Optimal asset allocation of pension fund under Value-at-Risk constraints

Liu Lei, Chen Lei

Watson Wyatt Worldwide (GRS Wuhan)

Abstract

In this paper, we consider the optimal asset allocation strategy of a defined-contribution pension scheme in both accumulation phase and decumulation phase. We model the optimal problem from the view of a defined-contribution pension plan member, who is aiming at maximizing his expected utility from consumptions and bequest. Assume that pension assets can be invested into three assets: cash, stock and bonds. The optimal problem is solved in the general case where asset returns and wage growth are all stochastic and correlated. Long-term Value-at-Risk is used as the risk constraint to ensure a minimum level of replacement ratio and solvency ratio under general conditions. We process a numerical simulation to find the optimal path for investment strategies without and with risk constraints and compare the differences between the optimal investment strategies in the two cases.

Key words: Value-at-Risk, Stochastic optimal control, Defined-contribution pension, Optimal asset allocation

1. Introduction

After suffering the end of contribution holiday for conventional defined-benefit (hereafter DB) and "perfect storm" at the beginning of 21st century, pension scheme around the world are undergoing the shift from defined-benefit scheme to defined-contribution (hereafter DC) scheme now, which means that the investment risk will be transferred meanwhile from pension plan sponsor to the individual member of the plan. Besides, due to long duration of pension plan, pension funds are exposed to not only long-term market risk but also longevity risk. So when dealing with the optimal assets allocation for pension fund, it is necessary to take such special risk profile of pension plan into consideration. In this paper, we examine a continuous time stochastic model for a DC pension fund. Our main purpose is analyzing optimal asset allocation strategy of pension fund under risk constraints to ensure the stability and security of pension plan from the view of pension fund member.

The classical dynamic lifetime portfolio selection in a continuous time model is developed by Merton (1969, 1971), who considers the market with a constant interest rate. Since pension fund would maintain for a long horizon, it is not realistic to assume the interest rate to be deterministic. After stochastic interest rate models were introduced, it

soon was widely applied into assets allocation problem for pension fund. Such as Boulier et al. (2001), they deal with the optimal assets allocation problem of DC pension fund with a guarantee given on benefits under stochastic interest rates.

Other than interest rate risk, salary risk, contribution risk, mortality risk and inflation risk are always considered in this framework. Cairns et al. (2000) introduce non-hedgeable salary risk into the optimal asset allocation problem in accumulation phase of a DC pension plan. Cairns (2000) develops a continuous-time stochastic pension fund model with Markov control strategies over the contribution rate and asset-allocation. He finds the optimal proportions invested in each risky asset are constant relative to one another. Battocchio and Menoncin (2002) use a stochastic dynamic programming approach to model a DC pension fund in a complete financial market with stochastic investment opportunities and two background risks, salary risk and inflation risk. They give a closed form solution to the asset allocation problem and analyze the behavior of the optimal portfolio with respect to salary and inflation.

Risk profile of pension plan also attracts more attention. Blake et al. (2001) summarize a range of asset-return models and asset-allocation strategies, and estimate Value-at-Risk of DC plans under different asset-allocation strategy. They find DC plan can be very risky relative to a DB benchmark and the estimation of Value-at-Risk will be very sensitive to the choice of asset-allocation strategy. Haberman and Vigna (2002) use a dynamic programming approach to derive a formula for optimal investment allocation in a DC scheme and compare three risk measures to analyze the final net replacement ratio achieved by members. They suggest that when the choice of investment strategy is determined, risk profiles of individual and different risk measures are both important factors which should be taken into consideration. Menoncin (2005) consider the asset allocation problem of a pension fund which follows a pay-as-you-go rule, and the members of pension fund include workers and pensioners. He finds that when the number of worker grows faster than pensioners, the optimal portfolio will be constantly riskier than Merton's portfolio. Menoncin and Scaillet (2006) analyze the asset allocation problem for a pension fund either with a DC or a DB pension scheme base on the assumption that both contributions and pensions can be perfectly spanned on the complete financial market. They prove that under a very special case, the optimal portfolio for defined-contribution is almost always less risky than the optimal portfolio for DB.

Compared with accumulation phase of pension fund, the decumulation phase is paid on less attention. There are few literature referred to this topic. For many countries, life annuity is the common instrument for defined contribution pension plan. Charupat and Milevsky(2002) derive a optimal utility-maximizing asset allocation between a risky and risk-free asset within a variable annuity contract, they find for a constant relative risk aversion preference and geometric Brownian motion dynamics, the optimal asset allocation during decumulation phase is identical to the accumulation phase. Blake et al. (2003) compare three different distribution programs for a male DC plan member retiring at age 65, Purchased life annuity, Equity-linked annuity with a level, life annuity purchased at age 75 and Equity-linked income-drawdown with a level, life annuity

purchased at age 75, they argue that the best program depend on the plan member's attitude to risk. Battocchio et al. (2003) show a unique optimal asset allocation of pension fund under mortality risk for both accumulation phase and decumulation phase. They prove that risky assets should decrease through time in accumulation phase and increase through time in decumulation and suggest that it is not optimal to manage the two phase separately.

In this paper, we extend the stochastic model for pension fund presented by Battocchio et al. (2003) and consider a case with stochastic interest rates and salary. Accumulation phase and decumulation phase of the pension plan are both taken into consideration. Our optimization problem is to maximize pension fund member's utility at the terminal of pension plan, which is measured by CRRA utility function with respect to the consumptions in decumulation phase and bequest. We assume the survival probabilities follow the Gompertz-Makeham mortality model. Then the objective function of our optimal problem can be expressed by the expected utility of pension member. We assume the pension asset can be invested into three assets, cash, stock and bond. As for the behavior of financial market, Vasicek model is adopted to describe the behavior of instantaneous riskless interest rate; Geometrical Brown Motion is used to describe the behavior of stock and bond price. The correlations between the three assets also are taken into consideration. For salary, we assume it is age-dependent, and the increase rate is a concave function and is also with random effect from the change of bond and stock's returns and volatility as well as some non-hedgeable shock.

As a risk measure, Value-at-Risk has clear disadvantages, such as it is not a coherent risk measure and ignores the severity of tail loss. However, due to its convenience of interpretation and calculation, it is widely accepted in practice. In this paper, we use Value-at-Risk as the risk measure to control the risk of pension fund on the purpose to ensure a minimum level of consumption in decumulation phase under general conditions. There are few papers dealing with the optimal assets allocation problem of pension fund under Value-at-Risk constraints. Blake et al. (2001) consider DC designs in the accumulation phase using simulations. Gabih and Wunderlich (2005) investigate the optimal portfolio strategies with bounded shortfall risk constraint measured by Value-at-Risk. They give the analytic expressions for the optimal terminal wealth and portfolio strategies with martingale representation method, which is developed by Cox and Huang (1989, 1991). Yiu (2004) solves the optimal portfolio problem with a Value-at-Risk constraint by applying dynamic programming techniques to derive the Hamilton-Jacobi-Bellman equation and using the method of Lagrange multipliers to tackle the risk constraint. Hainaut and Devolder (2006) quote the method of Yiu (2004) and propose a numerical method to solve the optimal asset allocation problem under a Value-at-Risk constraint. However, VaR is mainly used to measure the risks over short horizon, when dealing with long-term risks, there will be a problem that volatility is difficult to forecast over long horizons. In this paper we use the method supplied by Dowd et al. (2001) to measure the risk of pension fund with long-term Value-at-Risk.

The paper is organized as follows. Section 1 is introduction and literature review. Section 2 presents the framework of modeling including stochastic processes describing financial

market, the process of contribution and distribution, the dynamics of fund wealth, objective function and mortality model, as well as risk constraint for pension plan. Section 3 presents the stochastic optimal control problem. Section 4 shows main results of numerical simulations. Section 5 concludes.

2. The model

2.1 The behavior financial market

We consider a complete financial market with no arbitrage, frictionless and continuously open. The uncertainty involved by financial market can be described by a standard, independent, two-dimensional Brownian motion (Ws(t), Wr(t)), which is defined on a complete probability space(Ω,F, P). Here F = {F (t)}, t≥0 is the filtration generated by the Brownian motions. F (t) can be interpreted as the information available to the investor at time t. P is the historical probability measure.

Assume there are three kinds of assets available for pension fund in financial market: cash, equity and bond.

The return of cash is the risk-free rate, which is assumed to follow an Ornstein-Uhlenbeck process (Vasicek' s model)

$$\begin{aligned} &dr(t) = \alpha(\beta - r(t))dt + \sigma_r dW_r(t) \\ &r(t_0) = r_0 \end{aligned} \tag{1}$$

where the speed of mean reversion α, the mean level attracting the interest rate β and the volatility of interest rate σ_r , are constants.

