

Probability Transforms with Elliptical Generators*

EMILIANO A. VALDEZ[†]
SCHOOL OF ACTUARIAL STUDIES
FACULTY OF COMMERCE & ECONOMICS
UNIVERSITY OF NEW SOUTH WALES
SYDNEY, AUSTRALIA
E-MAIL: E.VALDEZ@UNSW.EDU.AU
TEL. +61-2-9385-3391
FAX +61-2-9385-1883

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Abstract

This paper introduces the notion of *elliptical transformations* for possible applications in constructing insurance premium principles. It is a common notion in actuarial science to transform or sometimes said, to distort the real probability measure of a risk X and then calculate premiums based on the expectation of the transformed distribution. Elliptical transformations are a way to distort the probability distribution of a risk and it is based on the relative ratio of a density generator of a member of the class of elliptical distributions to the density generator of the Normal distribution which is also well-known to belong to the class of elliptical distributions. The class of elliptical distribution models consists of distributions considered symmetric and provides a generalization of the class of Normal loss distribution models. We examine the premium principle implied by this elliptical transformation. We find that the elliptical transforms lead us to, as special cases, the Wang premium principle, the Wang Student-t distortion principle, as well as the Esscher premium principle. In several cases, the resulting premium principle has a replicating feature of the standard deviation premium principle. A numerical example has been elaborated to illustrate the practical implementation of this elliptical transformation.

*Keywords: probability transform, distortion, risk measures, premium principles, elliptical distributions, elliptical transforms.

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1 Introduction

Consider an insurance company that over some well-defined time reference, faces a random loss of X on a well-defined probability space $(\Omega, \mathcal{F}, \Pr)$. We can sometimes appropriately call X the risk faced by the insurer. For this random loss X , assume that its distribution function is denoted by $F_X(x)$ and its corresponding tail probability by $\bar{F}_X(x) = 1 - F_X(x)$. Assuming X is a continuous random variable, as is often the case, then the density function of X is given by

$$f_X(x) = \frac{dF_X(x)}{dx} = -\frac{d\bar{F}_X(x)}{dx}.$$

The task of the insurer is to assign a price tag on this risk X leading us to examine premium principles. As pointed out by Young (2004), that “loosely speaking, a premium principle is a rule for assigning a premium to an insurance risk”. The insurance premium is usually coined “risk-adjusted” to refer to the fact that it already incorporates model risks as well as parameter uncertainties. As advocated in this paper, these model and parameter risks are often handled by distorting the real probability measure. This paper introduces yet another method to transform distributions.

A premium principle π is a mapping from a set Γ of real-valued random variables defined on (Ω, \mathcal{F}) to the set R of real numbers. Effectively, we have

$$\pi : \Gamma \longrightarrow R$$

so that for every X belonging to Γ , $\pi[X] \in R$, is the assigned premium. To no surprise, premium principles are well-studied in the actuarial literature. See for example, Goovaerts, DeVijlder and Haezendonck (1984), Kaas, van Heerwaarden and Goovaerts (1994), and more recently Young (2004). A chapter on insurance premium principles is devoted in Kaas, Goovaerts, Dhaene and Denuit (2001).

Risk-adjusted premiums are often computed based on the expectation with respect to a transformed probability measure, say Q , such that

$$\pi[X] = E_Q[X] = E[\Psi X] \tag{1}$$

where Ψ is a positive random variable which technically is also the Radon-Nikodym derivative of the Q -measure with respect to the real probability measure, sometimes denoted by P . See Gerber and Pafumi (1998). This is also sometimes called, in the Finance literature, the pricing density. In several premium principles, the Ψ follows the general form defined by a relation

$$\Psi = \frac{h(X; \lambda)}{E[h(X; \lambda)]} \tag{2}$$

for some function h of the random variable X and a parameter λ often describing some level of risk-aversion to the insurer. This relation in (2) can also be justified in a decision theoretic framework as discussed in Heilman (1985) and Hürlimann (2004). Following Hürlimann (2004), for example, consider a loss function $L : R^2 \longrightarrow R$ which describes the loss incurred $L(x, x^*)$ for a realized outcome $X = x$ when the action taken by the decision maker has been x^* . Decisions on premiums are determined on the basis of minimizing the expectation of this loss function. Consider the loss function defined by

$$L(x, x^*) = h(x; \lambda) \times (x - x^*)^2. \tag{3}$$

It is straightforward to show then that the resulting premium principle follows the general form

$$\pi[X] = \mathbb{E} \left[\frac{h(X; \lambda)}{\mathbb{E}[h(X; \lambda)]} \cdot X \right].$$

Many familiar premium principles lead to this form. For example, the net premium principle can be derived with $h(X; \lambda) = 1$ and the variance premium principle can be derived with $h(X; \lambda) = 1 + \lambda X$. The celebrated Esscher premium principle can be derived by choosing $h(X; \lambda) = \exp(\lambda X)$, and the Karlsruhe premium principle discussed in Hürlimann (2004) can be derived by choosing $h(X; \lambda) = X$. Even the conditional tail expectation used as a risk measure can be derived with $h(X; \lambda) = I(X > F_X^{-1}(q))$ where $I(\cdot)$ is an indicator function and $F_X^{-1}(q)$ is the quantile of the distribution of X at a pre-determined probability level q .

In the univariate case, we have X belonging to the class of elliptical distributions if its density can be expressed as

$$f_X(x) = \frac{C}{\sigma} g \left[\left(\frac{x - \mu}{\sigma} \right)^2 \right] \quad (4)$$

for some so-called density generator g (which is a function of non-negative variables), and where C is a normalizing constant. Later in the paper, the definition of elliptical distributions is expressed in terms of its characteristic function because this function always exist for a distribution. For the familiar Normal distribution $N(\mu, \sigma^2)$ with mean μ and variance σ^2 , it is straightforward to show that its density generator for which we shall denote by g_N in this paper has the form

$$g_N(u) = \exp(-u/2). \quad (5)$$

In this paper, we define elliptical transformations by considering the functional $h(x; \lambda)$ to have the following form:

$$h(x; \lambda) = \frac{g \left[(\Phi^{-1}(\overline{F}_X(x)) + \lambda)^2 \right]}{g_N \left[(\Phi^{-1}(\overline{F}_X(x)))^2 \right]} \quad (6)$$

where g is the density generator of some member of the class of elliptical distributions. The family of elliptical distributions consists of symmetric distributions for which the Student- t , the Uniform, Logistic, and Exponential Power distributions are other familiar examples. This is a rich family of distributions that allow for a greater flexibility in modelling risks that exhibit tails heavier than that of a Normal distribution. This can be especially useful for capturing insurance losses for extreme events.

