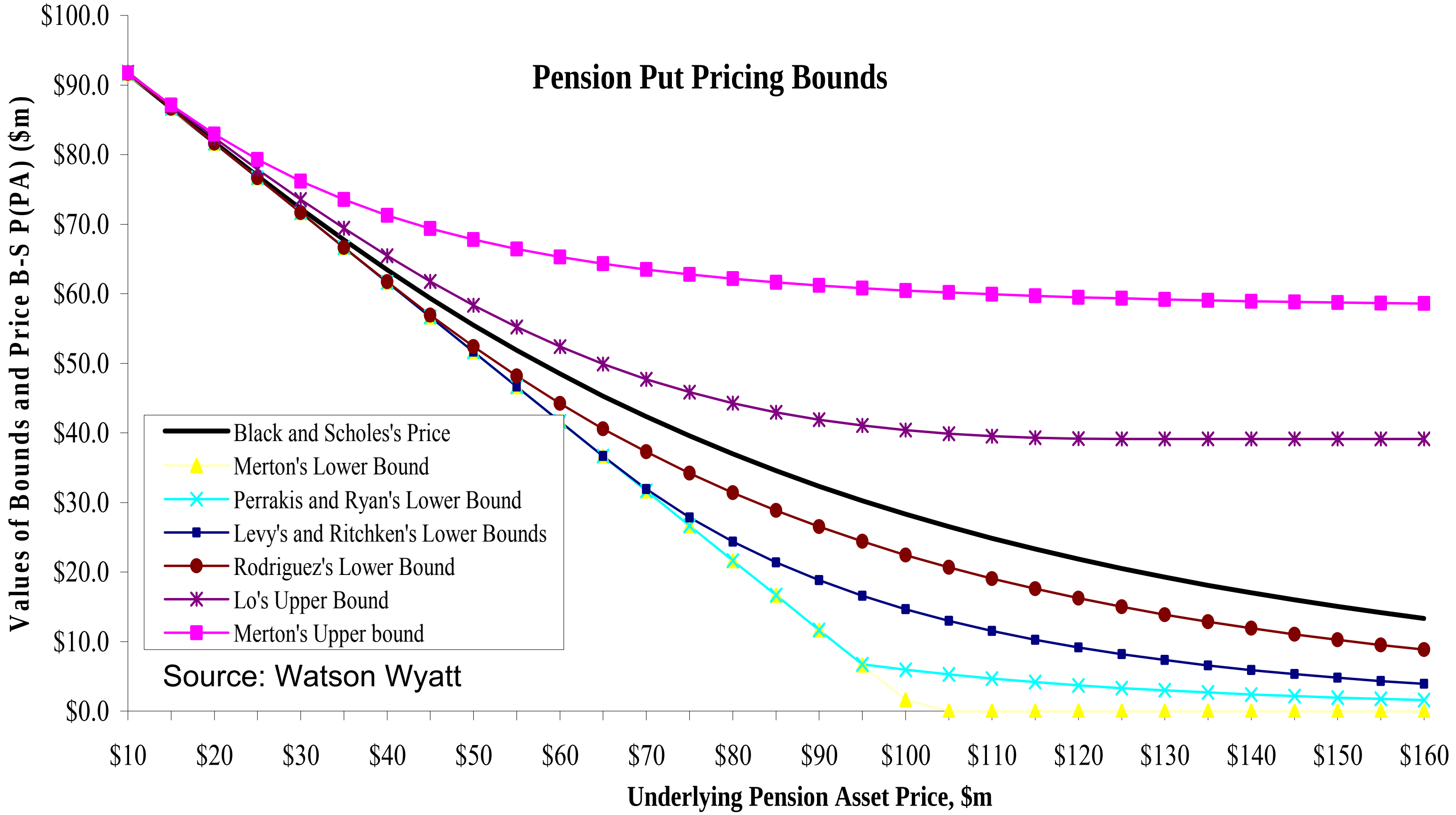
- Based on put-call parity rule Stoll (1969), the bounds of pension put options are:

$$P=C-S+XR_f$$

- Illustrative example of pension scheme:
  - The available riskless rate (r_f) of return is 6% and the return on pension assets is 10% per year and follow log normal distributions.
  - There is no dividend payout , q_i=0.
  - Mean term (duration) of pension liability is 15 years (T).
  - Actuarial value of pension liability at present is $100m, so the future pension liability at retirement date (T) is around $250m
  - The volatility (σ) of pension asset, liability and corporation values are 18%, 5% and 21% per year respectively.