The stochastic differential equation (1) has the solution:

$$r(t) = (r_0 - \beta)e^{-\alpha t} + \beta + \sigma \int_{t_0}^{t} e^{-\alpha(t-u)} dW_r(u) \tag{2}$$

Then the price process of cash $S_0(t)$ can be expressed in the following way:

$$\begin{aligned} &\frac{dS_0(t)}{S_0(t)} = r(t)dt \\ &S_0(t_0) = S_0^0 \end{aligned} \tag{3}$$

We assume the price of a zero-coupon bond B(t, s) satisfies the following stochastic differential equation:

$$\frac{dB(t,s)}{B(t,s)} = r(t)dt + \sigma_B(s-t)(dW_r(t) + \lambda_r dt) \tag{4}$$
$$B(s,s) = 1$$

where B(t, s) is the price of the bond at time t $\in$[0,s], s is maturity. λ_r is the risk premium, which we assume to be constant. $\sigma_B(t)$ is the volatility of the zero-coupon bond whose maturity is t. It can be proved that $\sigma_B(t)$ can be expressed in the following way: (El Karoui(1998) and Vasicek(1977))

$$\sigma_B(t) = \frac{1-e^{-\alpha t}}{\alpha}\sigma_r \tag{5}$$

Then differential equation (4) can be transferred into this form:

$$\frac{dB(t,s)}{B(t,s)} = r(t)dt + \frac{1-e^{-\alpha(s-t)}}{\alpha}\sigma_r(dW_r(t) + \lambda_r dt) \tag{6}$$
$$B(s,s) = 1$$

Because Vasicek's interest rate model is a one-factor model, $W_r(t)$ is the unique source of random, so a zero-coupon bond at any time can be used to replicate all the other bonds in the market. In this paper, we use "rolling bond" as the representative of bond market. Suppose $B_K(t)$ denotes the price of rolling bond, which means the price of bond with constant maturity K at time t, then

$$\frac{dB_K(t)}{B_K(t)} = r(t)dt + \frac{1-e^{-\alpha K}}{\alpha}\sigma_r(dW_r(t) + \lambda_r dt) \tag{7}$$

Boulier et al. (2000) have shown the relationship between B(t,s) and $B_K(t)$ through cash asset $S_0(t)$:

$$\frac{dB(t,s)}{B(t,s)} = (1-\frac{\sigma_B(s-t)}{\sigma_K})\frac{dS_0(t)}{S_0(t)} + \frac{\sigma_B(s-t)}{\sigma_K}\frac{dB_K(t)}{B_K(t)} \tag{8}$$

For stock market, we consider a simple case. Assume there is only a stock or a stock market index. Since our model is built in a complete financial market, this assumption won't loss generality. Assume stock price process S(t) follows Geometric Brownian motion and is also affected by interest rate:

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_S(dW_S(t) + \lambda dt) + \sigma_{Sr}(dW_r(t) + \lambda_r dt) \tag{9}$$

where σ_S is the stock inherent volatility, λ is the risk premium related to Ws(t), and σ_{Sr} is the volatility brought by the fluctuating of interest rate. Then the whole premium of stock risk is $\sigma_S \lambda + \sigma_{Sr} \lambda_r$.

2.2 Contribution and distribution

Contributions to DC schemes are by definition the product of salary L(t) and the contribution rate γ:

$$C(t) = \gamma L(t) \tag{10}$$

Assume a fixed contribution rate. The dynamic behavior of contributions can be expressed by the dynamic behavior of salary.

$$\frac{dC(t)}{C(t)} = \frac{dL(t)}{L(t)} \tag{11}$$

It implies that the growth rate of contribution equals to the growth rate of salary.

Here we follow the model of Battocchio and Menoncin (2002), which consider the non-hedgeable risk of salary as well as the link between salary growth and returns on bond and stock. The dynamic behavior of salaries is given by:

$$\begin{aligned} &\frac{dL(t)}{L(t)} = u_L(t)dt + l_r\sigma_r dW_r(t) + l_s\sigma_S dW_S(t) + \sigma_L dW_L(t) \\ &L(t_0) = L_0 \end{aligned} \tag{12}$$

where l_r and l_s are volatility scale factors reflecting the effect of bond and stock market on salaries. $W_L(t)$ is the non-hedgeable risk of salaries, which is a one-dimensional standard Brownian motion independent of $W_S(t)$ and $W_r(t)$. Here the drift term $u_L(t)$ is time-dependent, which can describe the age-dependent trend of salaries growth. We take a special form for $u_L(t)$, where $u_L(t)=\max\{0, 0.1-(t-40)^2/4000\}$. It is a concave function of age and will take a peak value at 40 years old. It implies that the increase rate of salary will ascend along with the accumulation of experience, but slow down from mid-life onwards, because experiences will less be relied on.

Thus the dynamic behavior of contribution is

$$\begin{aligned} &\frac{dC(t)}{C(t)} = (0.1 - \frac{(t-40)^2}{4000})dt + l_r\sigma_r dW_r(t) + l_s\sigma_S dW_S(t) + \sigma_L dW_L(t) \\ &C(t_0) = \gamma L_0 \end{aligned} \tag{13}$$

For the decumulation phase, we assume individual member of DC plan would make a consumption plan for decumulation phase at retirement rather than buy a life annuity. All

investment in decumulation phase will be based on this consumption plan. On one side, investment strategy should ensure the pension fund is in the position to disburse the consumption expense. On the other side, investment strategy should try to achieve as much bequest as possible. Suppose the consumption plan is consistent with a fixed immediate annuity brought at retirement, that is to say, the amount drawn from pension account for consumption every year equals to the expected payment from a fixed immediate annuity. The other part left in pension account would continue to be used for investment. When the member dies, the final wealth in pension account would be left to bequest. Suppose amount for consumption is denoted as P(t), which is drawn at the beginning of every year. It is usually determined by the total accumulated wealth in pension account at retirement and annuity factor of that time. That is,

$$P(t) = \begin{cases} 0 & \text{if } t \le T \\ \dfrac{F(T)}{{}_T a_T} & \text{if } t > T \end{cases} \tag{14}$$

where F(T) is the wealth of pension fund at retirement. ${}_Ta_T$ is annuity factor of a fixed immediate annuity at retirement. It is the fair price of a single-life life annuity with 1 dollar income every year starting from T.

$${}_T a_T(r) = \int_T^{\infty} e^{-r(t-T)} {}_t P_T dt \tag{15}$$

where ${}_tP_T$ is the conditional probability that an individual aged T will survive to age t.

2.3 Fund wealth

Suppose ω_0, ω_1, ω_2 denote the investment percentage of cash, stock and bond respectively, and $\omega_0+\omega_1+\omega_2=1$.Contributions and pensions are both paid at the beginning of every period, and pension member make investment strategy accordingly. So the fund wealth process F(t) is given by:

$$\begin{aligned} dF(t) &= F(t)\cdot[\omega_0 \frac{dS_0(t)}{S_0(t)} + \omega_1 \frac{dS(t)}{S(t)} + \omega_2 \frac{dB_K(t)}{B_K(t)}] + C(t)dt - P(t)dt \\ F(t_0) &= C(t_0) = \gamma L_0 \end{aligned} \tag{16}$$

The first term can be viewed as the change due to the fluctuation of asset price. The other two terms can be viewed as the change of portfolio composition, which should be financed either by member's new contribution or by the amount drawn from pension fund for consumption. Together with equations (3), (7) and (9), the SDE can be written as

$$dF(t) = F(t)[r(t) + \omega_1(\sigma_S\lambda + \sigma_{Sr}\lambda_r) + \omega_2 \cdot \frac{1-e^{-\alpha K}}{\alpha}\sigma_r\lambda_r]dt + F(t)\omega_1\sigma_S dW_S(t)$$
$$+ F(t)(\omega_1\sigma_{Sr} + \omega_2 \cdot \frac{1-e^{-\alpha K}}{\alpha}\sigma_r)dW_r(t) + C(t)dt - P(t)dt \tag{17}$$

For convenience, we define:

$$\omega = \begin{bmatrix}\omega_1 & \omega_2\end{bmatrix}$$
$$\Omega = \begin{bmatrix}\sigma_s\lambda + \sigma_{sr}\lambda_r \\ \lambda_r \dfrac{1-e^{-\alpha K}}{\alpha}\end{bmatrix}$$
$$W = \begin{bmatrix}W_S & W_r\end{bmatrix}'$$

$$\Lambda = \begin{bmatrix}\sigma_S & \sigma_{Sr} \\ 0 & \dfrac{1-e^{-\alpha K}}{\alpha}\end{bmatrix}$$

After simplification, equation can be written as

$$dF(t) = (F(t)\cdot\omega\cdot\Omega + F(t)r(t) + C(t) - P(t)dt + \omega\cdot\Lambda\cdot F(t)dW \tag{18}$$

2.4 Mortality model

There are many well-known models for mortality rates over the span of life. In 1825 Benjamin Gompertz found human mortality shown some exponential patterns for most age. In 1860, Makeham improved Gompertz's model by adjusting the distribution for higher age. Although the theory of mortality models has developed much after that, Gompertz-Makeham model is still one of the most popular mortality models nowadays. In this paper, we will use Gompertz-Makeham on the consideration that it is still used in actuarial science today and it serves as the foundation of later survival model. Under the framework of Gompertz-Makeham model, the force of mortality obeys:

$$M_x = M + \frac{1}{b}e^{(x-m)/b} \tag{19}$$

where positive constant M stands for the deaths not related with age, and m and b are modal and scaling parameters respectively. Thus, the survival probability equals to

$${}_tP_{t0} = \exp\{-M(t-t_0) - \frac{1}{b}\int_{t0}^{t0+t} e^{(s-m)/b}ds\} = \exp\{-M(t-t_0) + e^{(t0-m)/b}(1-e^{t/b})\} \tag{20}$$

From equation (20), we can obtain the probability density function of death time

$$f(t) = \exp\{-M(t-t_0) + e^{(t0-m)/b}(1-e^{t/b})\} \cdot (M + \frac{1}{b} e^{\frac{t_0 - m + t}{b}}) \tag{21}$$

Based on this Gompertz-Makeham formula, Charupat and Milevsky(2001) show the expression of annuity factor ${}_Ta_T$ has the following form:

$$ {}_T a_T = \frac{b\Gamma(-(M+r)b, b(M_T - M))}{\exp\{(m-T)(M+r) + b(M - M_T)\}} \tag{22}$$

where Γ(x,y) is defined as

$$\Gamma(x,y) = \int_y^{\infty} s^{x-1} e^{-s} ds$$

2.5 Objective function

In this paper, we are aiming at finding an optimal asset portfolio from the view of pension fund member. We solve the optimal problem by maximizing the expected utility of pension fund member, which is made up of two parts, one is consumption utility, and the other is bequest utility. The utility function is defined as follows:

$$U(F,t) = \int_{t0}^{t} e^{-\beta(s-t_o)} \frac{P(s)^{\delta}}{\delta} ds + B(F,t,\phi) \tag{23}$$

where β is the subjective discount rate of pension member. Actually, the former part is the aggregate utility of consumption in decumulation phase following a CRRA utility function. In order to keep the utility function increasing and concave, it is usually assume δ <1 and doesn't equal to 0. B(F, t,Φ) is a specified "bequest valuation function", which is also assumed to be concave in F. For convenience, we assume bequest valuation function is also with the same CRRA form as consumption utility. Φ is the scale of bequest utility. Since the attitude towards bequest varies from family to family. Compared with a single or a divorced, a person with many kids would have a stronger desire for bequest. So Φ is used to describe such different degree of bequest desire. Then the utility function is expressed as

$$U(F,t) = \int_{t0}^{t} e^{-\beta(s-t_o)} \frac{P(s)^{\delta}}{\delta} ds + \phi \frac{F(t)^{\delta}}{\delta}$$

Thus our optimal problem equals to find the proper amounts for cash, bond and equity so that the expected utility of final wealth at time t_0 when the pension plan starts is maximized. That is,

$$\max_{\omega} E_{t0}^{\tau} [U(F(\tau),\tau)]$$

where τ is the stochastic death time of the member.