As we shall observe in our discussion in the paper, the probability transform defined using the ratio of density generators in (6) effectively distort the distribution by putting heavier weights on both tails of the distribution. In insurance premium calculations, what this translates to is giving more penalty on large losses but at the same time, placing some emphasis on small or zero losses thereby encouraging small claims. The result is therefore a trade-off between small and large claims redistributing the claims distribution so that it leads to a more sensible and equitable

premium calculation. This trade-off is best parameterized by the parameter λ which also effectively introduces a shift or translation of the distribution.

The rest of the paper has been organized as follows. In Section 2, we briefly discuss the rich class of elliptical distributions in the univariate dimension, together with some important properties. Section 3 describes the notion of elliptical transformation. Several examples follow. In Section 4, we discuss the special case where the risk has a distribution that is location-scale. Elliptical distributions are special cases of location-scale families. In Section 5, using some claims experience data, we demonstrate practical implementation of introducing premium loadings using the ideas developed in the paper. Finally, in section 6, we provide some concluding remarks together with some remarks about possible direction for further work.

2 The Family of Elliptical Distributions

The class of elliptical loss distribution models provides a generalization of the class of normal loss models. In the following, we will describe this class of models first in the univariate dimension, and briefly extending it to the multivariate dimension. The class of elliptical distributions has been introduced in the statistical literature by Kelker (1970) and widely discussed in Fang, Kotz and Ng (1990). See also Landsman and Valdez (2003), Valdez and Dhaene (2003), and Valdez and Chernih (2003) for applications in insurance and actuarial science. Embrechts, et al. (2001) also provides a fair amount of discussion of this important class as a tool for modelling risk dependencies.

2.1 Definition of Elliptical Distributions

It is widely known that a random variable X with a normal distribution has the characteristic function expressed as

$$E[\exp(itX)] = \exp(it\mu) \cdot \exp\left(-\frac{1}{2}t^2\sigma^2\right), \quad (7)$$

where μ and σ^2 are respectively, the mean and variance of the distribution. We shall use the notation $X \sim N(\mu, \sigma^2)$. The class of elliptical distributions is a natural extension to the class of normal distributions.

Definition 1 *The random variable X is said to have an elliptical distribution with parameters μ and σ^2 if its characteristic function can be expressed as*

$$E[\exp(itX)] = \exp(it\mu) \cdot \psi(t^2\sigma^2) \quad (8)$$

for some scalar function ψ .

If X has the elliptical distribution as defined above, we shall conveniently write $X \sim E(\mu, \sigma^2, \psi)$ and say that X is elliptical. The function ψ is called the *characteristic generator* of X and therefore, for the normal distribution, the characteristic generator is clearly given by $\psi(u) = \exp(-u/2)$.

It is well-known that the characteristic function of a random variable always exists and that there is a one-to-one correspondence between distribution functions and characteristic functions. Note however that not every function ψ can be used to construct a characteristic function of an elliptical distribution. Obviously, this function

ψ should fulfill the requirement that $\psi(0) = 1$. A necessary and sufficient condition for the function ψ to be a characteristic generator of an elliptical distribution can be seen in Theorem 2.2 of Fang, Kotz and Ng (1990).

It is also interesting to note that the class of elliptical distributions consists mainly of the class of symmetric distributions which include well-known distributions like normal and Student- t . The moments of $X \sim E(\mu, \sigma^2, \psi)$ do not necessarily exist. However, it can be shown that if the mean, $E(X)$, exists, then it will be given by

$$E(X) = \mu \tag{9}$$

and if the variance, $Var(X)$, exists, then it will be given by

$$Var(X) = -2\psi'(0)\sigma^2, \tag{10}$$

where ψ' denotes the first derivative of the characteristic generator. A necessary condition for the variance to exist is

$$|\psi'(0)| < \infty, \tag{11}$$

see Cambanis, Huang and Simmons (1981).

In the case where $\mu = 0$ and $\sigma^2 = 1$, we have what we call a spherical distribution and the random variable X is replaced by a standard random variable Z . That is, we have $Z \sim E(0, 1, \psi)$ and the notation $S(\psi)$ for $E(0, 1, \psi)$ is more typically used and thus, we write $Z \sim S(\psi)$. It is clear that the characteristic function of Z has the form

$$E[\exp(itZ)] = \psi(t^2)$$

for any real number t .

Also, if we consider any random variable X satisfying

$$X \stackrel{d}{=} \mu + \sigma Z,$$

for some real number μ , some positive real number σ and some spherical random variable $Z \sim S(\psi)$, then it can be shown that $X \sim E(\mu, \sigma^2, \psi)$. Similarly, for any elliptical random variable $X \sim E(\mu, \sigma^2, \psi)$, we can always define the random variable

$$Z = \frac{X - \mu}{\sigma}$$

which is clearly spherical.

2.2 Densities of Elliptical Distributions

An elliptically distributed random variable $X \sim E(\mu, \sigma^2, \psi)$ does not necessarily possess a density function $f_X(x)$. In the case of a normal random variable $X \sim N(\mu, \sigma^2)$, its density is well-known to be

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right]. \tag{12}$$

For elliptical distributions, one can prove that if $X \sim E(\mu, \sigma^2, \psi)$ has a density, it will have the form

$$f_X(x) = \frac{C}{\sigma} g \left[\left(\frac{x - \mu}{\sigma} \right)^2 \right] \quad (13)$$

for some non-negative function $g(\cdot)$ satisfying the condition

$$0 < \int_0^\infty z^{-1/2} g(z) dz < \infty \quad (14)$$

and a normalizing constant C given by

$$C = \left[\int_0^\infty z^{-1/2} g(z) dz \right]^{-1}. \quad (15)$$

Also, the opposite statement holds: Any non-negative function $g(\cdot)$ satisfying the condition (14) can be used to define a one-dimensional density

$$\frac{C}{\sigma} g \left[\left(\frac{x - \mu}{\sigma} \right)^2 \right]$$

of an elliptical distribution, with C given by (15). The function $g(\cdot)$ is called the *density generator*. One sometimes writes $X \sim E(\mu, \sigma^2, g)$ for the one-dimensional elliptical distributions generated from the function $g(\cdot)$. A detailed proof of these results for the case of n -dimension, using spherical transformations of rectangular coordinates, can be found in Landsman and Valdez (2003).

From (12), one immediately finds that the density generators and the corresponding normalizing constants of the normal random variable $X \sim N(\mu, \sigma^2)$ are given by $g(z) = \exp(-z/2)$ and $C = \frac{1}{\sqrt{2\pi}}$, respectively.