Pricing Bounds of Pension Put

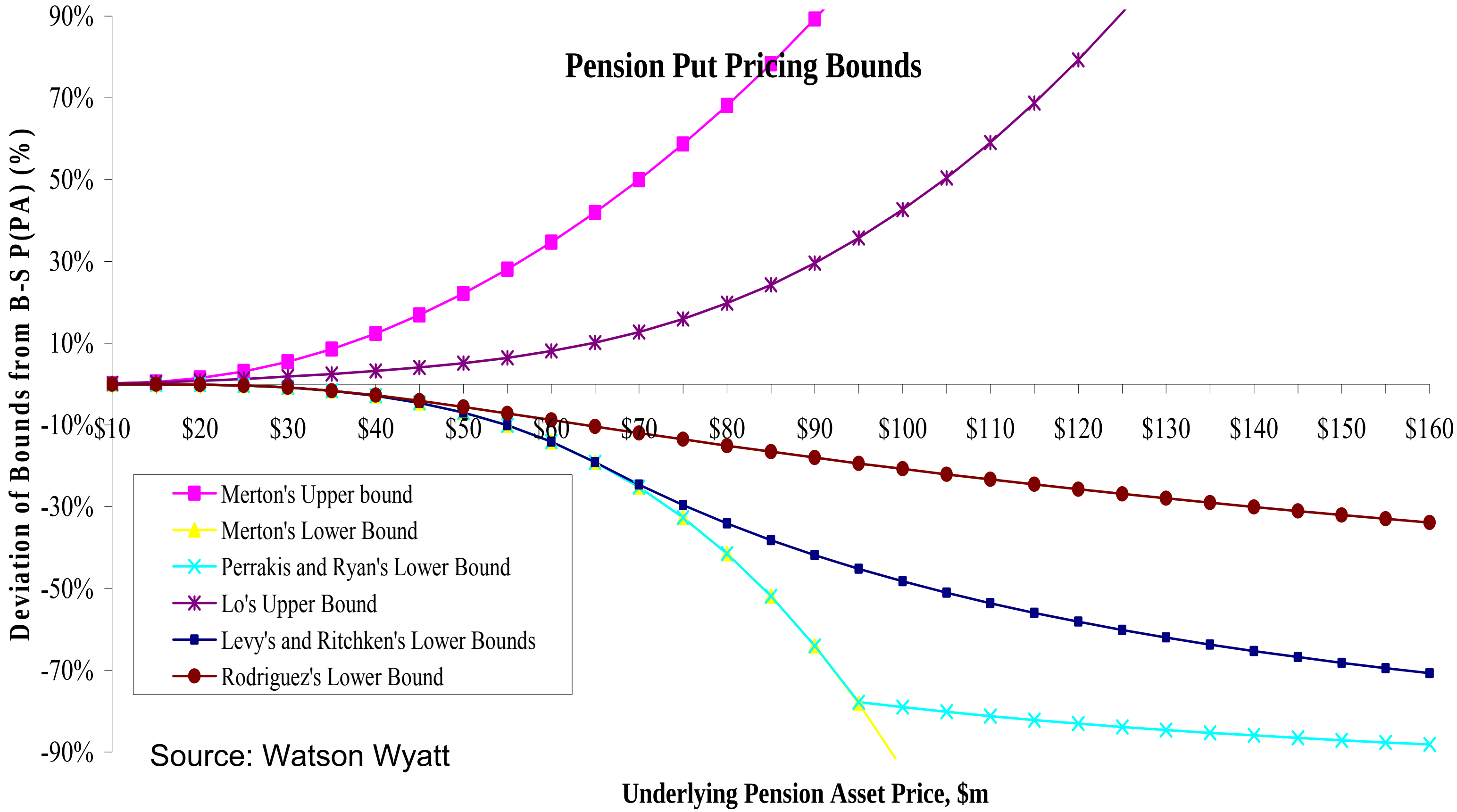
Case One: if pension liability is known



Scheme with 20%($20m) funding shortfall can impose risk to sponsor companies valued at between $35m to $45m