Assume mortality risk is independent of financial market risk, f(t) is the distribution function of death time following Gompertz-Makeham model. Then the objective function can be rewritten as:

$$\max_{\omega} E_{to}^{\tau}[U(F(\tau),\tau)] = \max_{\omega} E[\int_{t0}^{\infty} f(t)U(F(t),t)dt] \tag{24}$$

2.6 Risk constraints

Value-at-Risk is a statistical measure, generally defined as possible minimum revenue of the portfolio over a given holding period within a fixed confidence level. Formally,

$$VaR_{\mu} = \inf\{l \in R : P(L \le l) \le 1-\mu\} = \inf\{l \in R : F_L(l) \le 1-\mu\} \tag{25}$$

where μ is the confidence level, L stands for the random variable of revenue of the portfolio and $F_L()$ is its distribution function.

So far VaR is mainly used to measure the risks over short horizon in practice, when dealing with long-term risks, there will be a problem that volatility is difficult to forecast over long horizons. VaR literature itself pays less attention on longer-term risk measurement. Best-known advice is the square-root rule. However, the square-root rule will has more deviation from the real value as the periods grow. The case will be more complex if the variable to be forecasted has a trend and the horizon needed is quite long. Dowd et al. (2001) suggest that if forecasting with trends leads to explosive results over long horizons and if fluctuations around a zero trend tend to cancel out, then we might as well use a simplistic approach and take a view about the average long-term values of the relevant parameters. For a pension plan, the duration is always more than 30 years. So when we deal with the risk of pension fund, we adopt the idea of Dowd et al. (2001) and use yearly data.

We assume that fund member is a risk-aversion investor. When he decides the investment strategy, he has to make a trade-off between risk and return. Here we consider risk controls in accumulation phase and decumulation phase separately. In accumulation phase, risk control is aiming at ensuring a minimum consumption level no matter how financial market performs. It is something like a guarantee of insurance product. We define the replacement ratio q(t) as

$$q(T) = \frac{P(t)}{L(T)} \qquad t \le T \tag{26}$$

which represents the ratio of consumption level to final salary.

We quantify the requirement of a minimum consumption level after retirement by a random event {q(T)≤k}, k is a constant. So the constraint on minimum consumption level

equals to constraint on the probability of {q(T)≤k}. Assume that we require under 100μ% condition the pension fund should be in the position to take on a consumption level as much as k of final salary , which equals to constrain the probability P(q(T)≤k)≤1-μ. Expressing this idea with the method of Value-at-Risk, it holds $VaR_u(q(T))\geq k$.

However, there is some problem to choose a reasonable value for k. Jie Ren(2008) do a historical analysis of DC plan post-retirement assurance across OECD countries. He investigates four static investment strategies, all invested in stocks, all invested in bonds, 65% invested in stock and 35% invested in bond, and the lifestyle investment strategy (all fund are invested in stock in the first 30 years of accumulation phase and gradually shift to bond in the last 10 years). He argues a pure pension plan and annuity post retirement assurance system is quite risky. The nominal replacement ratio for representative retiree by purchasing fixed annuity would range from 36% to 106% even for the best performing strategy – lifestyle in UK and US, with the best behavior in financial market. So the choice of a reasonable value for $VaR_u(q(T))$ is really difficult. A too low level is meaningless, and a too high level is impossible to be met. Instead of a hard constraint on replacement ratio, we turn to a soft constraint. Suppose the expected replacement ratio of pension fund member is 2/3. Define the penalty function for utility function as

$$PF(q) = (\frac{2}{3} - VaR_{\mu}(q(T)) * E_{t0}(L(T))^{\delta} / \delta \tag{27}$$

It is interpreted as the additional utility for pension member resulted from a potential deviation from the expected replacement ratio. That is to say, there is a 1-μ likelihood that replacement ratio would fall to $VaR_u(q(T)$. If $VaR_u(q(T)$ is less than 2/3, then (2/3-$VaR_u(q(T))*E_{to}$ (L(T)) can be viewed as the potential reduction of consumption level from an expected one, and PF(q) measures the additional negative utility brought by this potential reduction of consumption. If $VaR_u(q(T)$ is more than 2/3, then (2/3-$VaR_u(q(T))*E_{to}$ (L(T)) can be viewed as the potential increase of consumption level from an expected one, and PF(q) is the additional positive utility brought by this potential increase of consumption.

During decumulation phase, risk control is aiming at ensuring the realization of his consumption plan and meanwhile gaining as much as possible for bequest. Such risk control is similar to control the solvency risk in a DB plan, so we just call it as solvency risk for convenience. As above assumption, the amount for consumption is determined by the wealth and annuity price on retirement. So we can define the solvency rate as

$$\rho(t) = \frac{F(t)}{\sum_{s=t_0}^{\tau} \frac{P(s)}{(1+r(t))^{s-t}}} = \frac{F(t)}{\sum_{s=T}^{\tau} \frac{F(T)}{{}_T a_{t0}(1+r(t))^{s-t}}} \qquad \mathrm{T} < \mathrm{t} \leq \tau \tag{28}$$

which is the ratio of current fund wealth to the total amount of wealth due to drawn from pension fund for consumption. We assume that we require under 100μ% condition the pension fund should be in the position to disburse at least 90% amount of consumption

expenditure, which equals to constrain $P(\rho(t)\leq 0.9)\leq 1-\mu$. Expressing this idea with the method of Value-at-Risk, it holds

$$\mathrm{VaR}_{\mu}(\rho(t)) \geq 0.9 \qquad T<t\leq\tau \tag{29}$$

3. The optimal portfolio

3.1 Optimization in the unconstrained case

First, we will analyze the optimal problem without risk constraints. The problem of optimal portfolio selection is formulated as

$$\begin{aligned} &\max_{\omega} E_s^{\tau}[U(F(\tau),\tau)] \\ &\text{s.t} \quad dF(t) = (F(t)\cdot\omega\cdot\Omega + F(t)r(t) + C(t) - P(t)dt + \omega\cdot\Lambda\cdot F(t)dW \quad F(t_0)=F_0 \end{aligned} \tag{30}$$

Merton (1969) provides a general method to deal with the problem of optimal portfolio selection in continuous time under a relative simple market structure without any background risks. In this subsection, we will follow the method supplied by him. For any time $s \in [t_0, \tau]$, define the optimal value function $J(F_s, s)$ as the optimal value that can be obtained starting at time s in state F_s . Thus,

$$\begin{aligned} &J(F_s,s) = \max_{\omega} E_s^{\tau}[U(F(\tau),\tau)] = \max_{\omega} E[\int_{ts}^{\infty} f(t)U(F(t),t)dt] \\ &= \max_{\omega} E[\int_s^{\infty}\int_s^{\tau} \exp\{-M(t-t_0) + e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot(M+\frac{1}{b}e^{\frac{t_0-m+t}{b}})\cdot e^{-\beta(s-t_o)}\frac{P(s)^{\delta}}{\delta}dtd\tau + \int_s^{\infty} B(F,t)f(t)dt] \\ &= \max_{\omega} E[\int_s^{\infty}\cdot\exp\{-M(t-t_0) + e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^{\delta}}{\delta}dt + \int_s^{\infty} B(F,t)f(t)dt] \\ &= \max_{\omega} E[\int_s^{\infty}\cdot\exp\{-M(t-t_0) + e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^{\delta}}{\delta} + B(F,t)f(t)dt] \\ &\text{s.t} \quad dF(t) = (F(t)\cdot\omega\cdot\Omega + F(t)r(t) + C(t) - P(t))dt + \omega\cdot\Lambda\cdot F(t)dW \quad F(s)=F_s \end{aligned} \tag{31}$$

We break up the integral in (31) as

$$\begin{aligned} &J(F_s,s) = \max_{\omega} E[\int_s^{s+\Delta t}\cdot\exp\{-M(t-t_0) + e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^{\delta}}{\delta} + B(F,t)f(t)dt \\ &+ \int_{s+\Delta t}^{\infty}\cdot\exp\{-M(t-t_0) + e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^{\delta}}{\delta} + B(F,t)f(t)dt] \end{aligned} \tag{32}$$

where Δt is taken to be a very small interval. Since the control function $\omega(t)$ should also be optimal for the problem starting at $s+\Delta t$ in state $F(s+\Delta t)=F(s)+\Delta F$, then

$$J(F_s,s)=\max_\omega E[\int_s^{s+\Delta t}\cdot\exp\{-M(t-t_0)+e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^\delta}{\delta}+B(F,t)f(t)dt$$
$$+\max_\omega\int_{s+\Delta t}^{\infty}\cdot\exp\{-M(t-t_0)+e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^\delta}{\delta}+B(F,t)f(t)dt]$$
$$\text{s.t } dF(t)=(F(t)\cdot\omega\cdot\Omega+F(t)r(t)+C(t)-P(t)dt+\omega\cdot\Lambda\cdot F(t)dW \quad \text{F(s)=F}_\text{s} \tag{33}$$