TABLE 1
SOME KNOWN ELLIPTICAL DISTRIBUTIONS WITH THEIR DENSITY GENERATORS

Family	Density generators $g(u)$
Bessel	$g(u) = (u/b)^{a/2} K_a \left[(u/b)^{1/2} \right]$, $a > -1/2, b > 0$ where $K_a(\cdot)$ is the modified Bessel function of the 3rd kind
Cauchy	$g(u) = (1 + u)^{-1}$
Exponential Power	$g(u) = \exp[-r(u)^s]$, $r, s > 0$
Laplace	$g(u) = \exp(- u)$
Logistic	$g(u) = \frac{\exp(-u)}{[1 + \exp(-u)]^2}$
Normal	$g_N(u) = \exp(-u/2)$
Student- t	$g(u) = \left(1 + \frac{u}{m}\right)^{-(m+1)/2}$, $m > 0$ an integer

As a special case, we find that from the results concerning elliptical distributions, if a spherical random variable $Z \sim S(\psi)$ possess a density $f_Z(z)$, then it will have the form

$$f_Z(z) = Cg(z^2),$$

where the density generator g satisfies the condition (14) and the normalizing constant C satisfies (15). Furthermore, the opposite also holds: any non-negative function $g(\cdot)$ satisfying the condition (14) can be used to define a one-dimensional density $Cg(z^2)$ of a spherical distribution with the normalizing constant C satisfying (15). One often writes $S(g)$ for the spherical distribution generated from the density generator $g(\cdot)$.

3 Elliptical Transforms

Suppose $g_Z(u)$ is the density generator of a univariate spherical random variable $Z \sim E(0, 1, g_Z)$, also written as $Z \sim S(g_Z)$. We have subscripted the density generator to emphasize the corresponding spherical random variable Z . Now we define the ratio of density generators as follows.

Definition 2 Let X be a random variable with tail probability function $\overline{F}_X(\cdot)$. We define the ratio of density generators g_Z and g_N to be the random variable

$$h_{g_Z}(X; \lambda) = \frac{g_Z\left[\left(\Phi^{-1}(\overline{F}_X(X)) + \lambda\right)^2\right]}{g_N\left[\left(\Phi^{-1}(\overline{F}_X(X))\right)^2\right]} \quad (16)$$

for some non-negative parameter $\lambda \geq 0$ and where $\Phi(\cdot)$ is the cumulative distribution function of a standard Normal random variable and g_N is the density generator of a Normal distribution.

Notice that in the definition we assume we have a random variable $X \in \Gamma$, belonging to the set of all risks, whose tail probability function is given by $\overline{F}_X(\cdot)$. As we shall observe later in the examples that follow, this non-negative parameter can be interpreted as a premium per unit of volatility (or risk). The following theorem gives an expression for the expectation of the ratio of density generators.

Theorem 1 Let X be a random variable with tail probability function $\overline{F}_X(\cdot)$ and let the cumulative distribution function of a standard Normal be denoted by $\Phi(\cdot)$. The expectation of the ratio of density generators g_Z and g_N can be expressed as

$$E[h_{g_Z}(X; \lambda)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_Z(z^2) dz. \quad (17)$$

Proof. First assuming the density of X is denoted by $f_X(\cdot)$, then we can write the expectation as

$$E[h_{g_Z}(X; \lambda)] = \int_{-\infty}^{\infty} \frac{g_Z\left[\left(\Phi^{-1}(\overline{F}_X(x)) + \lambda\right)^2\right]}{g_N\left[\left(\Phi^{-1}(\overline{F}_X(x))\right)^2\right]} f_X(x) dx.$$

Using the transformation $u = \bar{F}_X(x)$ so that $du = -f_X(x) dx$, then we have

$$\mathbb{E}[h_{g_Z}(X; \lambda)] = \int_0^1 \frac{g_Z[(\Phi^{-1}(u) + \lambda)^2]}{g_N[(\Phi^{-1}(u))^2]} du.$$

With another transformation $w = \Phi^{-1}(u)$ so that $du = \frac{1}{\sqrt{2\pi}} g_N(w^2) dw$, we then have

$$\mathbb{E}[h_{g_Z}(X; \lambda)] = \int_{-\infty}^{\infty} \frac{g_Z[(w + \lambda)^2]}{g_N(w^2)} \frac{1}{\sqrt{2\pi}} g_N(w^2) dw.$$

Applying one last transformation $z = w + \lambda$, we get the desired result. ■

Following the above Theorem, it is clear that for a spherical $Z \sim S(g_Z)$ with a normalizing constant C_Z , we have the following result (which will later be useful):¹

$$C_Z = \left[\int_{-\infty}^{\infty} g_Z(z^2) dz \right]^{-1} = \frac{1}{\sqrt{2\pi} \mathbb{E}[h_{g_Z}(X; \lambda)]}.$$

We now give some examples of ratios of density generators.

Example 3.1: *Normal-to-Normal Generators*

If g_Z is also the density generator of a Normal distribution, then we have

$$\begin{aligned} h_{g_N}(X; \lambda) &= \frac{g_N[(\Phi^{-1}(\bar{F}_X(X)) + \lambda)^2]}{g_N[(\Phi^{-1}(\bar{F}_X(X)))^2]} = \exp[-\lambda (\Phi^{-1}(\bar{F}_X(X)) + \frac{1}{2}\lambda)] \\ &= e^{-\lambda^2/2} \exp[-\lambda \Phi^{-1}(\bar{F}_X(X))]. \end{aligned} \quad (18)$$

It is easy to see that in this case, the expectation of (18) is equal to 1. Direct application of Theorem 1, for example, leads us to this. ■

Example 3.2: *Student-t-to-Normal Generators*

If g_Z is the density generator of a Student-t distribution with m degrees of freedom, then we have²

$$\begin{aligned} h_{g_Z}(X; \lambda) &= \frac{g_Z[(\Phi^{-1}(\bar{F}_X(X)) + \lambda)^2]}{g_N[(\Phi^{-1}(\bar{F}_X(X)))^2]} \\ &= \frac{\left[1 + \frac{1}{m} (\Phi^{-1}(\bar{F}_X(X)) + \lambda)^2\right]^{(m+1)/2}}{\exp\left[\frac{1}{2} (\Phi^{-1}(\bar{F}_X(X)))^2\right]}. \end{aligned} \quad (19)$$

Applying Theorem 1 to solve for the expectation, we have

$$\begin{aligned} \mathbb{E}[h_{g_Z}(X; \lambda)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_Z(z^2) dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(1 + \frac{1}{m} z^2\right)^{-(m+1)/2} dz. \end{aligned}$$

¹correction made: 1-May-2005.

²correction made: 1-May-2005.

