

Pension Options Valuation and Hedging Bounds

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Abstract

In this paper, various option pricing models are used to provide analytical solutions to valuing defined pension liabilities (or securitised parts of pension liabilities) in incomplete markets. Unlike when markets are complete, there is not a single arbitrage-free price for liabilities but instead a range of prices consistent with the absence of arbitrage. We analyse and compare application incomplete market variants of the following models to pension liabilities: i) the standard Black-Scholes(1973) option model ii) the Margrabe(1978) exchange options model; iii) the Stulz (1982) rainbow option model to price pension schemes liabilities. No matter which approach is used, we find the range of liability prices to be broad, implying it is difficult to put a precise market value on pension liabilities. However, we also find that the implication for strategic asset allocation is relatively minimal; strategic asset allocations, at least in the models we looked at, appear to be relatively robust to incomplete markets. This conclusion is more about risk exposure than the financial instruments to achieve a given level of risk exposure as the desired financial instruments may depend considerably on the degree of market completeness.

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Key Words: pensions, contingent claims analysis, option pricing, incomplete market, b pension funds, pension liabilities, Black Scholes, exchange option, rainbow option

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I. Introduction

Unfunded pension liabilities are a debt of the sponsoring employer and therefore can be valued using financial economics principle. As pension obligations are options-like, it was natural to use option-pricing analysis proposed by Black and Scholes (1973) to evaluate the economic cost of unfunded liabilities.

Sharpe (1976) used option pricing analysis to show that the value of the firm was unchanged when asset allocations changed because existing members bid up their wages. Subsequent papers such as Willinger (1985, 1992); Steekamp (1999); and Wilkie (1989) extend the Sharpe analysis in various directions.

But all these modelling efforts assumed complete markets – that all claims could be replicated by marketable securities. In the case of short term traded options this is acceptable but in the case of long-term securities such as pension liabilities which are essentially nontraded, it is not. Even where there are parts of pension liabilities that are tradable (such as through bulk buyouts), there are substantial transactions costs and trading is infrequent.

This paper extends the earlier papers in this literature to incomplete markets, showing how the results change when markets are incomplete. There are three approaches generally used in the academic literature:

The first approach is to find a super-replicating portfolio. This is a portfolio whose payoffs are always (in any state of the world) at least as big as the payoff of the derivatives. The value of the derivatives is then bounded by the value of the super-replicating portfolio. Although for some products, such as options, this may be a useful approach, for pension fund liabilities this approach is not very helpful as the unhedgable wage growth is in principle unbounded and no super-replicating portfolio can be found.

The second approach is based on the utility a representative agent can derive from investing in the nontraded claim. The investor in the pension liability assumes unhedgable risk, which will affect the probability distribution of his consumption and final wealth level, and hence his utility. The certainty equivalent wealth of the expected utility then is the value an investor is prepared to pay for the claim. Conversely, we can measure how much wealth the pension fund should invest in the market to give their members the same utility as a fully guaranteed wage indexation pension. This wealth then is the “shadow” market value of the unhedgable claim. The seminal paper in this field belongs to Hodges and Neuberger (1992). De Jong (2005) proposes a discount factor adjustment to calculate value of wage indexed pension liabilities using utility-based method. Kocken (2006) prices various embedded pension options value based on the assumptions of complete market.

The final approach stems from a no-arbitrage argument, to compare different portfolio strategies involving the underlying asset and options and work out the option bounds by

excluding the existence of any dominant strategies. In this paper we shall further develop the third approach for the specific case of pension liabilities. We show that pricing and hedging bounds can be derived for pension put in incomplete market.

We use three incomplete markets models in this paper: i) use standard Black-Scholes (1973) option model to price liabilities with nominal values and derive a tighter pricing and hedging bounds; ii) use Margrabe (1978) exchange options model to price varied liabilities and validate the results using Monte Carlo simulation model; iii) use Stulz (1982) rainbow option model to price pension scheme integrated with parent companies.

We begin with a review of various embedded options in a defined benefit (DB) pension fund and corresponding valuation theories in complete markets literature. The following three sections then go through and look at separate incomplete markets' cases. A final section concludes.

II. Valuing Pension Liabilities in Complete Markets: A Review of the Results

The embedded options in pension funds are among the most complex, interesting as well as economically relevant embedded options in current financial system. Because pension fund constructions are by their very nature established via a process of negotiations. These negotiations usually do not make use of explicit options valuation knowledge as part of the negotiation process. This process therefore may generate more complex contingent claims than in the case options are traded deliberately and determining the right value is difficult.

2.1 The pension put

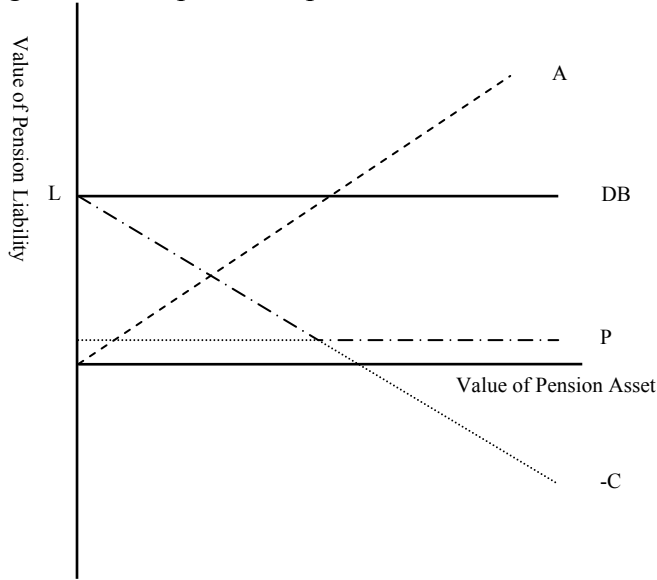
The seminal paper on financial economic valuation of pension liabilities is Sharpe (1976). Sharpe models pension liabilities as a series of contingent claims and he suggests a defined benefit plan can be replicated using a long put (P) and a short call (C) on pension fund assets (PA), both with the same exercise price (PL) -- pension liabilities. The put option is held by scheme members and written by the scheme sponsors, while the call option is written by the member and held by scheme sponsors. On the retirement date of the scheme member, which coincides with the expiry date of the options, one of the options is almost certain to be 'exercised'. If the value of the assets is less than the exercise price (liability value) so that the scheme is showing an actuarial deficit, the member will exercise the put option against parent companies who will then be required to make a deficiency payment ($PL_T - PA_T$). If, on the other hand, the value of the assets exceeds the exercise price (liability value), so that the scheme is showing an actuarial surplus, the sponsors will exercise the call option against member and recover the surplus ($PA_T - PL_T$).

