Probability Transforms with Elliptical Generators

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Introduction

• We introduce the notion of *elliptical transformations* for possible applications in constructing insurance premium principles.

• The idea is to distort the real probability measure of a risk $X$ based on the relative ratio of a density generator of a member of the class of elliptical distributions to that of the Normal distribution.

• We examine premium principle implied by this elliptical transformation:
  – Wang premium principle;
  – Esscher premium principle;
  – Wang student-t distortion principle.

• For location-scale families, this leads to the standard deviation principle.
Risk-adjusted premiums

- Premium principle $\pi$ is a mapping from $\Gamma$ of real-valued r.v's defined on $(\Omega, \mathcal{F})$ to the set of reals, $\pi: \Gamma \rightarrow R$, so that $\pi[X] \in R$, is the assigned premium.

- Risk-adjusted premiums are computed based on expectation w.r.t. $Q$:

$$\pi[X] = E_Q[X] = E[\Psi X]$$

where $\Psi$ is a positive r.v. (Radon-Nikodym derivative). See Gerber and Pafumi (1998).

- $\Psi$ has the form $\Psi = \frac{h(X; \lambda)}{E[h(X; \lambda)]}$ for some function $h$ of the r.v. $X$ and parameter $\lambda$.

Premium principles

- Premium principles are well-discussed in:
  - Kaas, van Heerwaarden and Goovaerts (1994)
  - Young (2004), Encyclopedia of Actuarial Science
  - Kaas, Goovaerts, Dhaene, and Denuit (2001)
Family of elliptical distributions

- $X$ is said to be elliptical with parameters $\mu$ and $\sigma^2$ if char. function is $E[\exp(itX)] = \exp(it\mu) \cdot \psi(t^2\sigma^2)$ for some scalar function $\psi$.

- Notation: $X \sim E(\mu, \sigma^2, \psi)$

- If the density exists, it has the form $f_X(x) = \frac{C}{\sigma}g\left[\left(\frac{x-\mu}{\sigma}\right)^2\right]$ for some non-negative function $g(\cdot)$ satisfying the condition $0 < \int_0^\infty z^{-1/2}g(z) \, dz < \infty$ and normalizing constant $C = \left[\int_0^\infty z^{-1/2}g(z) \, dz\right]^{-1}$.

- Any non-negative function $g(\cdot)$ can be used to define a one-dimensional density of an elliptical distribution.

- $g(\cdot)$ is called the density generator. One then sometimes writes $X \sim E(\mu, \sigma^2, g)$.
Normal and Student-t distributions

- Normal distribution: $X \sim N(\mu, \sigma^2)$
  - density generator: $g_N(u) = \exp(-u/2)$
  - normalizing constant: $C = \frac{1}{\sqrt{2\pi}}$

- Student-t distribution:
  - density generator: $g(u) = \left(1 + \frac{u}{m}\right)^{-\left(m+1\right)/2}$, $m > 0$ an integer
  - normalizing constant: $C = \frac{\Gamma \left(\left(m + 1\right)/2\right)}{\sqrt{m\pi}\Gamma\left(m/2\right)}$
Other known elliptical distributions

<table>
<thead>
<tr>
<th>Family</th>
<th>Density generators $g(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bessel</td>
<td>$g(u) = (u/b)^{a/2} K_a \left[(u/b)^{1/2}\right], \ a &gt; -1/2, \ b &gt; 0$ where $K_a(\cdot)$ is the modified Bessel function of the 3rd kind</td>
</tr>
<tr>
<td>Cauchy</td>
<td>$g(u) = (1 + u)^{-1}$</td>
</tr>
<tr>
<td>Exponential Power</td>
<td>$g(u) = \exp[-r(u)^s], \ r, s &gt; 0$</td>
</tr>
<tr>
<td>Laplace</td>
<td>$g(u) = \exp(-</td>
</tr>
<tr>
<td>Logistic</td>
<td>$g(u) = \frac{\exp(-u)}{[1 + \exp(-u)]^2}$</td>
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Elliptical transforms

- Let $g_Z(u)$ be density generator of spherical r.v. $Z \sim E(0, 1, g_Z)$.

- Ratio of density generators $g_Z$ and $g_N$:

$$h_{g_Z}(X; \lambda) = \frac{g_Z \left[ \left( \Phi^{-1}(\overline{F}_X(X)) + \lambda \right)^2 \right]}{g_N \left[ \left( \Phi^{-1}(\overline{F}_X(X)) \right)^2 \right]}$$

for some non-negative parameter $\lambda \geq 0$ and where $\Phi(\cdot)$ is the c.d.f of standard Normal and $g_N$ is the density generator of Normal.