Applying transformations, it can be shown that this expectation leads us to

$$\begin{aligned} \mathbb{E}[h_{g_Z}(X; \lambda)] &= \sqrt{\frac{m}{2\pi}} \int_0^1 z^{(m/2)-1} (1-z)^{(1/2)-1} dz = \sqrt{\frac{m}{2\pi}} B\left(\frac{m}{2}, \frac{1}{2}\right) \\ &= \sqrt{\frac{m}{2\pi}} \frac{\Gamma(m/2) \Gamma(1/2)}{\Gamma((m+1)/2)} = \sqrt{\frac{m}{2}} \frac{\Gamma(m/2)}{\Gamma((m+1)/2)}, \end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function and $\Gamma(\cdot)$ is the Gamma function. Here, we also note that we applied the fact that $\Gamma(1/2) = \sqrt{\pi}$. Now interestingly, the case where $m = 1$ leads us to the Cauchy-to-Normal generators and in which case, we would have

$$\mathbb{E}[h_{g_Z}(X; \lambda)] = \sqrt{\frac{1}{2\pi}} \frac{\Gamma(1/2) \Gamma(1/2)}{\Gamma(1)} = \sqrt{\frac{\pi}{2}}.$$

It can also be shown that in the limiting case where $m \rightarrow \infty$, this leads us back to the Normal-to-Normal density generators. ■

Example 3.3: *Exponential Power-to-Normal Generators*

Assuming now that g_Z is the density generator of an Exponential-Power distribution (see Table 1), then we have

$$\begin{aligned} h_{g_Z}(X; \lambda) &= \frac{g_Z\left[(\Phi^{-1}(\overline{F}_X(X)) + \lambda)^2\right]}{g_N\left[(\Phi^{-1}(\overline{F}_X(X)))^2\right]} \\ &= \exp\left\{-\left[r(\Phi^{-1}(\overline{F}_X(X)) + \lambda)^{2s} - \frac{1}{2}(\Phi^{-1}(\overline{F}_X(X)))^2\right]\right\}. \quad (20) \end{aligned}$$

Applying Theorem 1 to solve for the expectation, we have

$$\begin{aligned} \mathbb{E}[h_{g_Z}(X; \lambda)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_Z(z^2) dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-rz^{2s}} dz \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-rz^{2s}} dz \end{aligned}$$

where the third line follows because of symmetry. Let us derive an explicit form for this expectation. First, consider the transformation $u = z^{2s}$ so that $du = 2sz^{2s-1} dz = 2su^{(2s-1)/2s} dz$. This leads us to

$$\mathbb{E}[h_{g_Z}(X; \lambda)] = \frac{1}{s\sqrt{2\pi}} \int_0^{\infty} u^{(\frac{1}{2s})-1} e^{-ru} du.$$

Next then, consider the transformation $z = ru$ so that $dz = rdu$. We have

$$\begin{aligned} \mathbb{E}[h_{g_Z}(X; \lambda)] &= \frac{1}{s\sqrt{2\pi}} r^{1-(1/2s)} \int_0^{\infty} z^{(\frac{1}{2s})-1} e^{-z} dz \\ &= \frac{1}{s\sqrt{2\pi}} r^{1-(1/2s)} \Gamma\left(\frac{1}{2s}\right), \end{aligned}$$

where $\Gamma(\cdot)$ is the usual Gamma function. The case where $r = 1/2$ and $s = 1$ leads us to the Normal distribution case. ■

Consider again g_Z as the density generator of a univariate spherical random variable $Z \sim S(g_Z)$. Now we define what we meant by probability transformation using these elliptical density generators. These transformations we will, for simplicity, call *elliptical transformations*.

Definition 3 Let X be a random variable with tail probability function $\overline{F}_X(\cdot)$ and whose density function exists and is equal to $f_X(\cdot)$. We define the transformed random variable, denoted by X^* , to be one with a (transformed) density function given by

$$f_{X^*}(x) = C \times \frac{g_Z \left[(\Phi^{-1}(\overline{F}_X(X)) + \lambda)^2 \right]}{g_N \left[(\Phi^{-1}(\overline{F}_X(X)))^2 \right]} \times f_X(x) \quad (21)$$

for all x in the domain or range of X and where C is a normalizing constant.

By recognizing that $h_{g_Z}(X; \lambda) = \frac{g_Z \left[(\Phi^{-1}(\overline{F}_X(X)) + \lambda)^2 \right]}{g_N \left[(\Phi^{-1}(\overline{F}_X(X)))^2 \right]}$, the ratio of density generators, we simply note that the normalizing constant is

$$C = \frac{1}{\mathbb{E}[h_{g_Z}(X; \lambda)]}$$

for which can be easily evaluated from Theorem 1. We can, as a matter of fact, find an expression for the distribution function of the (transformed) random variable X^* . First, notice that

$$\overline{F}_{X^*}(x) = \int_x^\infty f_{X^*}(v) dv = C \int_x^\infty \frac{g_Z \left[(\Phi^{-1}(\overline{F}_X(v)) + \lambda)^2 \right]}{g_N \left[(\Phi^{-1}(\overline{F}_X(v)))^2 \right]} \times f_X(v) dv.$$

Applying the transformation $u = \overline{F}_X(v)$ so that $du = -f_X(v) dv$, this yields

$$\overline{F}_{X^*}(x) = C \int_0^{\overline{F}_X(x)} \frac{g_Z \left[(\Phi^{-1}(u) + \lambda)^2 \right]}{g_N \left[(\Phi^{-1}(u))^2 \right]} du$$

and then applying the transformation $w = \Phi^{-1}(u)$ so that $du = \frac{1}{\sqrt{2\pi}} g_N(w^2) dw$, we have

$$\begin{aligned} \overline{F}_{X^*}(x) &= \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(\overline{F}_X(x))} \frac{g_Z \left[(w + \lambda)^2 \right]}{g_N(w^2)} g_N(w^2) dw \\ &= \int_{-\infty}^{\Phi^{-1}(\overline{F}_X(x)) + \lambda} C_Z \times g_Z(z^2) dz \\ &= F_Z \left[\Phi^{-1}(\overline{F}_X(x)) + \lambda \right] \end{aligned} \quad (22)$$

where $F_Z(\cdot)$ is the distribution function of a spherical random variable with density generator g_Z . In short, we have $Z \sim S(g_Z)$.

In an independent work done by Landsman (2004), he also proposed to use a transformation of densities of elliptical random variables using a ratio of the density generator evaluated as

$$\frac{g\left[\left(\frac{x - \mu}{\sigma}\right)^2 - 2\lambda x\right]}{g\left[\left(\frac{x - \mu}{\sigma}\right)^2\right]}.$$

Landsman (2004) showed that this transformation is a generalization of the Esscher transform for elliptical distributions. He further demonstrated that this generalizes the variance premium principle applied again to elliptical distributions and he appropriately called these transformations *elliptical tilting*.