Figure 1 shows the option composition of defined benefits scheme, a defined benefit (DB) scheme is invested in a portfolio containing the underlying asset (A) plus a put option minus a call option on these assets.

Sharpe's model is:

$$PV(PL) = PV(PA) + PV(P) - PV(C) \quad (1)$$

Figure 1 The Option composition of a defined benefit scheme



2.1 Nominal pension liabilities

The options embodied in the DB scheme have the following characteristics. They are European options, since they cannot be exercised before the retirement date². In addition, the underlying asset does not make payouts prior to expiry date of the option, no dividends payments. The value of pension put in one period can be calculated, using Black-Scholes options pricing model, as shown in equation (2). This is a simple version of the formula, without dividend payments and a non-stochastic interest rate.

The model is

$$UPL = -PA * N \left[\frac{-\ln(PA / PL) - [r_f + (\sigma_{PA}^2 / 2)T]}{\sigma_{PA} \sqrt{T}} \right] + PL * e^{-r_f T} * N \left[\frac{-\ln(PA / PL) - [r_f - (\sigma_{PA}^2 / 2)T]}{\sigma_{PA} \sqrt{T}} \right]$$

(2)

where:

² Earlier terminations of the pension scheme because of redundancy or ill-health are not considered.

- UPL = the unfunded pension liability.
- PA = current market value of the pension fund market portfolio.
- PL = retirement date (T) value of accrued benefits as actuarially computed.
- T = mean term of pension liability.
- σ_{PA}^2 = variance of the pension assets
- N = cumulative standard normal distribution function.
- r_f^3 = risk-free market value of interest.

Equation (2) indicates that the option value of pension liabilities depends on the term to maturity of the liabilities, the volatility of pension assets and current funding level.

2.2 Stochastic pension liabilities

Pension entitlements are cost-of-living or “career” linked. In practice, the entitlements of the active members are in many cases linked to wage inflation (therefore linked to individual’s career), while indexation rights of the non-actives are dependent on the (consumer or retail) price inflation. In the past, indexation was something that was almost always granted in many DB plans. In the UK, a cap (LPI, limited price index) has been attached after the wave of price inflation in the seventies and eighties. Hence, the pension payoff in the LPI indexation option has similarities with an option termed a digital option (also “binary option” or “all or nothing” option). It is an option that pays either zero or a pre-agreed amount, depending on the stochastic underlying variable, in the case the price inflation.

In above Black-Scholes formula, there is only one source of uncertainty, the pension asset price, and the market is dynamically complete, since the payoff of the pension put can be attained by a trading strategy in the pension and the risk free assets. This is of course a very stylized setting and in practice, the payoff claims such as pension liabilities depend on a multitude of risk factors. Moreover, some of these factors can not be replicated by trading in financial market instruments. Pension liability, the strike price of pension put valuation, is not certain and follows a stochastic process, depending on the future course of inflation, wage growth and longevity. A more appropriate option valuation model is based on a modification model proposed by Fischer (1978) and Margrabe (1978)⁴ which recognises that the options in formula (1) are exchange options, i.e. options to exchange risky assets at an exercise price that is indexed to the uncertain value of the liabilities. This idea was first applied to analyse pension obligations by David Wilkie in July 1988.

The value of the exchange option from employee point of view is $\max(0, PL_T - PA_T)$, a put option on pension asset with strike price equals to the future value of pension liability at expired date T . Hence, the payoff of the pension put depends on changes of pension

³ An assumption of this model is that the short-term interest rate is known and constant over the life of the option. Pension liabilities are of a long duration and changes in the short-term interest rate may occur. However, Brennan and Schwartz (1980) and King (1980) provide analytical and empirical evidences that contingent claims models can appropriately value contingent claims of long duration.

⁴ Also known as “exchange-one-asset-for-another-options”.

assets (PA) on which the put is written relative to the change of pension liability (PL), rather than on absolute changes of pension assets assumed in formula (2). The variable of PA and PL are assumed to follow standard diffusion processes:

$$\frac{dPA}{PA} = a_{PA}dt + \sigma_{PA}dz_{PA} \quad \frac{dPL}{PL} = a_{PL}dt + \sigma_{PL}dz_{PL} \quad (3)$$

where a_{PA} , a_{PL} are expected rate of increase of pension assets and liability; σ_{PA} , σ_{PL} are standard deviation of pension asset and liability changes; dz_{PA} , dz_{PL} are standard Gauss Wiener process. The stated pension asset and liability have correlation coefficient ρ_{AL} :

$$dz_A dz_L = \rho_{AL} dt \quad (4)$$

The modified pension put option pricing formula is:

$$UPL = -PA * N(-d_1) + PL * N(-d_2) \quad (5)$$

where:

$$d_1 = \frac{\ln(PA/PL) + (\sigma^* / 2)T}{\sigma^* \sqrt{T}}$$

$$d_2 = d_1 - \sigma^* \sqrt{T}$$

$$\sigma^* = \sigma_{PA}^2 - 2\rho_{PA PL} \sigma_{PA} \sigma_{PL} + \sigma_{PL}^2$$

σ^* is standard deviation of pension surplus. For a constant strike price (pension liability is constant), $\sigma_{PL} = 0$, formula (5) reduces to (2) standard Black-Scholes formula.

A comparison of the equation (2) and (5) shows that the new option values do not depend explicitly on the riskless rate of interest as in the standard Black-Scholes model, and that the appropriate definition of risk is not the risk given by equation (2), attached to pension assets only, but the risk attached to pension surplus (σ^*). This is because the pension liabilities provide a natural hedge for the pension assets against both interest rate and growth rate risks. Risk free rate is normally approximated by government long-term bonds, which change constantly due to structural economic shifts and perturbations. The risk free rate at long end is particularly problematic to measure and predict, and is subject to manipulations based on different accounting standards.

2.3 Pension fund and corporation integrated

In case pension entitlement is guaranteed by the parent company to a previously stipulated and agreed amount (ratio to final or average salary), pension put entails the credit exposure of the beneficiaries (actives, deferred, pensioners) to the pension fund possibly defaulting on its commitments. In fact, it is an option on a “joint default” event to trigger “payments”: the pension fund must be in a state of deficit and the parent company must be in a state of corporate default, not able to fully pay for the shortfall in

the pension fund. The situation of corporate default is very likely to occur – from the pension fund perspective – as a sudden exogenous event, often modelled in the form of a Poisson process.

The payoff of pension put at the maturity date now depends on two underlying assets values – the value of pension assets PA_T and combined value of pension and corporation assets (PA_T+CA_T), *integrated value*. Hence, if the value of pension assets or integrated value at time T is no less than promised pension payment, the pension debt-holders will be paid fully. If the value of pension assets and integrated value at time T are less than the promised pension payments, the pension debt-holders are paid an amount equally to (PA_T+CA_T). Therefore, the payoff profile can be written as:

$$L_T - \max[0, L_T - \max(CA_T + PA_T, PA_T)] \quad (6)$$

This is a put option on the maximum of two risky assets with strike price L_T , also known as a two-colour rainbow exotic option.