Pricing Bounds of Pension Put

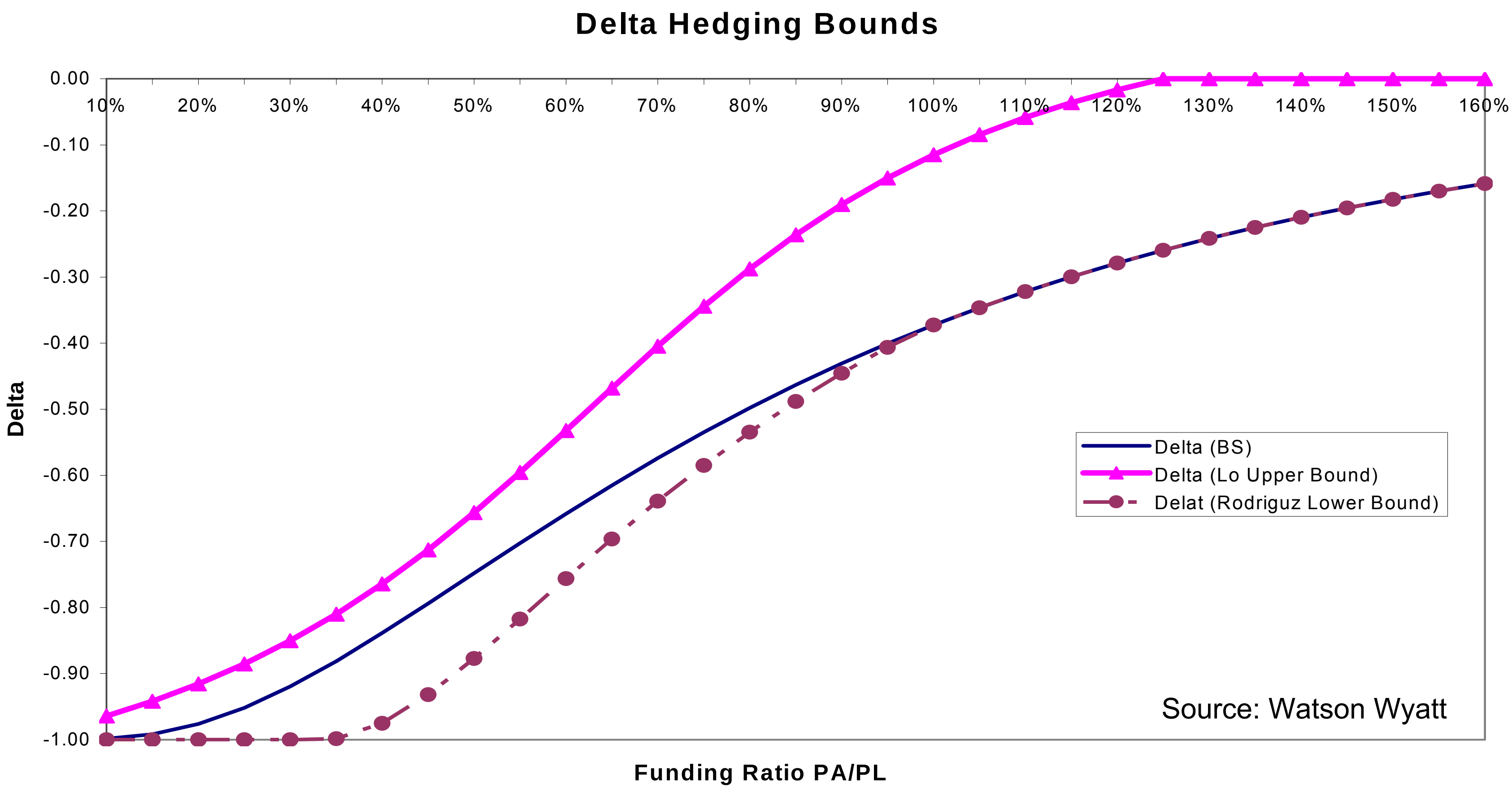
Case One: if pension liability is known



Comparing to value at complete market condition ($40m), the variations can account for -10% to 15%.