There are several differences between Landsman's elliptical tilting with those proposed in this paper. First, unlike Landsman (2004), we take the ratio of two different density generators, one of which is always the Normal density generator, and the other is an arbitrarily chosen one. In effect, we are transforming the distribution by taking the relative weights of the densities of an arbitrarily chosen elliptical random variable to the Normal random variable. The arbitrary selection allows the decision maker (which in this case is the insurer) some flexibility. Landsman chooses the density generator of the random variable being tilted which is always an elliptical random variable. Second, while both Landsman and this paper allows translation of the distribution by introducing a shift parameter λ , the manner in which the translation is accomplished differ in both respects. Landsman introduces the shift using $\left(\frac{x - \mu}{\sigma}\right)^2 - 2\lambda x$ while we, in this paper, introduce the shift via $\Phi^{-1}(\overline{F}_X(x)) + \lambda$. Third, Landsman limited its applications to exponential tilting of elliptical random variables while ours does not necessarily have such limitations. Lastly, in the case of a Normal distribution, Landsman generalizes the variance premium principle while we generalize the standard deviation premium principle. The standard deviation is more commonly utilized as a measure of the level of riskiness of a portfolio. See, for example, Markowitz (1952) and Merton (1990).

Definition 4 Let X be a random variable with tail probability function $\overline{F}_X(\cdot)$ and whose density function exists and is equal to $f_X(\cdot)$. Let X^* be the transformed random variable of X according to the elliptical transformation defined in (21). Then the expectation of X^* is defined to be the premium principle implied by the elliptical transformation:

$$\pi[X] = E(X^*) = E\left[\frac{h_{g_Z}(X; \lambda)}{E[h_{g_Z}(X; \lambda)]} \cdot X\right].$$

Observe that from (22), we can also derive the premium (or expectation of the transformed distribution) using

$$\pi[X] = E(X^*) = -\int_{-\infty}^0 F_Z[\Phi^{-1}(\overline{F}_X(x)) + \lambda] dx + \int_0^{\infty} \overline{F}_Z[\Phi^{-1}(\overline{F}_X(x)) + \lambda] dx$$

which reduces to just $\int_0^{\infty} \overline{F}_Z[\Phi^{-1}(\overline{F}_X(x)) + \lambda] dx$ for random variables with non-negative support.

We produce Figures 1 to 4 to help us visualize the transformation. All figures in this paper are attached as appendix following the list of references. For Figures

1 through 4, we assume the random variable X being transformed is Uniform on the interval $(0, 100)$. This choice has been made to help us more vividly visualize the effects of the transformation. The straight horizontal broken line found in these figures is the un-transformed (or original) Uniform density.

In Figure 1, we consider the case where we choose Z to be the Normal distribution and we present the various transformation for different values of λ . Generally, we see that using the Normal density generator, this puts more weights on the right-tail of the distribution. So if the risk are losses as in the case of insurances, this penalizes more heavily large amount of losses. The effect of λ introduces a shift of the distribution to the right for larger λ putting in an even heavier penalty on the tails.

In Figures 2, 3 and 4, we consider the case where we choose Z to be the Student-t distribution. Figure 2 examines the effect of increasing the degrees of freedom (fixing $\lambda = 1$) while Figure 3 examines the effect of increasing the risk aversion parameter λ (fixing $m = 5$). In Figure 4, we present the case where $\lambda = 0$ when there is no shift in the distribution. Here we observe equal penalty of both ends of the tail. The Student-t distortion has been first introduced by Wang (2004) in the pricing of catastrophe bonds. As pointed out in his paper, putting more weights to the tails of the underlying distributions is a way to reflect investor behavior and preferences that we often observe in the market. Investors generally fear large unexpected losses, but they also like large unexpected gains.

Interestingly as special cases, we recover some of the familiar premium principles as discussed below. This in some sense generalizes these premium principles.

Example 3.4: *Wang Premium Principle*

If g_Z is chosen to be the density generator of a Normal distribution, then it is immediate to get the Wang transformation:

$$\bar{F}_{X^*}(x) = \Phi[\Phi^{-1}(\bar{F}_X(x)) + \lambda].$$

See Wang (1996) and Wang (2000).

Example 3.5: *Wang's Student-t Distortion Premium Principle*

If g_Z is chosen to be the density generator of a Student-t distribution with m degrees of freedom, then we have as defined in Wang (2004)

$$\bar{F}_{X^*}(x) = Q[\Phi^{-1}(\bar{F}_X(x)) + \lambda],$$

where, following Wang's notation, $Q(\cdot)$ denotes the distribution function of a Student-t with m degrees of freedom. Wang actually has set $\lambda = 0$ and used $\bar{F}_{X^*}(x) = Q[\Phi^{-1}(\bar{F}_X(x))]$ instead.

Example 3.6: *Esscher Premium Principle*

Because $1 - \Phi(x) = \Phi(-x)$, then in the special case where X is Normally distributed say $N(\mu, \sigma^2)$, it is rather straightforward to see that the elliptical transformation leads to the Esscher transform. In this case, we have

$$\frac{h_{g_Z}(X; \lambda)}{\mathbb{E}[h_{g_Z}(X; \lambda)]} = \frac{e^{-\lambda^2/2} \exp[-\lambda \Phi^{-1}(\bar{F}_X(X))]}{\mathbb{E}[e^{-\lambda^2/2} \exp[-\lambda \Phi^{-1}(\bar{F}_X(X))]]} = \frac{\exp(\frac{\lambda}{\sigma} X)}{\mathbb{E}[\exp(\frac{\lambda}{\sigma} X)]}.$$

See Esscher (1932) and also more recent articles, for example, Gerber and Shiu (1994).

4 Location-Scale Families

By considering the risk X to belong to families of location-scale distributions, we can further simplify the premium principle implied by the elliptical transformation. Members of the elliptical distributions also belong to the larger family of location-scale distributions. As a matter of fact, in this case, we are able to recover a similar concept to the standard deviation premium principle. We give the following theorem.