Stulz (1982) and Rubinstein (1994) develop a closed for valuation solution for these two-colour rainbow options. The equation applied to pension models is:

$$UPL = L_T e^{-r_f T} - PA_0 e^{-r_f T} - (PA_0 + CA_0) e^{-r_f T} N(d) + PA_0 e^{-r_f T} N(d - \sigma_{rb} \sqrt{T}) + c_{rb}^{\max} \quad (7)$$

where,

- $c_{rb}^{\max} = (PA_0 + CA_0) e^{-r_f T} M(y_1, d; \rho_1) + PA_0 e^{-r_f T} M(y_2, -d + \sigma_{rb} \sqrt{T}; \rho_2) - L_T e^{-r_f T} [1 - M(-y_1 + \sigma_{CAPA} \sqrt{T}, -y_2 + \sigma_{PA} \sqrt{T}; \rho_{CAPA})]$
- $d = \frac{\ln(1 + CA / PA) + (\sigma_{rb} / 2)T}{\sigma_{rb} \sqrt{T}}$
- $y_1 = \frac{\ln(PA_0 + CA_0) / L_T + (\sigma_{CAPA}^2 / 2)T}{\sigma_{CAPA} \sqrt{T}}$
- $y_2 = \frac{\ln(PA_0 / L_T) + (\sigma_{PA}^2 / 2)T}{\sigma_{PA} \sqrt{T}}$
- $\sigma_{rb} = \sqrt{\sigma_{CAPA}^2 + \sigma_{PA}^2 - 2\rho_{CAPA}\sigma_{CAPA}\sigma_{PA}}$ ⁶
- $\rho_1 = \frac{\sigma_{CAPA} - \rho_{CAPA}\sigma_{PA}}{\sigma_{rb}}$
- $\rho_2 = \frac{\sigma_{PA} - \rho_{CAPA}\sigma_{CAPA}}{\sigma_{rb}}$

⁵ $M()$ is the value of cumulative bivariate Normal distribution

⁶ σ_{CAPA} is the volatility of corporate and pension assets, and ρ_{CAPA} is the correlation between pension asset and corporation assets.

2.4 Main risk factors of pension put

Table below summarise the relation between pension put value and main risk factors based on above three different models. We identified nine factors which are important for the valuation of pension liabilities: funding ratio (PA/PL), the size of the pension fund relative to corporation value (PA/CA), the duration of the pension liabilities (T), volatilities of pension assets, liabilities and corporation assets (σ_{PA} σ_{PL} σ_{CA}), correlation between pension assets and liability, and pension assets and corporation assets (ρ_{PAPL} ρ_{PACA}), and risk free rate (r_f). Most of these value-factors seem intuitively logical, and the results are also validated in illustrative samples in following sections.

Table 1 Relation between pension put value and explanatory variables

	Standalone nominal (standard option)	Standalone varied (exchange option)	Integrated (rainbow option)
PA/PL	–	–	–
PA/CA	*	*	+
σ_{PA}	+	+	+
σ_{CA}	*	*	+
σ_{PL}	*	+	*
ρ_{PAPL}	*	–	*
ρ_{PACA}	*	*	+
r_f	–	*	–
T	–	+	+

Note: “*” means no correlation, “–” means negative correlated and “+” means positive correlated

2.5 Problems with complete market

Analytical pension put pricing solutions presented above rely on several crucial hypotheses. First, the trader must have the possibility of trading continuously and he must be a “price taker” or “small” compared to the size of the market. Second, market is frictionless, without taxed and other transaction costs. In reality, every rebalancing implies a cost in the form of commissions or bid-ask spreads. Third, the Black-Scholes-Merton model assumes constant volatility and correlation, but this is not supported by empirical evidence. In general, the implied volatility and correlation tend to increase as the underlying asset price goes down and decrease in other cases. Fourth, there are other aspects that may pose dangers to such pricing approach: jump movements in asset prices and uncertainty regarding future interest rates in long term.

For the case of pension liabilities, the additional source of market incompleteness is the absence of financial market instruments that closely replicate the price or wage index that the pension are linked to. Although there may be products that correlate strongly with the price level, such as index linked bonds, this is definitely not true for wage-linked payoffs.

The following sections will go through and look at solutions separately in incomplete markets based on an illustrative example. A very stylised market model is assumed, with following external variables:

- The available riskless rate (r_f) of return is 6% and return of pension asset is 10% per year.
- There is no dividend payout⁷, $q_i=0$.
- Mean term (duration) of pension liability is 15 years (T).
- Actuarial value of pension liability at present is known with value of \$100m, so future pension liability at retirement date (T) is around \$250m⁸
- The volatility (σ_{PA} standard deviation) of pension asset, liability and corporation values are 18%, 5% and 21% per year respectively.

III. Pension Liabilities in Incomplete Markets: The Black-Scholes Model

Because complete market assumptions used in Black Scholes model are not valid in many circumstances⁹, particularly for pension put valuation. Therefore, in incomplete markets, instead of a single arbitrage-free price there appears an “arbitrage-free” interval $[h_{low}, h_{up}]$ which contains the complete BS markets price. Academic literatures provide several approaches to derive upper and lower pricing bounds on options valuation. A full review on methodologies used in this paper can be found in Hao (2007).

3.1 Pension put pricing bounds

Figure below shows the results of Black and Scholes put option price and corresponding bounds on contingent claims against pension parent in illustrative example at expiry date T . If the current pension asset value is \$85m, \$15m less than actuarial computed present liability, the put value of pension liability is \$35m in a complete market according to Black-Scholes formula, however, the value can vary between \$30m to \$40m in reality based on Rodriguez’s lower and Lo’s upper bounds. The value of pension put increases when the funding ratio declines, and the gap between lower and upper bounds narrows. When the funding status improves, the value of pension put diminishes but at slower paces.

⁷ Regular contributions are paid to scheme by individuals and employers, which can be seen as a negative dividends payout instead.

⁸ Future pension liability value at present is about \$100m, which is calculated at constant risk free rate r_f for T -periods on the basis of continuous compounding.

⁹ For further discussion on option pricing in incomplete market, see Hao (2007).