- Expectation of this ratio can be expressed as

$$\mathbb{E}[h_{g_Z}(X; \lambda)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_Z(z^2) \, dz.$$
Some illustrative examples

- **Example 3.1:** Normal-to-Normal Generators

\[ h_{g_N}(X; \lambda) = \exp[-\lambda (\Phi^{-1}(\overline{F}_X(X)) + \frac{1}{2} \lambda)] = e^{-\lambda^2/2} \exp[-\lambda \Phi^{-1}(\overline{F}_X(X))]. \]

It is easy to see that in this case, \( E[h_{g_Z}(X; \lambda)] = 1. \)

- **Example 3.2:** Student-t-to-Normal Generators

\[ h_{g_Z}(X; \lambda) = \frac{\exp\left[\frac{1}{2} \left(\Phi^{-1}(\overline{F}_X(X))\right)^2\right]}{\left[1 + \frac{1}{m} \left(\Phi^{-1}(\overline{F}_X(X)) + \lambda\right)^2\right]^{(m+1)/2}}. \]

Applying Theorem 1 to solve for the expectation, we have

\[ E[h_{g_Z}(X; \lambda)] = \sqrt{\frac{m}{2}} \frac{\Gamma(m/2)}{\Gamma((m + 1)/2)}. \]
Now interestingly, the case where $m = 1$ leads us to the Cauchy-to-Normal generators:

\[
E[h_{gZ}(X; \lambda)] = \sqrt{\frac{1}{2\pi}} \frac{\Gamma(1/2) \Gamma(1/2)}{\Gamma(1)} = \sqrt{\frac{\pi}{2}}.
\]

Also, in the limiting case where $m \to \infty$, this leads us back to the Normal-to-Normal density generators.

**Example 3.3**: *Exponential Power-to-Normal Generators*

\[
h_{gZ}(X; \lambda) = \exp \left\{ - \left[ r \left( \Phi^{-1}(F_X(X)) + \lambda \right)^{2s} - \frac{1}{2} \left( \Phi^{-1}(F_X(X)) \right)^2 \right] \right\}.
\]

Applying Theorem 1 to solve for the expectation, we have

\[
E[h_{gZ}(X; \lambda)] = \frac{1}{s\sqrt{2\pi}} r^{1-(1/2s)} \Gamma \left( \frac{1}{2s} \right).
\]

The case where $r = 1/2$ and $s = 1$ leads us to the Normal distribution case.
Transformed distributions

- Define the transformed r.v. $X^*$ to be one with a (transformed) density:

$$f_{X^*}(x) = C \times \frac{g_Z \left[ \left( \Phi^{-1}\left( \frac{F_X(X)}{\Phi^{-1}\left( F_X(X) \right)} \right) + \lambda \right)^2 \right]}{g_N \left[ \left( \Phi^{-1}\left( F_X(X) \right) \right)^2 \right]} \times f_X(x)$$

- By recognizing that $h_{g_Z}(X; \lambda) = \frac{g_Z \left[ \left( \Phi^{-1}(F_X(X)) + \lambda \right)^2 \right]}{g_N \left[ \left( \Phi^{-1}(F_X(X)) \right)^2 \right]}$, the normalizing constant $C = \frac{1}{\mathbb{E}[h_{g_Z}(X; \lambda)]}$.

- We can, as a matter of fact, find an expression for the d.f. of the (transformed) r.v. $X^*$:

$$F_{X^*}(x) = \int_x^\infty f_{X^*}(v) \, dv = F_Z \left[ \Phi^{-1}\left( \frac{F_X(x)}{\Phi^{-1}\left( F_X(X) \right)} \right) + \lambda \right].$$
The work of Landsman (2004)

- Landsman (2004) [elliptical tilting] also proposed to use transformation of densities of elliptical r.v.’s:

\[ g \left( \frac{(x - \mu)}{\sigma}^2 - 2\lambda x \right) \]

- Generalizes Esscher transform for elliptical distributions.
- Generalizes variance premium principle applied to elliptical distributions.