Theorem 2 *Let X be a random variable belonging to a location-scale family so that $\bar{F}_X(x) = \bar{F}_{Z^*}\left(\frac{x - \mu}{\sigma}\right)$ for some $Z^* = \frac{X - \mu}{\sigma}$ whose distribution is independent of the location parameter μ and scale parameter σ . Then the premium principle implied by the elliptical transformation can be expressed as*

$$\pi[X] = \mu + E_Z \left[\bar{F}_{Z^*}^{-1}(\Phi(Z - \lambda)) \right] \times \sigma.$$

Proof. First assuming the density of X exists and is denoted by $f_X(\cdot)$, then we can write the expectation as

$$\pi[X] = C \int_{-\infty}^{\infty} x \frac{g_Z \left[(\Phi^{-1}(\bar{F}_X(x)) + \lambda)^2 \right]}{g_N \left[(\Phi^{-1}(\bar{F}_X(x)))^2 \right]} f_X(x) dx.$$

Applying the transformation $u = \bar{F}_X(x)$ so that $du = -f_X(x) dx$, this yields to

$$\pi[X] = C \int_0^1 \bar{F}_X^{-1}(u) \frac{g_Z \left[(\Phi^{-1}(u) + \lambda)^2 \right]}{g_N \left[(\Phi^{-1}(u))^2 \right]} du.$$

With another transformation $w = \Phi^{-1}(u)$ so that $du = \frac{1}{\sqrt{2\pi}} g_N(w^2) dw$, we then have

$$\pi[X] = C \int_{-\infty}^{\infty} \bar{F}_X^{-1}(\Phi(w)) \frac{g_Z \left[(w + \lambda)^2 \right]}{g_N(w^2)} \frac{1}{\sqrt{2\pi}} g_N(w^2) dw.$$

Applying one last transformation $z = w + \lambda$, we get

$$\pi[X] = \int_{-\infty}^{\infty} \bar{F}_X^{-1}(\Phi(z - \lambda)) C_Z g_Z(z^2) dz$$

where $C_Z = \left[\int_{-\infty}^{\infty} g_Z(z^2) dz \right]^{-1}$ is the normalizing constant of the spherical random variable Z . Thus, we see that

$$\pi[X] = E_Z \left[\bar{F}_X^{-1}(\Phi(Z - \lambda)) \right]$$

where the expectation E_Z is evaluated with respect to the spherical random variable Z . Now, since X is location scale, we can re-write

$$\bar{F}_X^{-1}(\Phi(Z - \lambda)) = \mu + \bar{F}_{Z^*}^{-1}(\Phi(Z - \lambda)) \times \sigma$$

and the desired result immediately follows. ■

This result gives us a way of interpreting the parameter λ , which is actually a risk premium per unit of risk (or volatility) as measured by the standard deviation. In the case where X is normally distributed $N(\mu, \sigma^2)$ and we choose Z to be the standard Normal, then we get

$$\mathbb{E}_Z \left[\overline{F}_{Z^*}^{-1}(\Phi(Z - \lambda)) \right] = \mathbb{E}_Z(\lambda - Z) = \lambda - \mathbb{E}_Z(Z) = \lambda,$$

so that the resulting premium principle leads us to the familiar form of the standard deviation premium principle:

$$\pi[X] = \mu + \lambda\sigma.$$

Therefore, we have $\lambda = \frac{\pi[X] - \mu}{\sigma}$, a risk premium per unit of risk as measured by σ . Furthermore, it can also be shown that by using a first-order Taylor's series expansion, we can have the following approximation:

$$\mathbb{E}_Z \left[\overline{F}_{Z^*}^{-1}(\Phi(Z - \lambda)) \right] \approx \overline{F}_{Z^*}^{-1}(\Phi(-\lambda)) = \overline{F}_{Z^*}^{-1}(\overline{\Phi}(\lambda)),$$

where $\overline{\Phi}$ denotes the tail probability function of a standard normal.

5 Practical Implementation

In this section, we illustrate how to practically implement the premium principles developed in this paper using the empirical losses investigated in Frees and Valdez (1998). The experience consisted of a random sample of 1,500 claims from a general insurance liability portfolio provided by the Insurance Services Office, Inc. For our purposes, we only include in this analysis the observations from claims, for which we denote here as the risk X . Unfortunately, the experience data we consider here consists of observations that generated claims arising from an insurance portfolio. In an ordinary insurance portfolio, the case is usually one where there is a mass of zero claims, some observations that never generated claims. In the pricing of insurance, one would take into account the probability mass of zero claims. For the lack of available insurance claims data, we resort to this experience data, however, we ask the reader to exercise caution that normally in insurance pricing, the probability mass of zero claims must be taken into account. This mass has been ignored in the ensuing analysis.

Summary statistics can be found in Frees and Valdez (1998), but just to reiterate these statistics, the average loss was 41,208, the median was 12,000 and the standard deviation was 102,748. In fitting for the best distribution for this experience loss data, censoring had to be considered because claims have policy limits and if the actual loss exceeded the policy limit, only the policy limit was recorded. At any rate, according to Frees and Valdez (1998), the best fitting loss distribution model for the data has the Pareto form described by

$$\overline{F}_X(x) = \left(\frac{\lambda}{\lambda + x} \right)^\theta, \text{ for } x > 0.$$

Its density function therefore has the form

$$f_X(x) = \theta\lambda^\theta \left(\frac{1}{\lambda + x} \right)^{\theta+1}.$$

The maximum likelihood parameter estimates are $\hat{\lambda} = 14,453$ and $\hat{\theta} = 1.135$, with respective standard errors of 1,397 and 0.066. In Figure 5, we provide a frequency histogram of the logarithm of the observed losses (called logloss) with the fitted Pareto density function of the logloss. This figure ignores the censored observations.

For simplicity, in constructing the premium principle used to illustrate, we have considered the choice of a Student-t density generator with $m = 3$ degrees of freedom. There is no empirical basis for this choice, except the Student-t appears to be the most sensible as it allows flexibility with the additional parameter for degrees of freedom. Thus, from Example 3.2, we can write³

$$h_{gz}(X; \lambda) = \frac{\left[1 + \frac{1}{3} \left(\Phi^{-1}\left(\left(\frac{\lambda}{\lambda+X}\right)^\theta\right) + \lambda\right)^2\right]^2}{\exp\left[\frac{1}{2} \left(\Phi^{-1}\left(\left(\frac{\lambda}{\lambda+X}\right)^\theta\right)\right)^2\right]}.$$

The purpose here is therefore to develop risk-adjusted premiums. The results are summarized in Figures 6 and 7. Figure 6 provides the risk-adjusted premiums for varying values of λ and it is not surprising to see that the risk-adjusted premiums increase with increasing λ . For the different risk-adjusted premiums implied by the different λ , we computed then the probability of having losses smaller than these risk-adjusted premiums, under the original probability measure. This level of probability we denote here by q , and in Figure 7, we show the risk-adjusted premiums as a function of this probability level. Again, there is higher risk-adjusted premiums associated with larger q . Notice that the risk-adjusted premium does not always yield a larger value than the unadjusted premium (that is, the net premium equal to $\lambda/(\theta - 1)$ in the Pareto case), but because of its increasing nature, for some level λ and correspondingly for some level q , beyond which there will be a positive risk premium. Numerically, these results are summarized as Table 2 for convenience. Here, the net premium level is 107,059, and at $\lambda = 1.57204$ (or correspondingly $q = 0.9108$), this premium level will equal the risk-adjusted premium. See Table 2.