Hedging Bounds of Pension Put

Case One: if pension liability is known



The variations of value in incomplete market has limited impacts on delta hedging bounds, [-0.6 -0.35] and 0.55.

Pricing Bounds of Pension Put

Case Two: if pension liability is unknown

- Both PA and PL are assumed to follow standard diffusion processes

$$\frac{dPA}{PA} = a_{PA}dt + \sigma_{PA}dz_{PA} \qquad \frac{dPL}{PL} = a_{PL}\,dt + \sigma_{PL}\,dz_{PL}$$

and correlation coefficient is

$$dz_A\,dz_L = \rho_{AL}\,dt$$

- Value of pension put is

$$UPL = -PA * N(-d_1) + PL * N(-d_2)$$

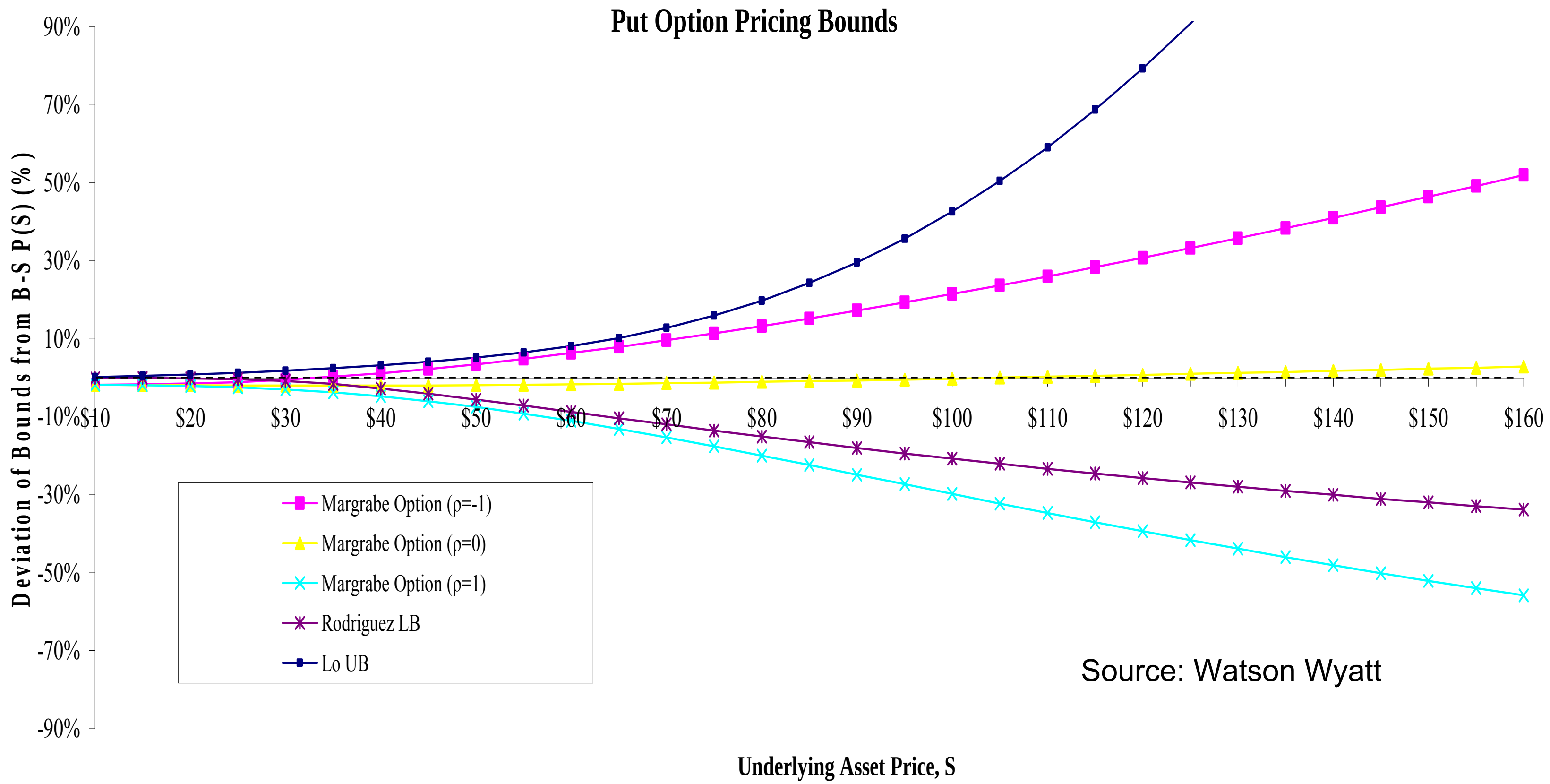
where

$$d_1 = \frac{\ln(PA/PL) + (\sigma^{*2}/2)T}{\sigma^*\sqrt{T}} \qquad d_2 = d_1 - \sigma^*\sqrt{T} \qquad \sigma^* = \sigma_{PA}^2 - 2\rho_{PAPL}\sigma_{PA}\sigma_{PL} + \sigma_{PL}^2$$