Then we obtain,

$$J(F_s,s)=\max_\omega E\{[\exp\{-M(t-t_0)+e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^\delta}{\delta}+B(F,t)f(t)]\Delta t$$
$$+J(Fs+\Delta F,s+\Delta t)\} \tag{34}$$

Assume J is twice differentiable, we expand the function on the right around (F_s, s),

$$J(F_s+\Delta F,s+\Delta t)=J(F_s,s)+J_F(F_s,s)\Delta F+J_t(F_s,s)\Delta t+\frac{1}{2}J_{FF}(F_S,s)\Delta F^2+h.o.t \tag{35}$$

Recall the differential equation in (30), we obtain,

$$\Delta F=(F(t)\cdot\omega\cdot\Omega+F(t)r(t)+C(t)-P(t)\Delta t+\omega\cdot\Lambda\cdot F(t)\Delta W$$

$$\Delta F^2=(F(t)\cdot\omega\cdot\Omega+F(t)r(t)+C(t)-P(t)^2\Delta t^2+(\omega\cdot\Lambda)(\omega\cdot\Lambda)'\cdot F(t)^2\Delta W^2$$
$$+2(\omega\cdot\Omega+F(t)r(t)+C(t)-P(t))(\omega\cdot\Lambda\cdot F(t))\Delta t\Delta\omega \tag{36}$$
$$=(\omega\cdot\Lambda)(\omega\cdot\Lambda)'\cdot F(t)^2\Delta t+h.o.t$$

Substitute from (36) into (35) and put the result into (34) to get

$$J(F_s,s)=\max_\omega E\{[\exp\{-M(t-t_0)+e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^\delta}{\delta}+B(F,t)f(t)]\Delta t$$
$$+J(F_s,s)+J_F(F_s,s)(F(t)\cdot\omega\cdot\Omega+F(t)r(t)+C(t)-P(t)\Delta t+J_F(F_s,s)\cdot\omega\cdot\Lambda\cdot F(t)\Delta W \tag{37}$$
$$+J_t(F_s,s)\Delta t+\frac{1}{2}J_{FF}(F_S,s)(\omega\cdot\Lambda)(\omega\cdot\Lambda)'\cdot F(t)^2\Delta t+h.o.t\}$$

The only stochastic term in (37) is ΔW and its expectation is zero, then we divide the both sides of (37) by Δt and let Δt→0, thus

$$0=\max_\omega[\exp\{-M(t-t_0)+e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^\delta}{\delta}+B(F,t)f(t)+J(F_s,s)$$
$$+J_F(F_s,s)(F(t)\cdot\omega\cdot\Omega+F(t)r(t)+C(t)-P(t)+J_t(F_s,s)+\frac{1}{2}J_{FF}(F_S,s)(\omega\cdot\Lambda)(\omega\cdot\Lambda)'\cdot F(t)^2] \tag{38}$$

Since equation (38) holds for all s $\in[t_0, \tau]$, we can obtain Hamilton-Jacobi-Bellman equation,

$$0=\max_{\omega}[\exp\{-M(t-t_0)+e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^{\delta}}{\delta}+B(F,t)f(t)+J_F(F_s,s)(F(t)\cdot\omega\cdot\Omega$$
$$+F(t)r(t)+C(t)-P(t)+J_t(F_s,s)+\frac{1}{2}J_{FF}(F_S,s)(\omega\cdot\Lambda)(\omega\cdot\Lambda)'\cdot F(t)^2] \tag{39}$$

Take first order condition with respect to ω',

$$J_F\cdot F(t)\cdot\Omega'+J_{FF}F(t)^2\cdot\omega\Lambda\Lambda'=0 \tag{40}$$

Since Λ is invertible, we can obtain

$$\omega^*=-\frac{J_F}{J_{FF}F(t)}\Omega'(\Lambda\Lambda')^{-1} \tag{41}$$

Take ω* into Hamilton-Jacobi-Bellman equation and carry out some simplification,

$$0=\exp\{-M(t-t_0)+e^{\frac{to-m}{b}}(1-e^{\frac{t}{b}})\}\cdot e^{-\beta(t-t_o)}\frac{P(t)^{\delta}}{\delta}+B(F,t)f(t)+J_t(F,t)$$
$$+J_F\cdot(F(t)r(t)+C(t)-P(t))-\frac{1}{2}\frac{J_F{}^2}{J_{FF}}(\Lambda^{-1}\Omega)'(\Lambda^{-1}\Omega) \tag{42}$$

The sufficient conditions for (39) is

$$J_{FF}\cdot F(t)^2\cdot\Lambda\Lambda'<0 \tag{43}$$

So meeting such sufficient conditions equals to require the value function is strictly concave.

Then solve partial differential equation (42) to get an explicit form of a strictly concave value function, and we will have the analytic expression for optimal investment strategy. However, it is the most challenging task in the stochastic dynamic control approach. In section 4 we will process a numerical simulation to show the result instead.

3.2 Optimization meeting risk constraints

The problem of optimal portfolio selection is formulated as

$$\max_{\omega}\ E_s^{\tau}[U(F(\tau),\tau)]-PF(q)$$
$$\text{s.t}\quad dF(t)=(F(t)\cdot\omega\cdot\Omega+F(t)r(t)+C(t)-P(t)dt+\omega\cdot\Lambda\cdot F(t)dW\quad F(t_0)=F_0 \tag{44}$$
$$\mathrm{VaR}_{\mu}(\rho(t))\geq 0.9\quad T<t\leq\tau$$

The purpose of fund member is to choose an optimal investment strategy to maximize the expected utility function under a soft risk constraint and a hard constraint. The soft constraint implies that the pension member has an expected replacement ratio and under a 1-u worse condition, the potential deviation from the expected replacement ratio would have an additional effect on his utility. In order to maximize the expected utility, investment strategy should ensure the distribution of replacement ratio in a range as high as possible. The hard risk constraint is at least 90% of his consumption plan should be achieved in more than 100μ% cases.

Even though VaR is widely adopted method to measure risk in practice, there are few papers using this method together with stochastic control problem. One of the reasons comes from the disadvantage of VaR, that is, if a random variable doesn't follow a standard distribution, its VaR won't be expressed in an analytic way. The distribution of fund wealth is a mixture of lognormal distribution and normal distribution and the two kinds of distribution are related each other. So the VaR of fund wealth is hard to express in an analytic way. Instead of showing an analytic solution, we will count on the numerical simulation to get the results.

4. A numerical simulation

This section will process a numerical simulation to illustrate the dynamic path of optimal portfolio investment strategy. Suppose the representative member is a male. He joins the pension plan since 25 years old and he will retire at 65. The following table shows the parameters in our model, which describe the behavior of financial market, defined-contribution process, mortality and utility function. Parameters of financial market are calibrated by the historical data of UK from 1900 to 2007. We cite the parameters of mortality model estimated in Milevsky (2001) based on the individual Annuity Mortality 2000 table.

Interest rate		**Stock**	
Speed of mean reversion, α	0.32	Inherent volatility of stock, σs	0.15
Mean rate, β	0.05	Volatility brought by interest rate, σsr	0.05
Volatility of interest rate, σ_r	0.01	Risk premium of stock, λ	0.31
Initial rate, r_0	0.05	Initial value of stock, s0	1
Rolling bond		**Mortality**	
Maturity, K	10	Death time not related with age, M	0
Risk premium of bond, λr	0.07	Modal parameter, m	88.18
Initial value of bond, bo	1	Scale parameter, b	10.5
Defined-contribution process		**Utility**	
Volatility scale factor of bond on salary, l_r	0.5	Parameter of absolute risk aversion, δ	0.05
Volatility scale factor of stock on salary, l_s	0.5	Confidence level, μ	90%
Non-hedgeable volatility of salary, σ_L	0.01	Subjective discount rate, β	0.05
Initial salary, L_0	100	Bequest scale	1
Contribution rate, γ	12%		

Figure 2 shows the survival probability of Gompertz model for a 25 years old person and 65 years old person. Based on our above parameters, the life expectancy of a 65 years old person is 82. So we set the end of our decumulation phase as 82 years old, which can be viewed as an approximation of death time. In every simulation, we generate 200 scenarios to describe the different situations of financial market and salary process. Based on these scenarios, we use Solver in Excel to solve our optimal problem respectively under the assumptions that pension member adjusts his investment strategy every year or every 5 years. Such procedure is repeated 250 times.

Figure 3 and 4 presents the distributions of optimal investment strategies with and without VaR constraints based on simulation results. It is shown that theses weights have covered a broad area, especially for stock weights, which implies that the optimal investment strategy is quite sensitive to the behavior of financial market and there is not a general optimal investment strategy.