TABLE 2
THE RISK-ADJUSTED PREMIUMS AND THEIR PROBABILITY LEVELS

λ	unadjusted premium $E(X)$	risk-adjusted premium $\pi[X]$	probability level q
0.0	107,059	4,540	0.2665
0.5	107,059	11,848	0.4932
1.0	107,059	31,784	0.7328
1.5	107,059	91,126	0.8953
2.0	107,059	290,017	0.9685
2.5	107,059	1,051,629	0.9924
3.0	107,059	4,462,059	0.9985

³correction made: 1-May-2005.

6 Final Remarks

This paper introduces the notion of elliptical transformation leading to a premium principle which in some sense generalizes the familiar Wang transformation as well as the premium principle introduced by Wang (2004) using the Student-t distortion function. We also note that in the special case of transforming random variables belonging to the location-scale families, the resulting premium principle is the standard deviation premium principle. Furthermore, we observe from the figures presented in the paper that in some sense, the transformation introduces a heavy penalty on the extreme right tails of the distribution but also encourages small losses by placing relatively reasonable weights on the extreme left tails of the distribution. How much this penalty is can depend upon the choice of the density generator together with the parameter λ which in a sense gives a measure of aversion to the level of risk of the insurer. This parameter also introduces a shift in the distribution of the risk. In the Normal distribution case, the same parameter leads to an interpretation of a risk premium per unit of volatility, or risk. We are also able to show that the elliptical transformation recovers many other familiar premium principles. The notion of elliptical transformation introduced in this paper can also be applied as a risk measure to compute economic capital. It will be interesting how this risk measure may be extended to the case where there is a need to aggregate several risks or to re-allocate the total economic capital into various constituents. Moreover, in the future, it will be an interesting work to examine some properties of this premium principle.

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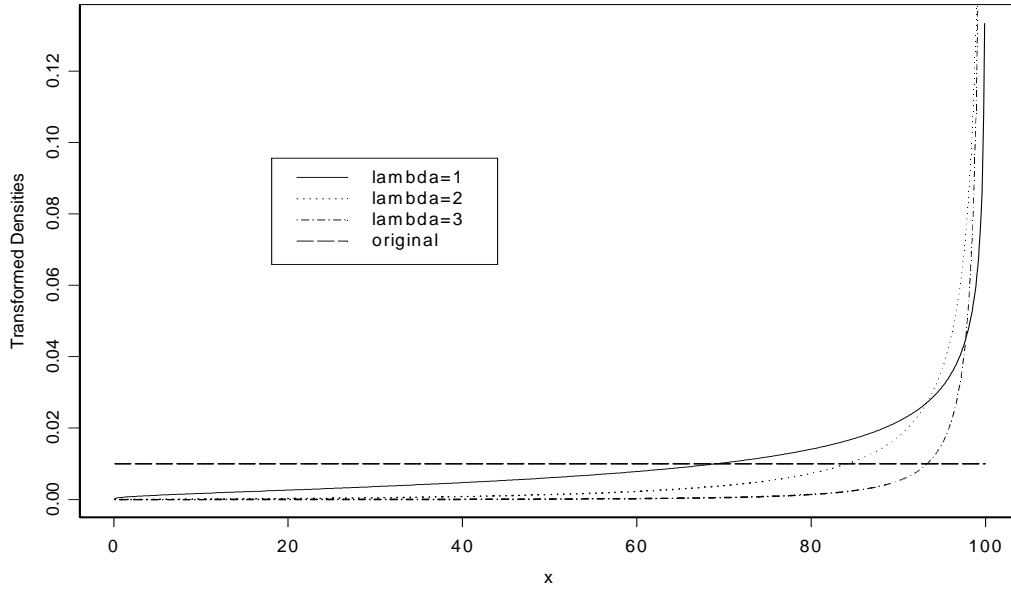


FIGURE 1: ELLIPTICAL TRANSFORMATION USING THE NORMAL DENSITY GENERATOR. THIS TRANSFORMATION EFFECTIVELY RESULTS IN THE WANG TRANSFORMATION SHOWING LARGE PENALTY ON THE RIGHT TAIL.

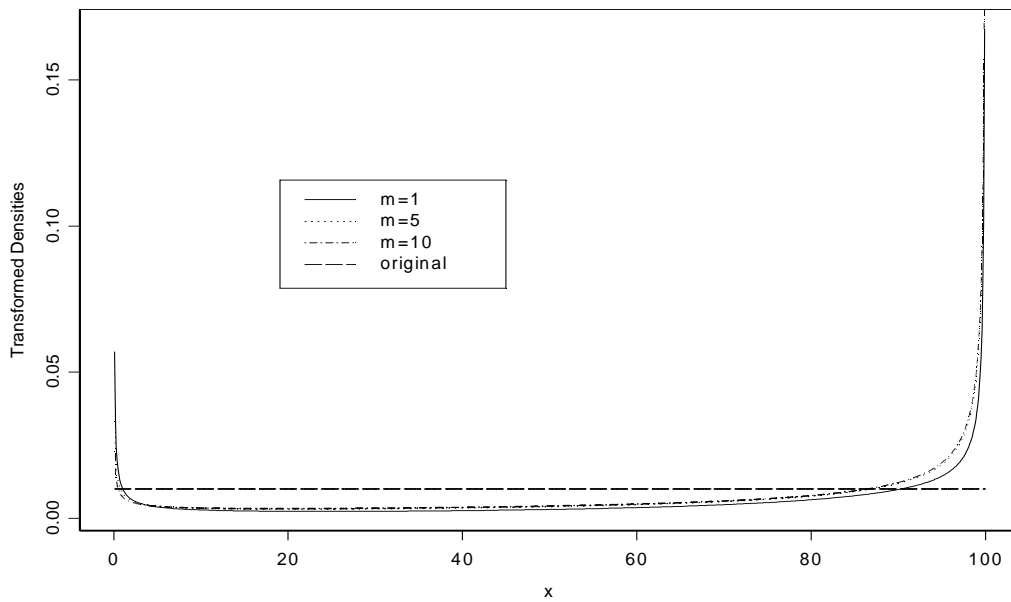


FIGURE 2: ELLIPTICAL TRANSFORMATION USING THE STUDENT-T DENSITY GENERATOR, VARYING THE DEGREES OF FREEDOM. THIS TRANSFORMATION EFFECTIVELY PENALIZES BOTH EXTREMES OF THE DISTRIBUTION, BUT MORE SO WITH RIGHT TAIL.