- The assumption on risk free rate is irrelevant anymore.

Pricing Bounds of Pension Put

Case Two: if pension liability is unknown



The variations of value in incomplete market are determined by the correlation between pension investment strategy and liabilities profile.

Pricing Bounds of Pension Put

Case Two: if pension liability is unknown

Table 2 | Value of put option of stochastic pension liability as a function of correlation, funding ratio or time to maturity, ($m)

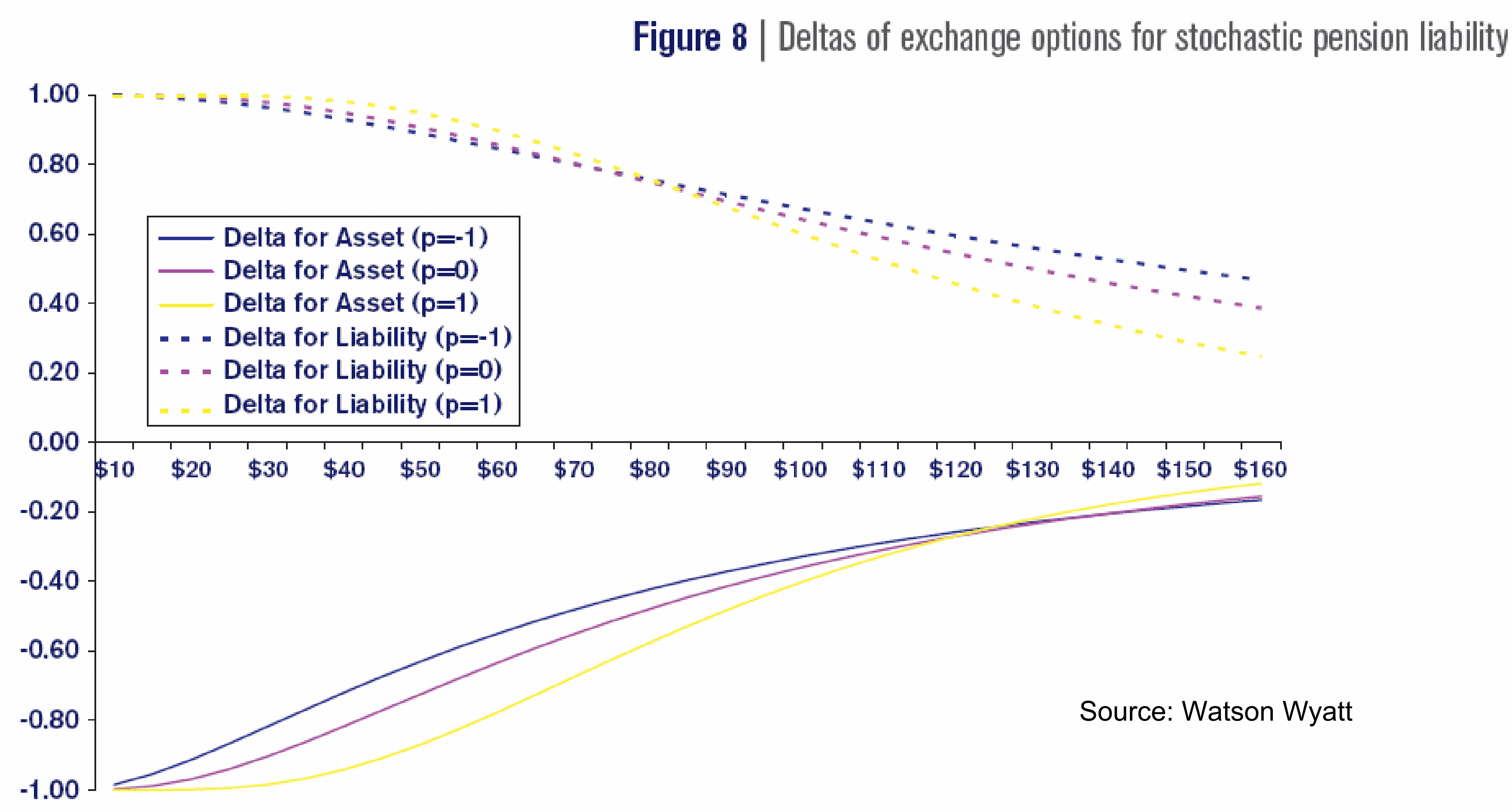
	T (years)	1	5	10	20	30	40
ρPAPL	(PA/PL)%						
-1.0	50%	$50	$51	$54	$60	$65	$69
	100%	$9	$20	$28	$39	$47	$53
	150%	$0.4	$8	$16	$28	$37	$44
0	50%	$50	$50	$52	$56	$60	$63
	100%	$7	$17	$23	$32	$39	$45
	150%	$0.1	$4	$10	$20	$27	$34
1.0	50%	$50	$50	$51	$52	$54	$56
	100%	$5	$12	$16	$23	$28	$32
	150%	$0	$1	$4	$10	$15	$19