Figure 2 Pensions Put Option Pricing Bounds, \$million ($X=\250, $T=15$ years, $r_f=0.06$, $\mu=0.1$, $\sigma=0.18$)

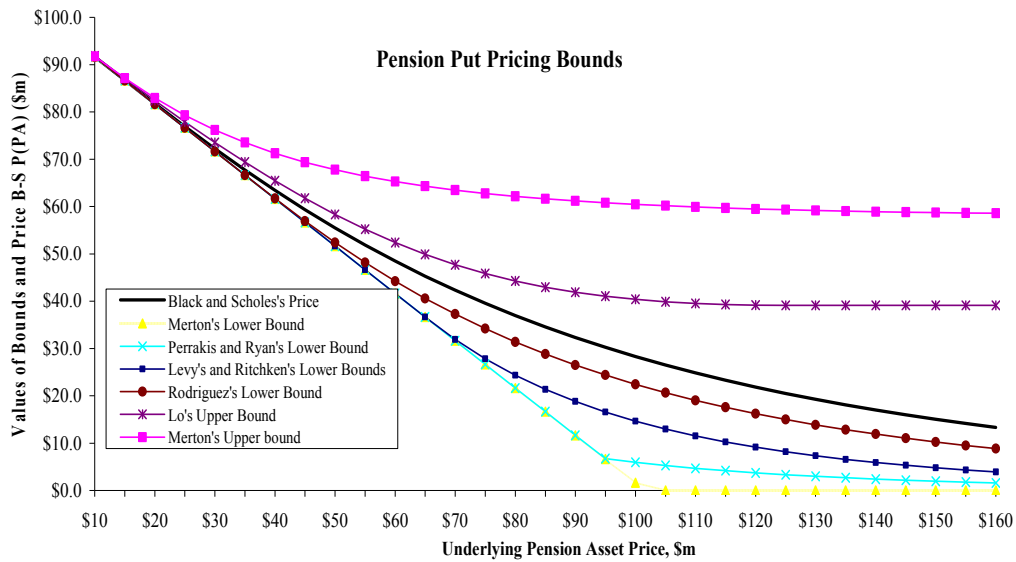


Chart 3 and 4 show the deviation of each pension put bounds from that Black-Scholes value in real and percentage terms. Both results confirm that the deviations of lower and upper bounds are progressively tighter when pension asset price decreases (PA), the pension put becomes more valuable. Hence, when pension put is in-the-money ($PA < PL$), Black-Scholes value provides reasonable good approximation in incomplete market condition. However, the variations enlarge rapidly when pension assets increases, particularly for upper bounds, for instances the upper bounds rises twice as much high as Black-Scholes price when pension plan is 30% over funded. On the other hand, the lower bounds tracks Black Scholes price more closely, the variation stands at about \$5m (30% less) when pension assets increase.

The width of bounds is larger, at-the-money ($PA = PL$) than it is far in-the-money or out-of-the-money. Pension put is hardest to hedge at the money because the nonlinearity of the option payoff as a function of stock price is greatest here. However, the width of the bounds is a much larger fraction of put option value for out-of-the-money options on the right hand side of the chart. In this sense, as well as when translated to implied volatilities, the bounds are wider for out-of-the money options.

Figure 3 Pensions Put Option Pricing Bounds, deviation from B-S put value, \$million

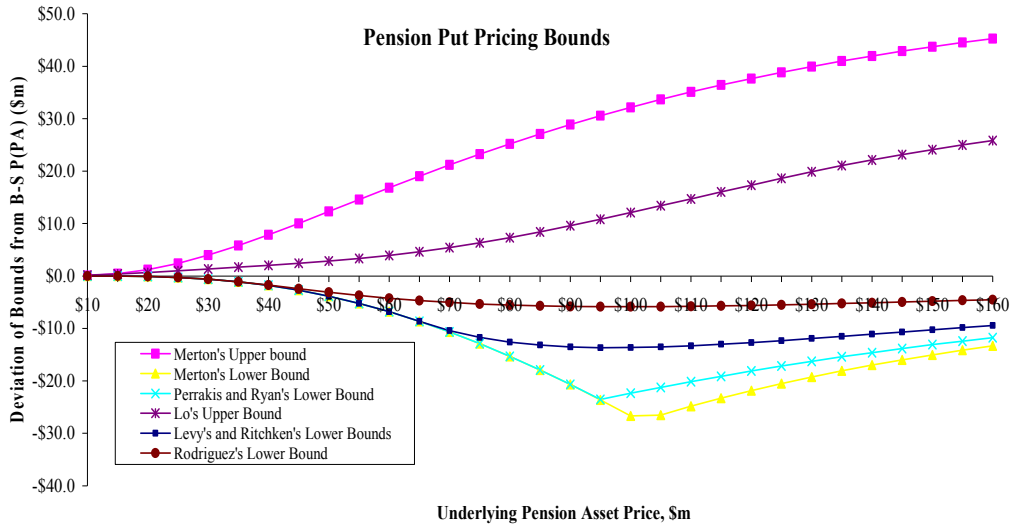
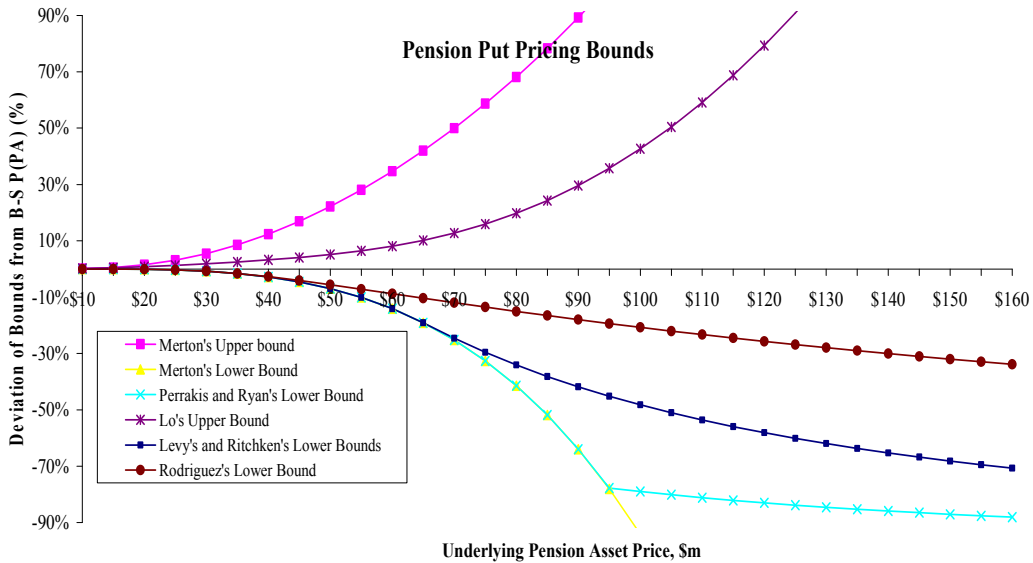


Figure 4 Pension Put Option Pricing Bounds, deviation from B-S put value, %



The highest and lowest pension put values estimated by Lo's and Rodriguez's bounds suggest that the potential contingent claims of unfunded pension liability against parent companies can vary considerably from Black-Scholes prices in a incomplete market, ranges between 20% to over 100%. Therefore a static hedging portfolio to minimise pension surplus risk based on Black-Scholes price can be far from optimal in reality.