Figure 5 illustrates the average weights for three assets based on the results of 250 simulations, from which we can find such trend: during the first several years of accumulation phase, a relatively conservative investment strategy is preferred. A diversified investment strategy is chosen with similar weights on cash, bond and a little higher weight on stock. As time passed by, investment strategy is gradually shifting to be more and more aggressive. When the pension fund has accumulated for more than 20 years, when it can be viewed as a relative mature fund, stocks will take up most of the weights. Such trend is consistent with the general rules of fund management. At retirement, there is a significant change of investment strategy. Because investment strategy in accumulation phase affect utility just by affecting the accumulation of fund wealth until retirement and then determining the consumption level. So the investment is aiming at accumulating fund wealth as much as possible until retirement. At the time of retirement, the wealth in pension fund equals to all the money that will be drawn from pension account for consumption in decumulation phase. Then the investment goal in decumulation phase is to ensure the achievement of consumption plan at first, and then consider a bequest. Hence, the desire for security takes the place of profit. That is to say, the investment goal has changed since retirement, so does the investment strategy. After retirement, pension fund member returns to relatively conservative investment strategy. Particularly, the weights of stock will suffer a sharp decrease at retirement, but after that, the weights of stock will increase gradually as pension plan close to terminal.

By comparing the investment strategy adjusted every year and adjusted every 5 years, we find the investment strategies adjusted every 5 years and every year are almost the same in accumulation phase. But the replacement ratios are different. Figure 6 illustrates that the distribution of average replacement ratio and VaR of replacement ratio adjusted every year all lie on the right side of the ones adjusted every 5 years. It implies that the investment strategy adjusted every year has a better performance on the whole and would suffer the bad financial situation with a smaller probability. In decumulation phase, the situation is complex. But we find an interesting phenomenon that for the investment strategy adjusted every year, pension fund member will prefer a quite conservative investment strategy with enough diversified concern and a low weight of stock when the

pension fund is close to terminal, while for the investment strategy adjusted every 5 years, it still stably follows the gradually more aggressive trend.

The comparison of the solutions with and without risk constraints reveals that before retirement, the investment strategies are almost the same. But after retirement, the investment strategies with risk constraints become much less risky than those without risk constraints. We define Delta as the ratio of stock weight under risk constraints to stock weight without risk constraints. From figure 7, we find the ratio is smaller for investment strategy adjusted every year than the one with every 5 years adjustment. It means the effect of risk constraint on investment strategy adjusted every year would be more significant than the effect on investment strategy adjusted every 5 years. Because the investment strategy which is adjusted every 5 years is less sensitive to the financial market behavior and can't response promptly, it actually bears more risk resulted from non-liquidity. So under the same ability to bear risk, the choice of investment strategy adjusted every 5 year would be less risky under the consideration of a liquidity risk. When the ability to bear risk is different, we use different CRRA coefficients of utility function to distinguish such cases. Suppose the CRRA coefficient can be 0.99, 0.95 and 0.5. 0.99 means the risk preference is most risk-aversion. 0.5 means a least risk-aversion case. And 0.95 is between the two cases. From figure 7, we find the more risk-averse the risk preference is, the less effect of risk constraint would be.

In the end, we investigate the effect of different bequest attitudes on investment strategy. Suppose the scale of bequest utility function can be 10, 1 and 0.1, which stand for different bequest desire. We find in accumulation phase the investment strategy would be more risky if the bequest desire is stronger no matter whether there are risk constraints. In decumulation phase, investment strategies without risk constraints still follow such trend, while the investment strategies under risk constraints are almost the same.

5. Conclusions

In this paper, we consider the optimal assets allocation strategy of a defined-contribution pension scheme in both accumulation phase and decumulation phase. We model the optimal problem from the view of a DC pension plan member, who is aiming at maximizing his expected utility which is the sum of bequest utility of terminal wealth in pension fund and aggregated consumptions utility in decumulation phase. Assume pension assets can be invested into three assets: cash, stock and bond. The optimal problem is discussed in a general background where interest rate, the return of bond and stock, the growth of salary are all stochastic and correlated. Long-term Value-at-Risk is used as the risk constraints to ensure a minimum level of solvency ratio under general conditions. We process a numerical simulation to find the optimal path for investment strategies without and with risk constraints under the assumption that investment strategy can be adjusted every year or every 5 years.

There are some interesting results revealed from our simulations. First, there are some significant trends for investment strategy. On the whole, during the first several years of accumulation phase, a relatively conservative and diversified investment strategy is preferred. As time passed by, investment strategy is gradually shifting to be more and

more aggressive and stocks will take up most of the weights. After retirement, pension fund member will soon return to relatively conservative investment strategy. Particularly, the weights of stock will suffer a sharp decrease at retirement, but after that, the weights of stock will increase gradually as pension plan close to terminal. Secondly, the frequency of investment strategy adjustment results in differences in optimal portfolio selection. In accumulation phase, the investment strategies adjusted every 5 years and every year are almost the same. But the investment strategy adjusted every year has a better performance on the whole and would suffer the bad financial situation with a smaller probability. In decumulation phase, for the investment strategy adjusted every year, pension fund member will prefer a quite conservative investment strategy with enough diversified concern and a low weight of stock when the pension fund is close to terminal, while for the investment strategy adjusted every 5 years, it still stably follows the gradually more aggressive trend. Thirdly, we find after retirement the investment strategies under risk constraints would be less risky than those without risk constraints. And the effect of risk constraint on investment strategy adjusted every year would be more significant than the effect on investment strategy adjusted every 5 years. Meanwhile, if the investor is less risk averse, the effect of risk constraints would also be more significant. Last, the attitudes towards bequest also have effect on the choice of investment strategy. We find in accumulation phase the investment strategy would be more risky if the bequest desire is stronger no matter whether there are risk constraints. In decumulation phase, investment strategies without risk constraints still follow such trend, while the investment strategies under risk constraints are almost the same.

Reference

Battocchio, P., Menoncin, F., 2004. Optimal pension management under stochastic interest rates, wages and inflation. Insurance: Mathematics and Economics 34,79-75

Battocchio, P., Menoncin, F., Scaillet, O., 2003. Optimal asset allocation for pension funds under mortality risk during the accumulation and decumulation phases. Annals of Operations Research.

Blake, D., Cairns, A., Dowd, K., 2000. Optimal Dynamic Asset Allocation for Defined-Contribution Pension Plans. The Pension Institution, London, Discussion Paper PI 2003

Blake, D., Cairns, A., Dowd, K., 2001. PensionMetrics: stochastic pension plan design and Value-at-Risk during the accumulation phase. Insurance: Mathematics and Economics 29,187-215

Blake, D., Cairns, A., Dowd, K., 2003. PensionMetrics2: stochastic pension plan design during the distribution phase. Insurance: Mathematics and Economics 33,29-47

Milevsky, M., 1998. Optimal asset allocation towards the end of the life cycle: to annuities or not to annuities? Journal of Risk and Insurance 65, 401-426

Boulier, J.- F., Huang, S.J, Taillard, G., 2001. Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund. Insurance: Mathematics and Economics 28,173-189

Brown, J. R., Mitchell, O. S. and Poterba, J. M., 2000. The role of real annuities and indexed bonds in an individual accounts retirement program, Risk Aspects of Social Security Reform, University of Chicago Press, Chicago

Cairns A.J.G., 2000. Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time. ASTIN Bulletin 30, 19-55

Chang F.-R., 2004.Stochastic Optimization in Continuous Time, 1st ed., Cambridge University Press, Cambridge

Charupat, N., Milevsky, M., 2002. Optimal asset allocation in life annuities: a note. Insurance: Mathematics and Economics 30,199-209

Gabih A., Wunderlich R., 2005. Optimal portfolios with bounded shortfall risks. Stochastic Analysis and Applications 23, 579-594

Haberman, S.,1997. Stochastic investment returns and contribution rate risk in a defined benefit pension scheme. Insurance: Mathematics and Economics 19,127-139

Hainaut D., Pierre D., 2006. MANAGEMENT OF A PENSION FUND UNDER A VAR CONSTRAINT. http://www.gloriamundi.org/picsresources/dhpd_mpf.pdf

Menoncin, F., Scaillet, O.,2006. Optimal asset management for pension funds. Managerial Finance 32, 347-374

Merton, R.C., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 372-413

Milevsky M., 2001. Optimal Annuitization Polices: Analysis of The Options. http://www. yorku.ca/ milevsky/Papers/NAAJ2001B.pdf

Φksendal, B, 2000. Stochastic Differential Equations- An introduction with Applications, 5th ed., Springer, Berlin

Vasicek, O. 1977. An equilibrium characterization of the term structure. Journal of Economic Theory 5, 177-188

Jie.Ren., 2008. Defined contribution pension plan post-retirement assurance: a historical analysis across OECD countries. Working paper.

Vigna, E., Haberman S., 2001. Optimal investment strategy for defined contribution pension schemes. Insurance: Mathematics and Economics 28,253-262
Yiu K.F.C., 2004. optimal portfolios under a value at risk constraint. Journal of Economic Dynamics & Control 28,1317-1334