Source: Watson Wyatt

The variations of value in incomplete market are determined by the correlation between pension investment strategy and liabilities profile.

Hedging Bounds of Pension Put

Case Two: if pension liability is unknown



The variations of value in incomplete market has limited impacts on delta hedging bounds of assets and liabilities as well.

Pricing Bounds of Pension Put

Case Three: if pension liability is fully guaranteed by company

- Stulz (1982) and Rubinstein (1994) develop a closed-form-valuation solution for these two-colour rainbow options. The equation applied to pension models is:

$$UPL = L_T e^{-r_f T} - PA_0 e^{-r_f T} - (PA_0 + CA_0) e^{-r_f T} N(d) + PA_0 e^{-r_f T} N(d - \sigma_{rb}\sqrt{T}) + c_{rb}^{\max}$$

Pricing Bounds of Pension Put

Case Three: if pension liability is fully guaranteed by company

Table 4 | Value of at-the-money pension put option[11] as a function of correlation, ratio of market value of corporation to pension assets or time to maturity, ($m)

ρ_{PACA}	T (years) / (PA/CA)%	1	5	10	15	20	25
-1.0	10%	$0	$0	$0	$0	$3	$26
	50%	$0	$0	$0	$5	$43	$125
	100%	$0	$0	$0	$35	$111	$215
0	10%	$0	$0	$0.08	$3	$15	$50
	50%	$0	$0.15	$7	$32	$84	$170
	100%	$0	$2	$19	$59	$125	$222
1.0	10%	$0	$0	$0.23	$4	$22	$65
	50%	$0	$1	$17	$54	$118	$212
	100%	$0.0043	$7	$36	$86	$160	$262

Note: numbers are rounded.

Conclusions

Table 1 | Relation between the pension put's value and explanatory variables

	Standalone nominal (standard option)	Standalone varied (exchange option)	Integrated (rainbow option)
PA/PL	–	–	–
PA/CA	*	*	+
PA	+	+	+
CA	*	*	+
PL	*	+	*
PAPL	*	–	*
PACA	*	*	+
rf	–	*	–
T	–	+	+

Note: "*" means no correlation, "–"means negatively correlated and "+" means positively correlated

Conclusions

- Unfunded pension liabilities can be explained in a put options valuation framework, but unique fair value isn’t valid because of market incompleteness.
- No matter which approach is used, we find the range of liability prices to be broad (between 10 to 30%), implying it is difficult to put a precise market value on pension liabilities.
- However, we also find that the implication for strategic asset allocation is relatively minimal. Strategic asset allocations, at least in the models we looked at, appear to be relatively robust to incomplete markets.
- This conclusion is more about risk exposure than the financial instruments available to achieve a given level of risk exposure, as the desired financial instruments may depend considerably on the degree of market completeness.