3.2 Pension put hedging bounds

Pension put price and bounds derived above can be used to quantify pension risk exposure by measuring how much a pension put value would change if pension assets value changed, due to extra contributions made by parent companies or strong investment performance. This sensitivity is known as the option's *delta*.

Assume that the delta of the “at-the-money” pension put option discussed before is -0.4. This means when the pension asset changes by a small amount, the pension put value changes by approximated 40% of that amount. Suppose the pension put price is \$10 and the pension asset price is \$100, the sponsored companies who has sold 20 put options contract, that is, options to sell 2,000 shares with strike price equals to predefined pension liabilities. The sponsor's position could be hedged by short $-0.4 \times 2000 = -800$ shares. The gain (loss) on the options positions would tend to be offset by the loss (gain) on the pension asset. For example, if the pension asset price goes down by \$1 (producing a gain of \$800 on the shares short), the option price will tend to go down by $\$1 \times 0.4 = \0.4 (producing a loss of \$800 on the options written). Hence, the total position (short 2,000 put options and short 800 shares) is zero. The delta of the position in the underlying pension assets offsets the delta of the option position. A position with a delta of zero is referred to as being delta neutral, one type of dynamic hedging.

It is important to realise that the pension fund position remains delta hedged for only a relatively short period of time (Δt). This is because delta changes with both changes in the underlying assets price and the passage of time. In practice, when delta hedging is implemented, the hedge has to be adjusted or “rebalanced” periodically.

The existence of an “arbitrage-free” price interval, instead of a single Black-Scholes price, suggests that the delta hedging ratio is not unique. The ratio corresponding to lower and upper bounds are calculated below.

Figure 5 Delta hedging bound of pension put options

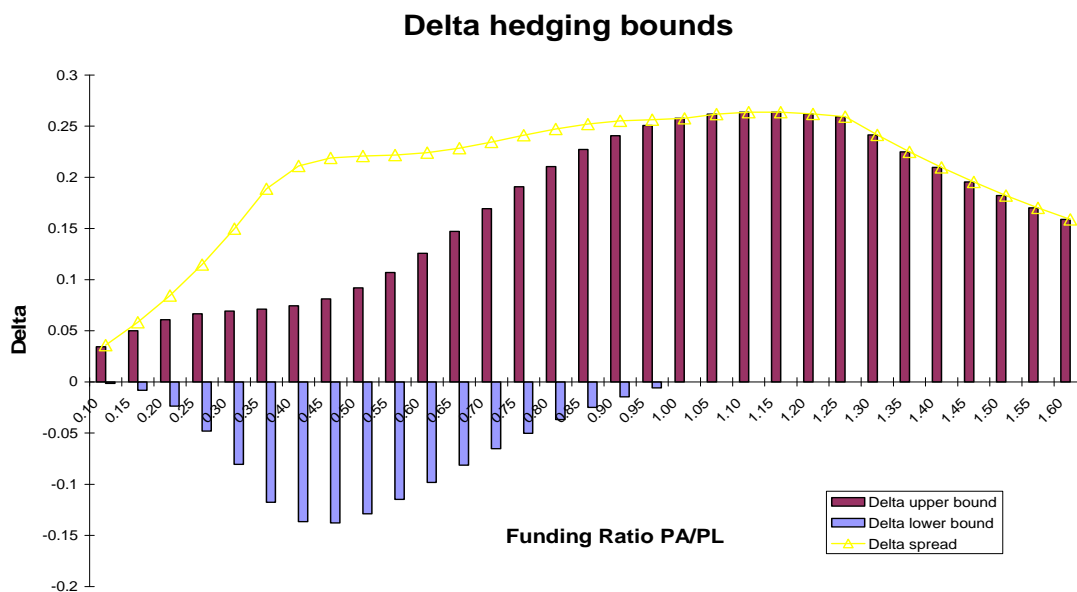


Figure 5 shows hedging bounds of corresponding Lo's (1987) upper and Rodriguez's (2003) lower bounds over Black-Scholes delta. The delta value of put options, which is negative between 0 and -1, decreases with decreasing funding ratio and converges to -1 at levels below roughly 10%. The lower hedging bound is much tighter than upper bound. The width of the hedging bounds follows the similar trends as price bounds, larger for close-to-the-money options. Because the duration of pension put can be extreme long and the size of pension assets tend to be very large, delta hedging only considering small price changes linearly appeared to be difficult to implement in practice.

IV. Pension Liabilities in Incomplete Markets: The Margrabe Model

In Black Scholes model, exercise price (pension liabilities) is assumed to be constant, a riskless hedge portfolio (delta hedge) can be constructed by holding underlying assets. On the contrary, Margrabe (1978) assumes that exercise price (pension liabilities) also follows a Geometric Brownian Motion, hence pension liabilities are hedged by holding in the portfolio assets whose returns are correlated to changes in the exercise price, i.e. with the change in the value of pension liability.

4.1. Margrabe options pricing

The value of pension put is affected by time to maturity (T), correlation between pension asset and liability, funding ratio (PA/PL), summarised in Table 1 based on illustrative sample¹⁰. The result clearly indicates that put values of pension liabilities diminish if either the funding ratio or correlation increases, or time to maturity decreases. However, the correlation is less important if the scheme is deeply under-funded or the liability matures in near future. The longer the term to maturity, the higher sensitivity of pension put value responds to current funding status and correlation between pension assets and liabilities.

Table 2: Value of put option of stochastic pension liability as a function of correlation, funding ratio or time to maturity, (\$m)

	T (years)	1	5	10	20	30	40
ρ_{PAPL}	$(PA/PL)\%$						
-1.0	50%	\$50	\$51	\$54	\$60	\$65	\$69
	100%	\$9	\$20	\$28	\$39	\$47	\$53
	150%	\$0.4	\$8	\$16	\$28	\$37	\$44
0	50%	\$50	\$50	\$52	\$56	\$60	\$63
	100%	\$7	\$17	\$23	\$32	\$39	\$45