Figure 1: Age-dependent trend of salaries growth

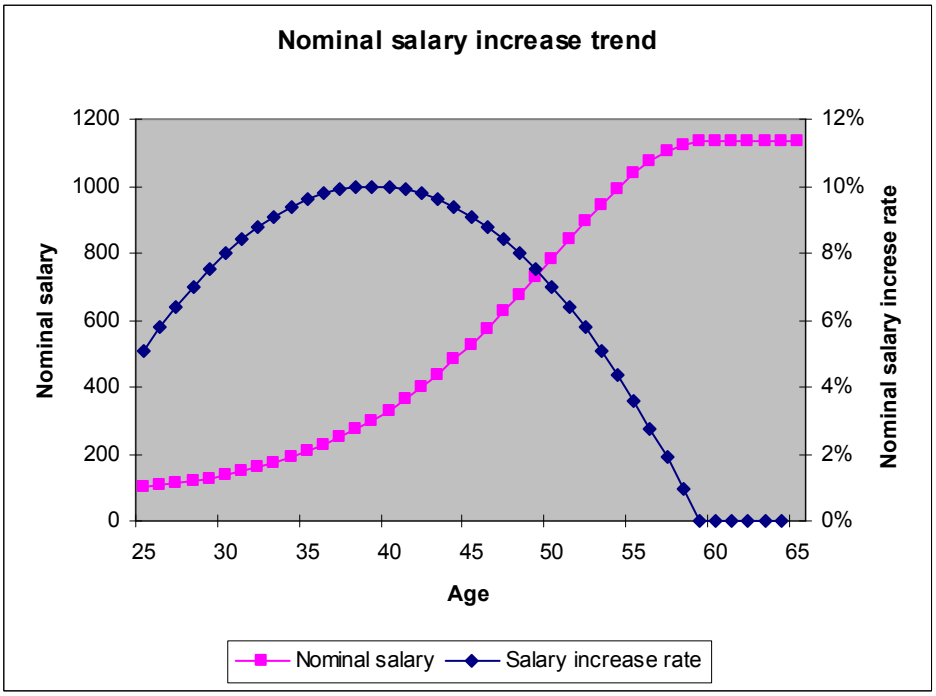
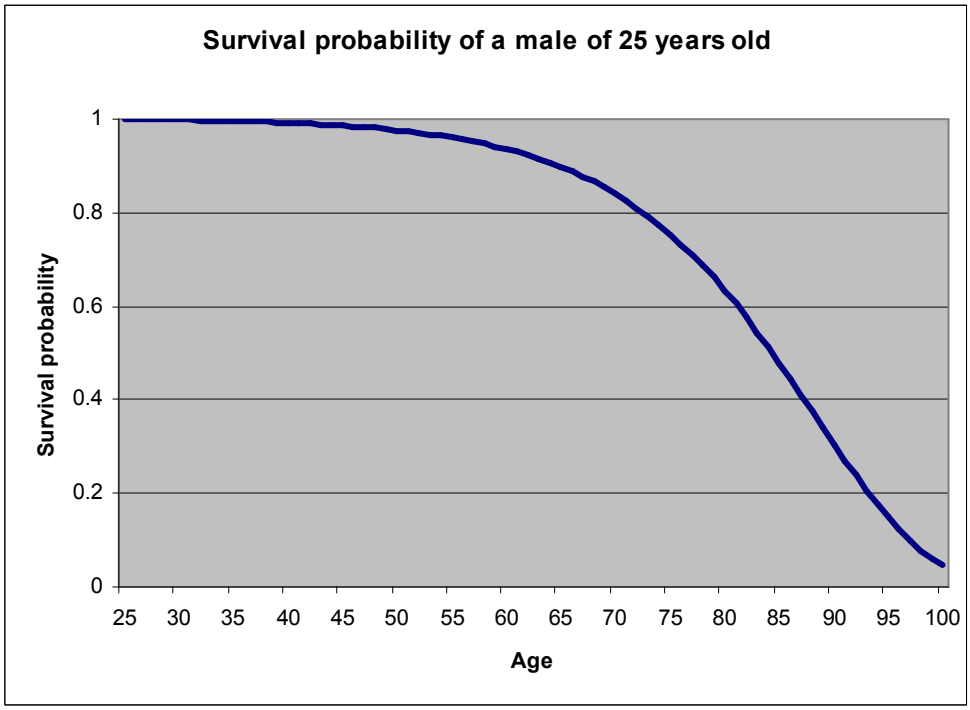


Figure 2: Survival probability of Gompertz's model



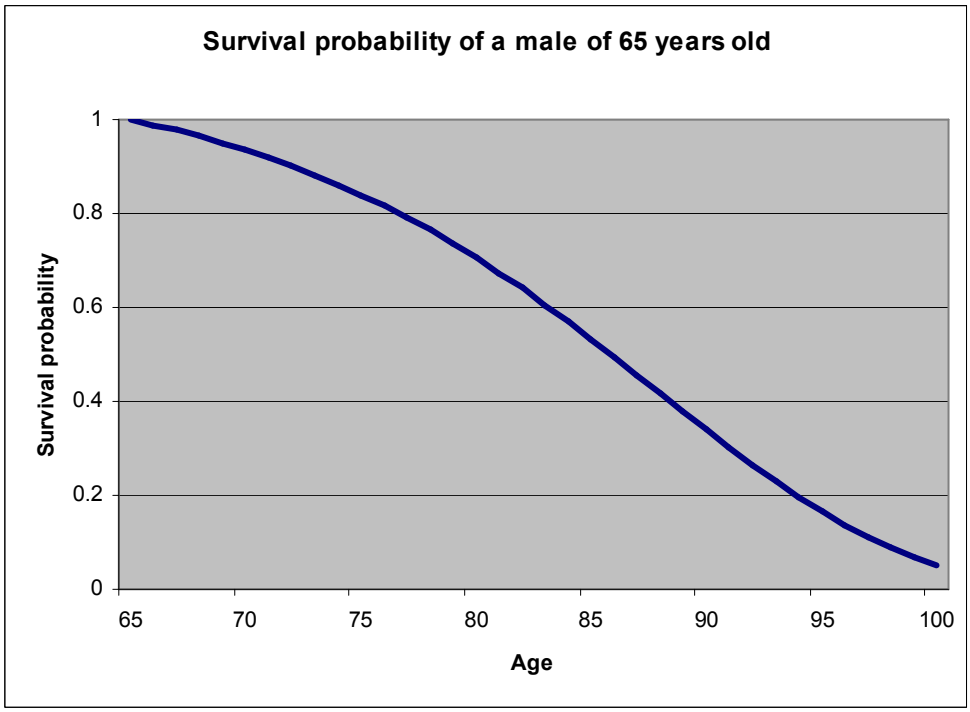


Figure 3: Distribution of optimal weights of portfolio (adjusted every 5 years)

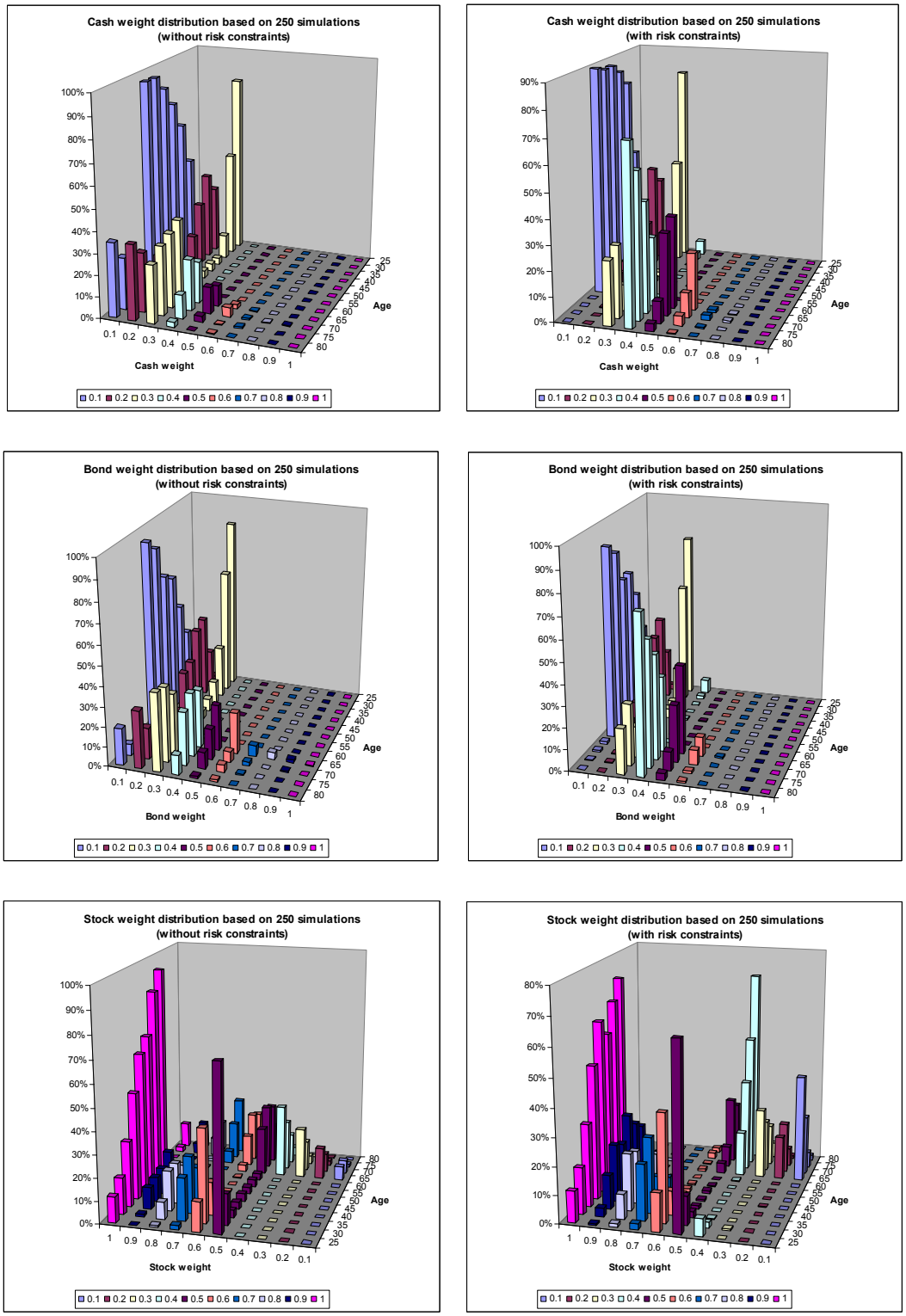


Figure 4: Distribution of optimal weights of portfolio (adjusted every year)