- Several differences between Landsman’s elliptical tilting with what we propose:
  - Two different density generators
  - Translation of the distribution by introducing a shift parameter \( \lambda \)
  - Not limited to transforming elliptical
  - Variance versus standard deviation premium principle
**Premium principle implied by the elliptical transformation**

- Let $X^*$ be the transformed random variable of $X$ according to the elliptical transformation. Then the expectation of $X^*$ is defined to be the premium principle implied by this transformation:

$$
\pi[X] = E(X^*) = E \left[ \frac{h_{gZ}(X; \lambda)}{E[h_{gZ}(X; \lambda)]} \cdot X \right].
$$

- Observe that we can also derive the premium (or expectation of the transformed distribution) using

$$
\pi[X] = - \int_{-\infty}^{0} F_Z[\Phi^{-1}(\overline{F}_X(x)) + \lambda] \, dx + \int_{0}^{\infty} \overline{F}_Z[\Phi^{-1}(\overline{F}_X(x)) + \lambda] \, dx
$$

which reduces to just $\int_{0}^{\infty} \overline{F}_Z[\Phi^{-1}(\overline{F}_X(x)) + \lambda] \, dx$ for r.v.’s with non-negative support.
Figure 1. Elliptical transformation using the Normal density generator
Figure 2. Elliptical transformation using the Student-t density generator
Figure 3. Using the Student-t density generator but varying lambda
Figure 4. The case of $\lambda = 0$
Recovering familiar premium principles

• **Example 3.4:** *Wang Premium Principle*  Choose $g_Z$ to be the density generator of a Normal:

\[
F_{X^*}(x) = \Phi[\Phi^{-1}(F_X(x)) + \lambda].
\]


• **Example 3.5:** *Wang’s Student-t Distortion Premium Principle*  Choose $g_Z$ to be the density generator of a Student-t distribution with $m$ d.f., then we have as in Wang (2004)

\[
F_{X^*}(x) = Q[\Phi^{-1}(F_X(x)) + \lambda],
\]

where, following Wang’s notation, $Q(\cdot)$ denotes the d.f. of a Student-t with $m$ degrees of freedom. Wang actually has set $\lambda = 0$ and used $F_{X^*}(x) = Q[\Phi^{-1}(F_X(x))]$ instead.
**Example 3.6: Esscher Premium Principle** In the special case where $X$ is $N(\mu, \sigma^2)$, the elliptical transformation leads to the Esscher transform:

\[
\frac{h_{gZ}(X; \lambda)}{E[h_{gZ}(X; \lambda)]} = \frac{e^{-\lambda^2/2} \exp[-\lambda \Phi^{-1}(F_X(X))]}{E[e^{-\lambda^2/2} \exp[-\lambda \Phi^{-1}(F_X(X))]]} = \frac{\exp(\frac{\lambda}{\sigma} X)}{E[\exp(\frac{\lambda}{\sigma} X)]}.
\]

See Esscher (1932) and also more recent articles, for example, Gerber and Shiu (1994).
Location-scale families

• Let $X$ belong to location-scale family so that $F_X(x) = F_{Z^*}(\frac{x - \mu}{\sigma})$ for some $Z^* = \frac{X - \mu}{\sigma}$, independent of $\mu$ and $\sigma$. Then

$$\pi[X] = \mu + \mathbb{E}_Z \left[ F_{Z^*}^{-1}(\Phi(Z - \lambda)) \right] \times \sigma.$$ 

• In case $X$ is $N(\mu, \sigma^2)$ and we choose $Z$ to be the standard Normal:

$$\mathbb{E}_Z \left[ F_{Z^*}^{-1}(\Phi(Z - \lambda)) \right] = \mathbb{E}_Z(\lambda - Z) = \lambda - \mathbb{E}_Z(Z) = \lambda,$$

and the resulting premium principle leads to: $\pi[X] = \mu + \lambda \sigma$.

• Here we have $\lambda = \frac{\pi[X] - \mu}{\sigma}$, risk premium per unit of risk, $\sigma$. 
Location-scale families

• This paper introduces notion of elliptical transformation leading to a premium principle which in some sense generalizes the familiar Wang transformation as well as the premium principle introduced by Wang (2004) using the Student-t distortion function.

• We also note that in the special case of transforming location-scale families, the resulting premium principle is the standard deviation premium principle.

• Transformation introduces a heavy penalty on the extreme right tails of the distribution but also encourages small losses by placing relatively reasonable weights on the extreme left tails of the distribution.

• How much this penalty depends on the choice of the density generator together with the parameter $\lambda$ which in a sense gives a measure of aversion to the level of risk of the insurer. This parameter introduces a shift in the distribution of the risk.
• We also show that the elliptical transformation recovers many other familiar premium principles.

• Elliptical transforms may also be applied as a risk measure to compute economic capital.

• In the future, it will be an interesting work to examine some properties of this premium principle.
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