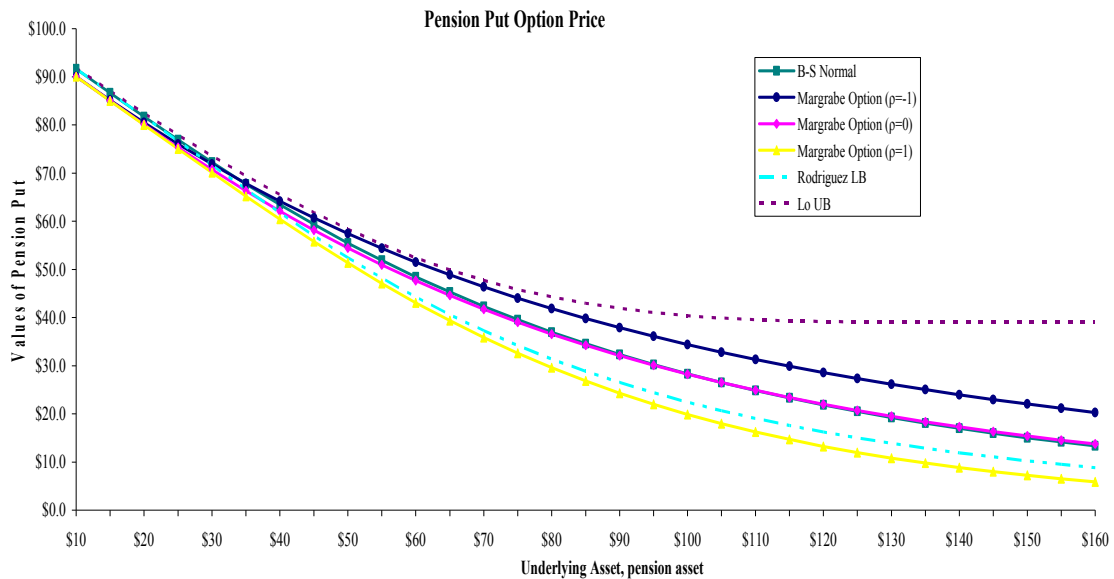
¹⁰ The annual volatility of pension liabilities is assumed to be 5%, accounting for volatilities of real wage growth, inflation and mortality.

	150%	\$0.1	\$4	\$10	\$20	\$27	\$34
1.0	50%	\$50	\$50	\$51	\$52	\$54	\$56
	100%	\$5	\$12	\$16	\$23	\$28	\$32
	150%	\$0	\$1	\$4	\$10	\$15	\$19

Figure 6 shows how the exchange option value by different funding level and correlation rates, and compares with Black-Scholes pension put value and its corresponding price bounds derived in section III.

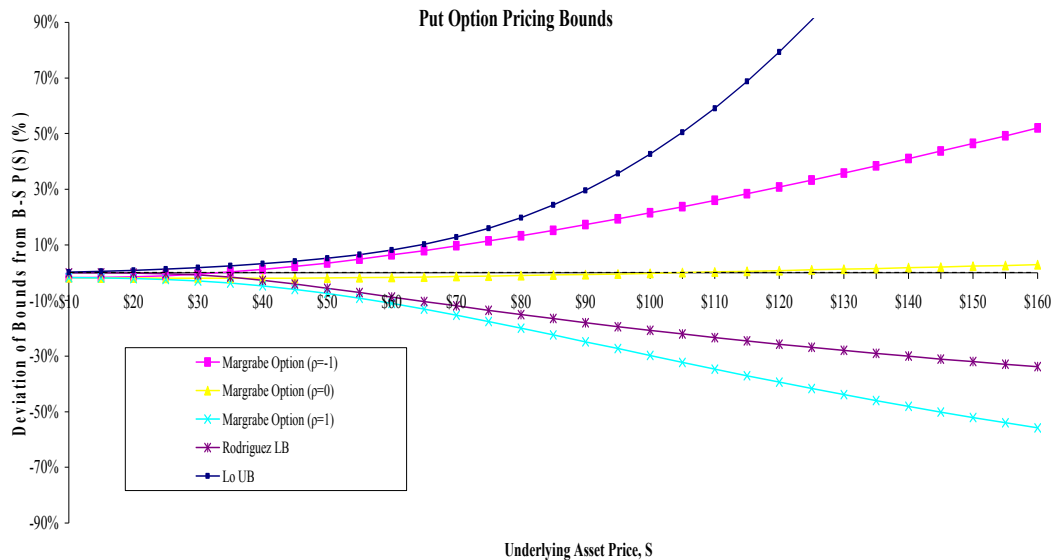
The value of pension put decreases along correlation between pension assets and liability. The pension put derived from exchange option approach in formula (5) isn't far from previous derived Black-Scholes pension put value in formula (2). Furthermore exchange put curve coincides with Black-Scholes put curve, when the asset and liability is uncorrelated at all. This is consistent with the assumptions applied in Black-Scholes formula that pension liability value is fixed.

Figure 6 Value of put option of varied and nominal pension liability, (\$m)



The divergence of Margrabe pension put value over Black-Scholes put value increases in line with funding ratio, illustrated in figure 7, when the put option is deep out-of-the-money. Margrabe option curves also provide good alternatives for options pricing bounds – tightest upper bound if pension liability and asset are negatively correlated ($\rho=-1$) and second best lower bound when they are perfectly correlated ($\rho=1$).

Figure 7 Divergence of value of put option for varied and nominal pension liability, (%)



Therefore, if pension assets and liabilities values respond to shocks in a similar way, for example, an unexpected increase in yields reduces the present values of both assets and liabilities, while an unexpected increase in growth rates has the opposite effect. A perfect riskless hedge requires an immunising portfolio replicating the changes of liabilities closely, when $\rho=1$. However, such perfect hedging portfolio does not exist in practice given the complex features of liabilities.

4.2 Margrabe options hedging

As far as the hedging of Margrabe option is concerned, Margrabe (1978) proposed a solution that involves the deltas of both assets. Maintaining a position in both underlying assets implies that correlation between them is of major interest. It is the reason why these options have been called “first order correlation dependent options”. Moreover, since correlation is not a traded asset, it is extremely hard, if not impossible, to think of a static hedging.

We assume that there is a correlation, $\rho \in [-1, +1]$, between pension asset and liability, but no correlation between their values and volatilities. Dynamic hedging of the exchange option is different from the one described in section 3.2, since there are two assets involved, and therefore two deltas. The first and second deltas are the derivatives of the exchange option price with respect to pension asset and liability respectively. We have used the pricing formula from Zhang (1998), thus:

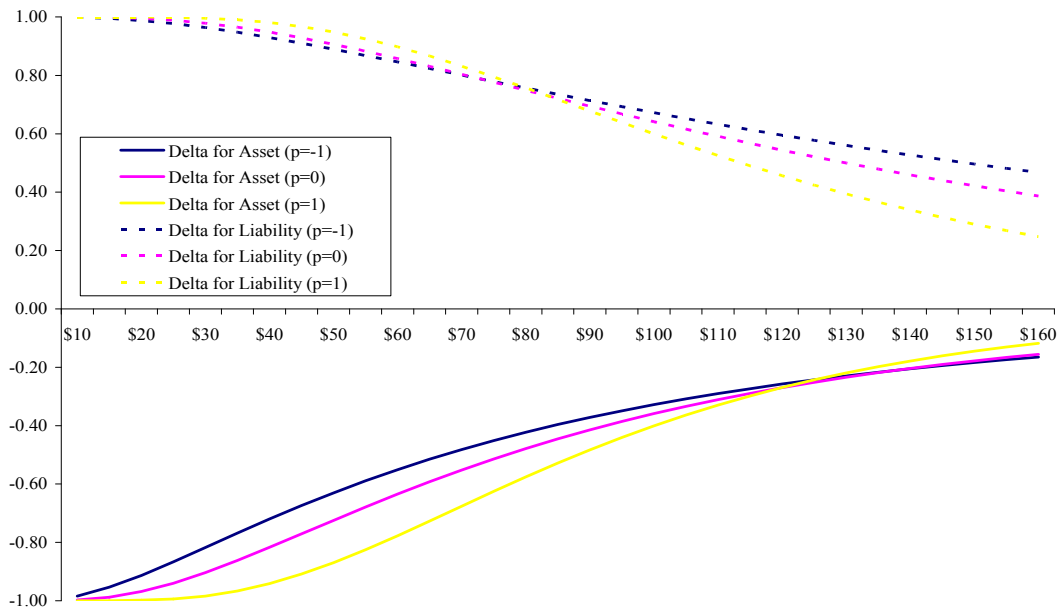
$$\frac{\partial PriceExch}{PA} = -N(-d_1)$$

$$\frac{\partial PriceExch}{PL} = N(-d_2) \quad (8)$$

where d_1 and d_2 refer to formula (5). Hence, a hedging strategy can be constructed by buying a certain amount of “pension assets” like portfolio and selling another amount of “pension liability” like portfolio according to their specific deltas. For instance, the above illustrative example assuming the correlation between pension asset and liability is $+1$, the sponsored companies who has sold 20 at-the-money exchange options contracts (value of exchange option is about \$20), that is, option to sell 2,000 shares of “pension asset portfolios” with strike price equals to “pension liability portfolios”. The sponsor’s position could be hedge by short 800 ($= -0.4*2000$) shares of pension asset portfolio and long 1200 ($= 0.6*2000$) shares of pension liability portfolio.

Figure 8 suggests that when the exchange option is out-of-the-money, value of pension asset is lower than pension liability, the magnitudes of delta changes are higher which suggests that more frequent transactions are required to maintain “delta-neutral” if the correlation between pension asset and liability is low ($\rho=-1$). Hence, the exchange option is far more sensitive to transaction cost and hedging errors than its simple counterpart in section 3.2.

Figure 8: Deltas of exchange options for stochastic pension liability



4.3 Monte Carlo simulations testing

The analytical formulas (5) are obviously more efficient in this case since they are simple to implement. However, Monte Carlo becomes more useful when the option involves in complex features, i.e. basket options consisting more than two underlying assets, asset prices following complicated process such as jump diffusions and etc.

The payoff of pension put discussed in section IV is a European exchange option, and the typical simulation approach is to price the option as the expectation value of discounted cash-flows:

$$-e^{-r_f T} E_Q[\max(PA_T - PL_T, 0)] \quad (9)$$

where “ $-\max(PL_T - PA_T, 0)$ ” denotes the payoff (pension shortfall) at expiration time T and the probability Q is the risk-neutral probability for the pricing problem.

So, for the risk-neutral version of the equations (3), it's to replace the expected rates of return a_{PL} and a_{PA} by the risk-free interest rate r_f plus the premium-risk, namely $a_i = r_f + \lambda_i \sigma_i$, for $i = PA$ and PL , λ_i is the Sharpe ratio of the asset and liability. So, we can obtain the risk-neutral stochastic equations:

$$\begin{aligned} \frac{dPA}{PA} &= r_f dt + \sigma_{PA}(dz_{PA} + \lambda_{PA} dt) \\ \frac{dPL}{PL} &= r_f dt + \sigma_{PL}(dz_{PL} + \lambda_{PL} dt) \end{aligned} \quad (10)$$

The Brownian process $dz_{PA}^* = dz_{PA} + \lambda_{PA} dt$ and $dz_{PL}^* = dz_{PL} + \lambda_{PL} dt$ are new Geometric Brownian Motions, hence the equation of funding ratio ($P=PA/PL$) simulation is:

$$\frac{dP}{P} = (\sigma_{PL}^2 - \sigma_{PL}\sigma_{PA}\rho_{PAPL})dt + \sigma_{PA}dz_{PA}^* - \sigma_{PL}dz_{PL}^* \quad (11)$$

As approved by Villani (2007), the equation can be rewritten as $\frac{dP}{P} = \sigma^* dz^P$, where $\sigma^* = \sqrt{\sigma_{PL}^2 + \sigma_{PA}^2 - 2\rho_{PAPL}\sigma_{PA}\sigma_{PL}}$ and z^P is a Geometric Brownian motion under new risk neutral measure \tilde{Q} . Hence expectation value of discounted cash-flow of put options, equation (6), is

$$\begin{aligned} &-e^{-r_f T} E_Q[\max(PA_T - PL_T, 0)] \\ &= -e^{-r_f T} E_Q[\max PL_T (\frac{PA_T}{PL_T} - 1, 0)] \\ &= -e^{-r_f T} E_Q[\max PL_T (P - 1, 0)] \\ &= PL_0 E_{\tilde{Q}}[\max(p - 1, 0)] \\ &= PL_0 E_{\tilde{Q}}[g(P_T)] \end{aligned} \quad (12)$$

The simulation of risk-neutral price P is:

$$P(t + dt) = P(t) \exp[(-0.5\sigma^{*2})t + \sigma^* \varepsilon(t)\sqrt{dt}] \quad (13)$$

where $\varepsilon(t) \sim N(0,1)$ is a standard normal distribution. Therefore, if we know the value of σ_{PL} , σ_{PA} , ρ_{PAPL} and P_0 , it's possible to compute, at any time t , the funding ratio P under the risk-neutral probability \tilde{Q} simulating the standard Normal distribution $\varepsilon(t)$.

Finally, it is possible to implement the Monte Carlo simulation to approximate

$$E_{\tilde{Q}}[g(P_T)] \approx \frac{1}{n} \sum_{i=1}^n g(P_i) \quad (14)$$

where n is the number of simulated-paths effected, P_i for $i = 1 \ 2 \ \dots \ n$ are the simulated values and $g(P_i) = \max(0; P_i - 1)$. We assume the stochastic process ε_i follows $N(I, I)$ to derive the evolution of funding ratio P (PA/PL) in software R.

Now we report the results of numerical simulations of Margrabe pension option. To compute the simulations we have assumed that the number of simulated-paths n is equal to 50,000, and the parameters values are the same as previous example: $\sigma_{PA}=18\%$, $\sigma_{PL}=5\%$, $\rho_{PAPL}=0.5$, $T=15$.