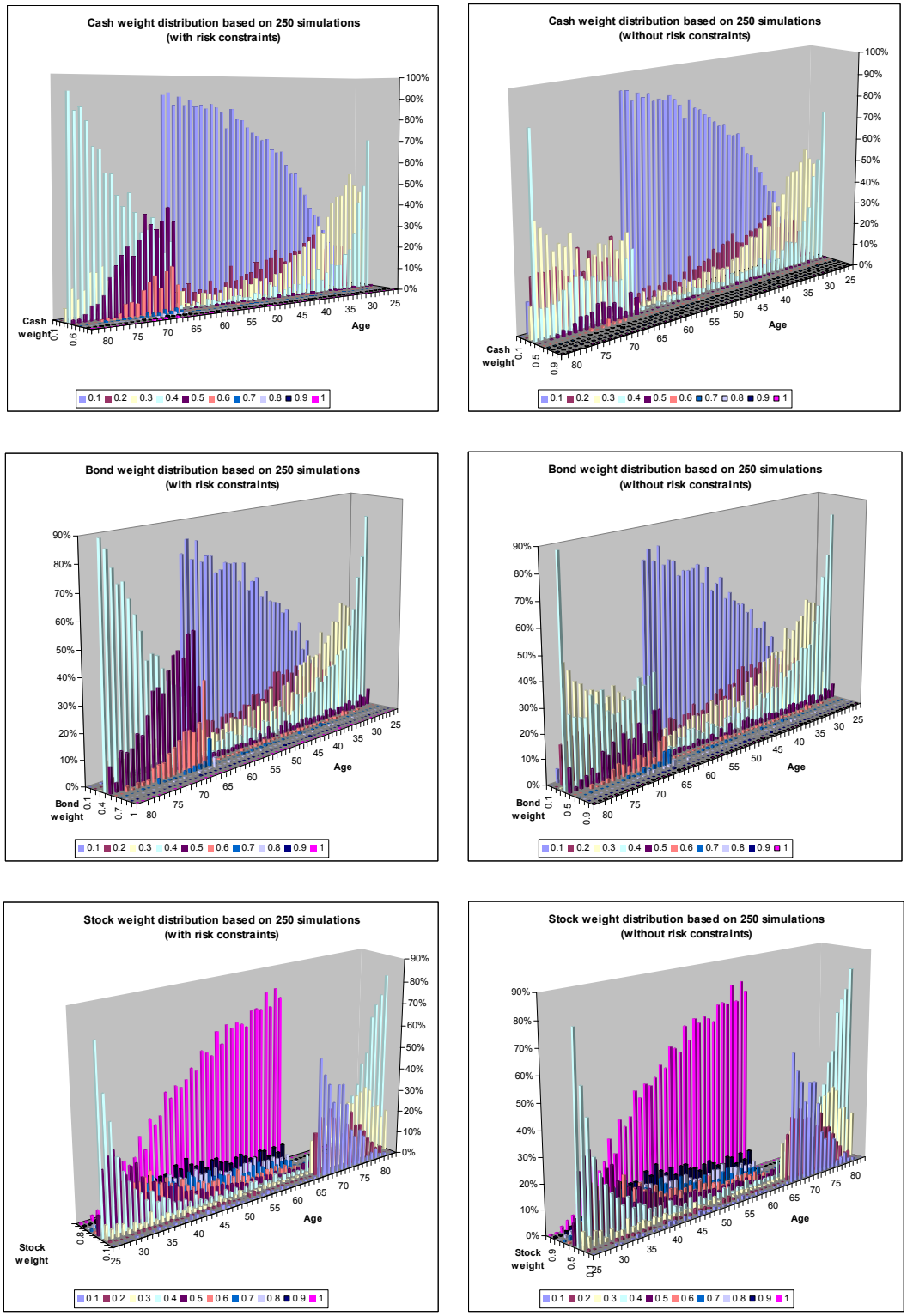
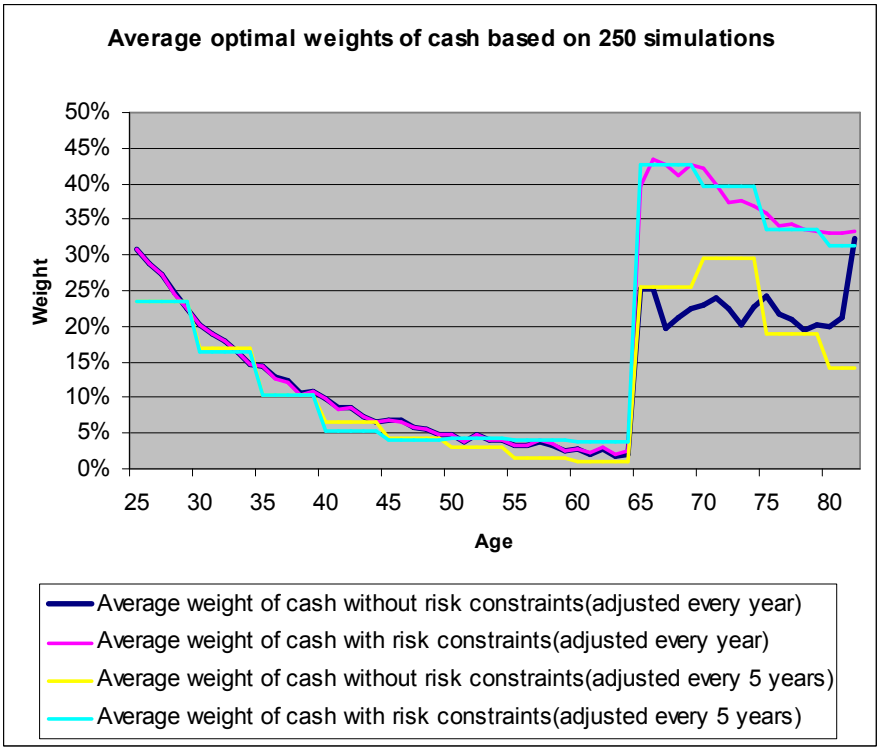
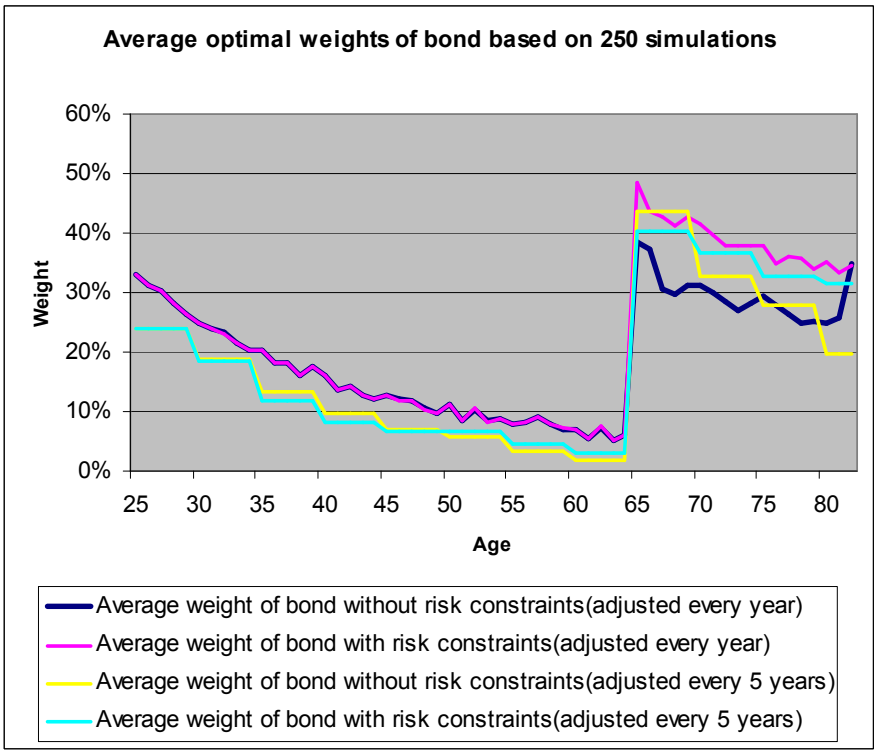