Table (3) summarizes the results of pension put simulations. The first column and the second one indicate the values of optioned asset PA and delivery asset PL while the third column gives the exchange prices using Margrabe (1978) formula (5). For each option we have reported four results given by Monte Carlo's simulation and we can observe that the simulated values are very close to true ones based on analytical methods. The last column presents the Standard Average Error (SAE) between the four simulated prices and the true value. The SAE is:

$$SAE = \frac{\sum_{i=1}^k \frac{|True - Sim|}{True}}{k} \quad (15)$$

where $k=4$ the number of simulation conducted. We can observe that the error ranges from 0.12% up to 3.68%. Moreover, we denote by bold type the simulations that are closer then others to true value.

Table 3: Simulation Prices of European Pension Exchange Option, \$m

PA	PL	Margrabe	1 st MC Sim	2 nd MC Sim	3 rd MC Sim	4 th MC Sim	SAE
50	100	52.86	52.89	52.78	52.80	52.95	0.0012
80	100	33.37	33.47	33.22	33.50	33.24	0.0038
100	100	24.47	24.47	25.00	24.12	24.48	0.0092
120	100	18.02	17.67	17.50	18.01	18.01	0.0124

150	100	11.54	12.05	12.41	11.23	11.55	0.0368
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This result shows the good approximation obtained with Monte Carlo simulation that validates the methodology presented previously. In addition, the Monte Carlo method used here can be very helpful to an increasing literature that use the contingent claim approach to value complexities embedded in pension liabilities, i.e. longevity improvements or early retirements, similar as an American exchange option.

V. Pension Liabilities in Incomplete Markets: The Stulz Model

Finally we consider previous example assuming the pension fund is integrated with the sponsored corporation. The table below summaries how the value of at-the-money pension put varies in respect to the characteristic of corporation business, given the illustrative example

Table 4: Value of at-the-money pension put option¹¹ as a function of correlation, ratio of market value of corporation to pension assets or time to maturity, (\$m)

	T (years)	1	5	10	15	20	25
P_{PACA}	(PA/CA)%						
-1.0	10%	\$0	\$0	\$0	\$0	\$3	\$26
	50%	\$0	\$0	\$0	\$5	\$43	\$125
	100%	\$0	\$0	\$0	\$35	\$111	\$215
0	10%	\$0	\$0	\$0.08	\$3	\$15	\$50
	50%	\$0	\$0.15	\$7	\$32	\$84	\$170
	100%	\$0	\$2	\$19	\$59	\$125	\$222
1.0	10%	\$0	\$0	\$0.23	\$4	\$22	\$65
	50%	\$0	\$1	\$17	\$54	\$118	\$212
	100%	\$0.0043	\$7	\$36	\$86	\$160	\$262

Notes: numbers are rounded.

The pension put values increases if either the correlation between corporation business and pension assets, or the ratio of pension asset to corporation value, or term to maturity¹² increases. An important difference with the stand-alone case in previous section is that the value of the pension put depends on the connection between corporate and pension assets. When the values of pension asset and corporate assets move towards the same direction, the pension put has value even in immediate future (one year) and the

¹¹ We assume the volatility of corporation assets is 21% (σ_{CA}), nominal pension asset value at time t_0 is \$100m, and pension liability at expiry date (strike price) is adjusted according to time to maturity between 1 and 25 years.

¹² The result shows the pension put value goes downwards after some long term threshold eventually.

value increases along with the term to maturity. The higher proportion of pension assets accounted for corporation assets, the higher value of pension put – corporation are more likely to default on pension liabilities. If the connection between pension and company assets diminishes, the value of the put option decreases.

The equation (14) and table above show that other factors are also important for the valuation of the pension put, they are:

- ❖ The relative size in market value between underlying corporation and pension fund. When the value of the company increases relative to the value of the pension fund, the value of the collateral increases. This will enhance the safety of pension fund, all other things equal, the value of the rainbow put option is lower.
- ❖ The volatility of corporate assets and corporate value. In general, an increasing volatility, both of corporate assets and pension assets, lead to high put option values,
- ❖ The correlation between corporate and pension assets. If the correlation between corporate assets and pension assets increases, the value of the put option will also increase. In the case of a low or even negative correlation, there will be a high probability that the put option will end up out of money. The payoff depends on the maximum of both corporate assets and pension assets. It is logical given that there is a greater chance that a low value of pension assets goes with a high value of corporate assets, thus giving better protection against default. For example, companies like in banking or insurance industries, with fixed corporate assets characteristic, may actually reduce its overall risk by allocating higher percentage shares in their pension fund portfolios.

Rainbow option pricing model is helpful to provide a parametric overview of risk imposed on the whole corporation incorporating untradable and long term pension obligation. This method can also be used by public pension insurance schemes (Pension Protection Fund in the UK and Pension Benefit Guaranty Corporation in the US) to set appropriate risk premium arising from parent companies' defaulting.

In contrast to pension value from standalone Black Scholes and Margrabe valuation models¹³, results from Stulz model integrating corporation asset, are more volatile given the proportion of pension asset over corporation asset is significant. For instance, if the pension asset accounts for half of corporation asset, the pension put value can vary ten-fold depending on the correlation between pension asset and corporation assets values. Usual business practice when dealing with rainbow options is to take constant correlation coefficients - and this despite empirical analyses unambiguously showing that “there is no such thing as constant volatility and correlation”. Considering the correlation between pension asset and corporation value changes constantly and is difficult to predict¹⁴ in

¹³ The pension put values of illustrative example calculated in section III and IV (15 years liability) for Black Scholes and Margrabe models are \$28 and \$20-\$35 respectively.

¹⁴ The correlation is affected by pension asset investment strategy and performance, and corporation business strategy, market conditions and etc.

practice, therefore the pricing bounds (if exist) can be so wide which yields little value to determine hedging strategy accurately.

VI. Conclusions

In this paper, various option pricing models are used to provide analytical solutions to valuing defined pension liabilities (or securitised parts of pension liabilities) in incomplete markets. Unlike when markets are complete, there is not a single arbitrage-free price for liabilities but instead a range of prices consistent with the absence of arbitrage. We analyse and compare application incomplete market variants of the following models to pension liabilities: i) the standard Black-Scholes(1973) option model ii) the Margrabe(1978) exchange options model; iii) the Stulz (1982) rainbow option model to price pension schemes liabilities. No matter which approach is used, we find the range of liability prices to be broad, implying it is difficult to put a precise market value on pension liabilities. However, we also find that the implication for strategic asset allocation is relatively minimal; strategic asset allocations, at least in the models we looked at, appear to be relatively robust to incomplete markets. This conclusion is more about risk exposure than the financial instruments to achieve a given level of risk exposure as the desired financial instruments may depend considerably on the degree of market completeness.

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