Figure 5: Average optimal weights of portfolio





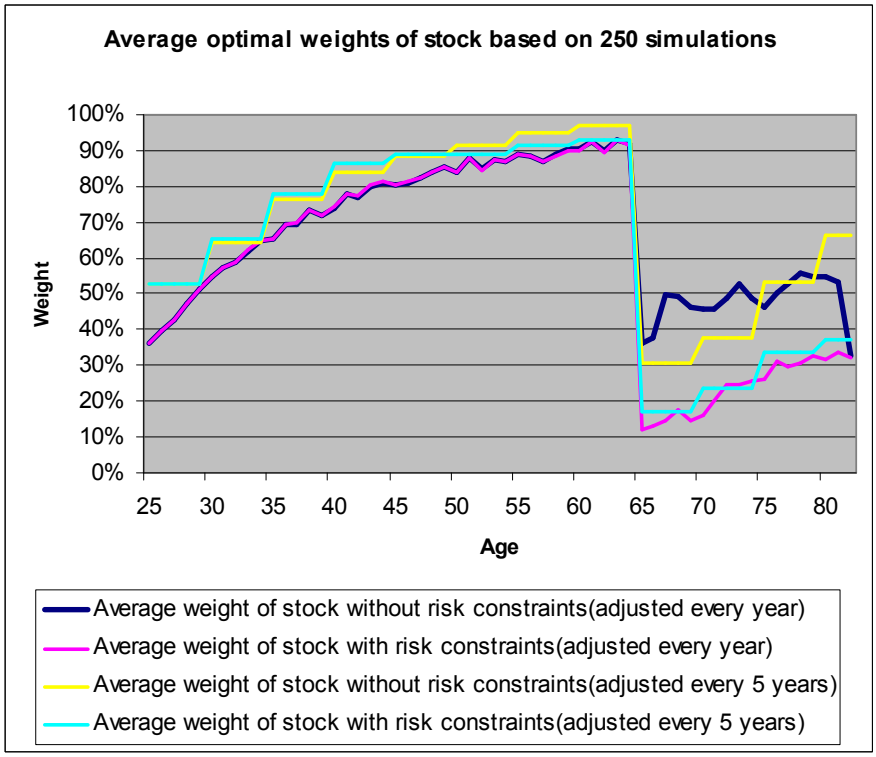
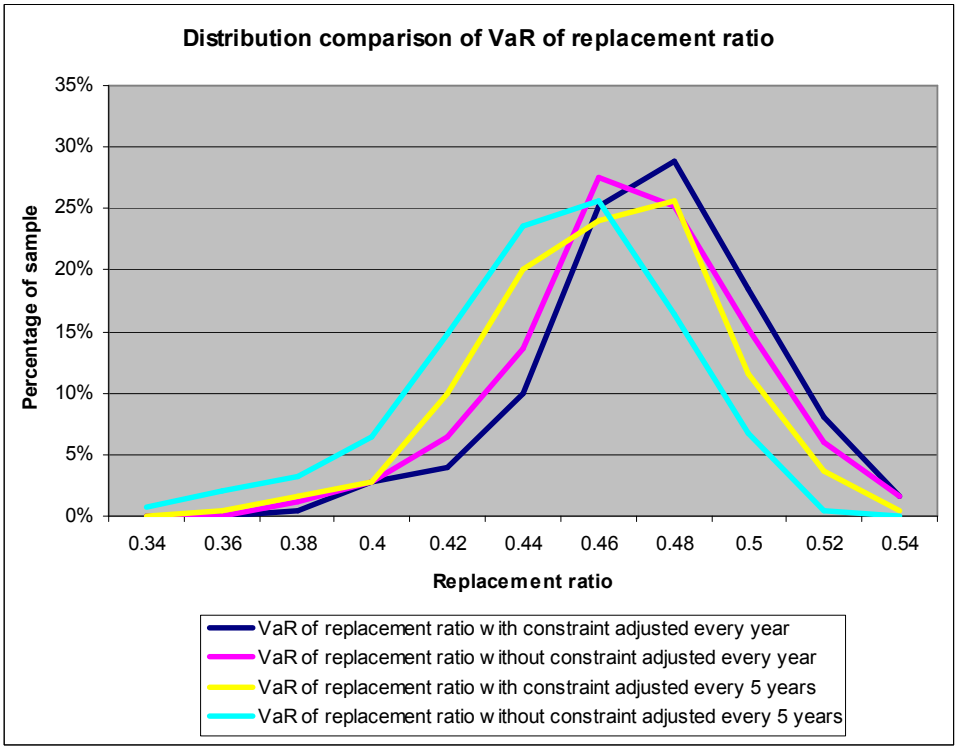

Average optimal weights of stock based on 250 simulations
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Weight
25
30
35
40
45
50
55
60
65
70
75
80
Age
Average weight of stock without risk constraints(adjusted every year)
Average weight of stock with risk constraints(adjusted every year)
Average weight of stock without risk constraints(adjusted every 5 years)
Average weight of stock with risk constraints(adjusted every 5 years)

Figure 6: Distribution of replacement ratio



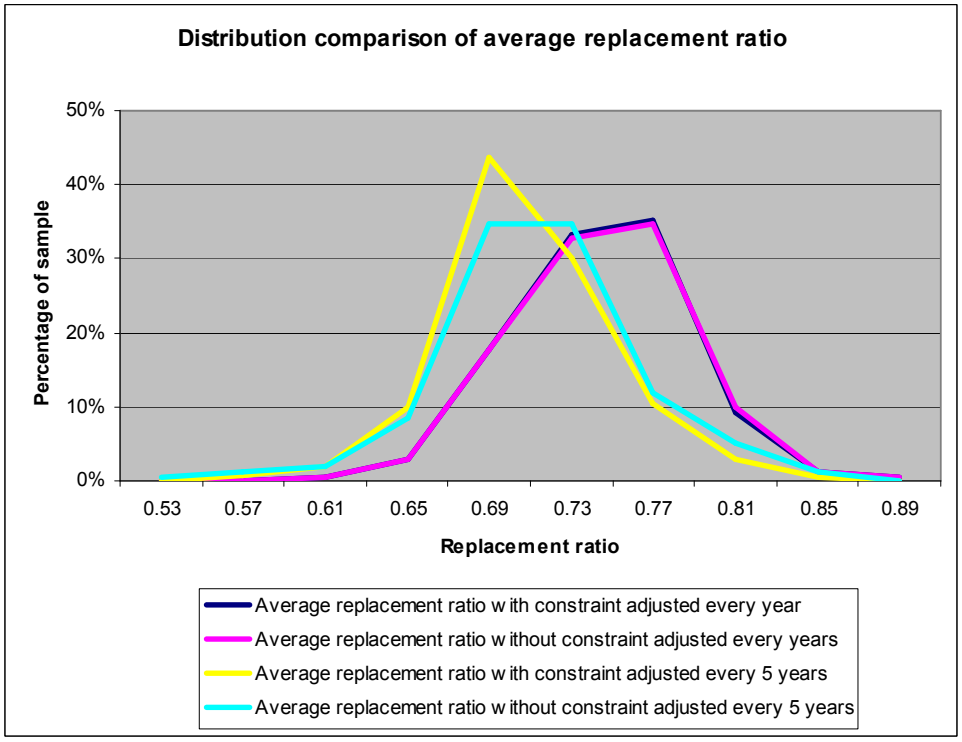
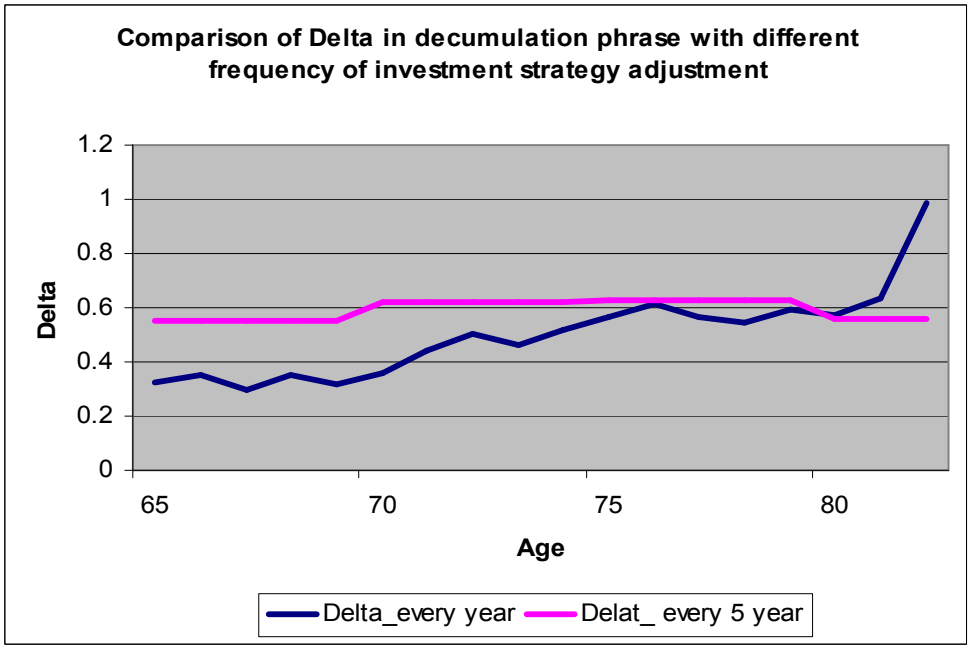


Figure 7: Comparison of Delta in decumulation phase



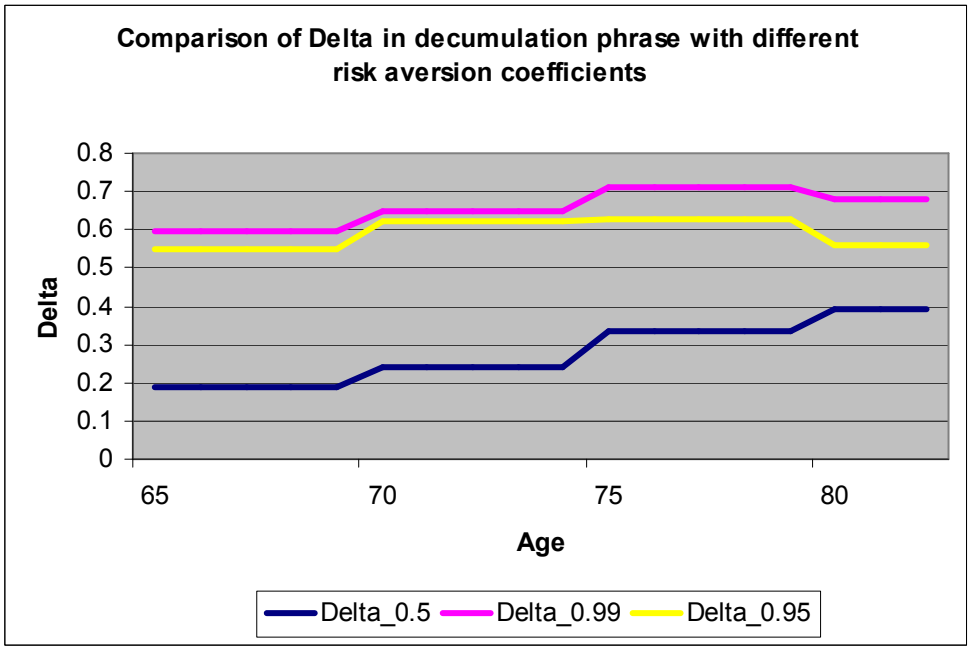


Figure 8: Comparison of investment strategy with different bequest scales