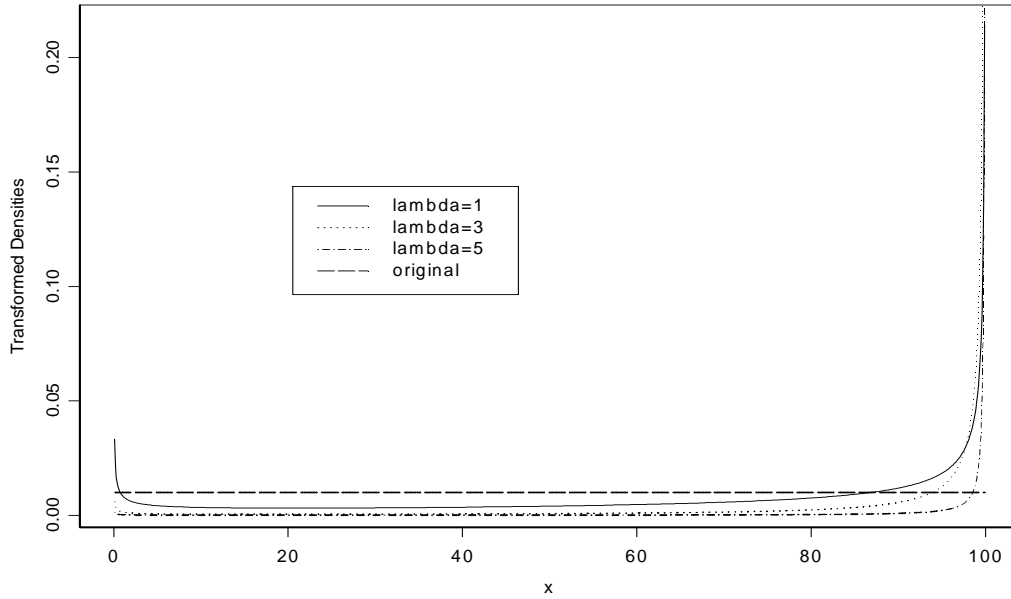


FIGURE 3: ELLIPTICAL TRANSFORMATION STILL USING THE STUDENT-T DENSITY GENERATOR, BUT VARYING THE PARAMETER λ . HERE WE HAVE FIXED THE DEGREES OF FREEDOM TO 5.

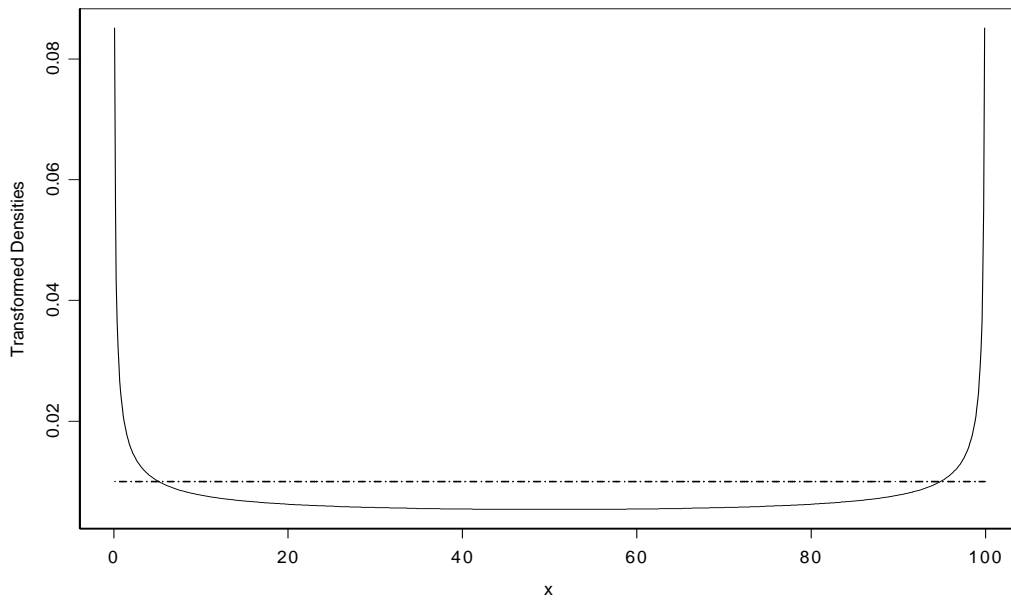


FIGURE 4: ELLIPTICAL TRANSFORMATION USING THE STUDENT-T DENSITY GENERATOR. THIS IS THE CASE WHERE $\lambda = 0$ AND DEGREES OF FREEDOM $m = 5$. IT SHOWS THAT EQUAL PENALTIES ARE IMPOSED AT EXTREMES OF THE DISTRIBUTION.

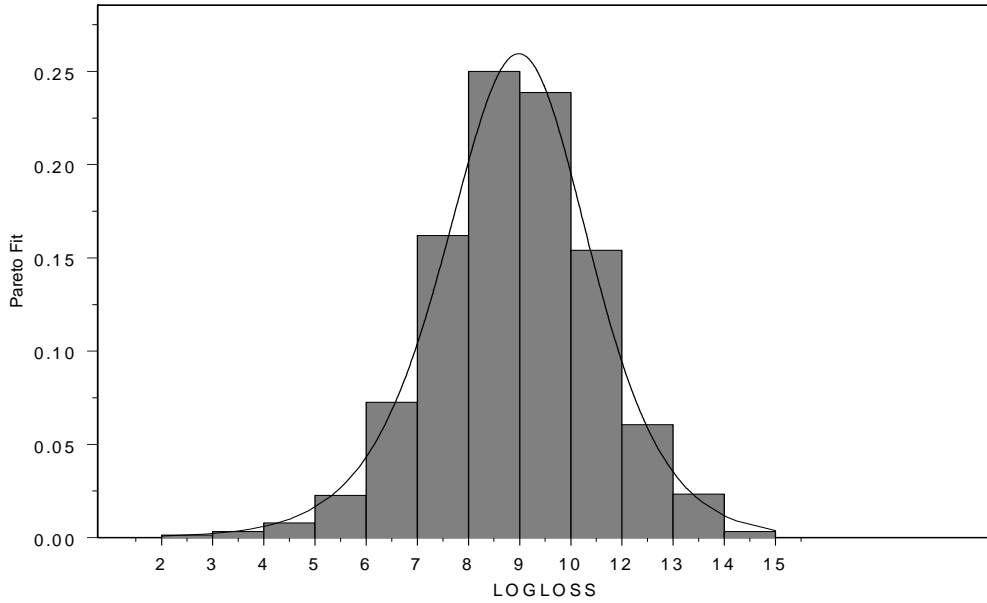


FIGURE 5: FREQUENCY HISTOGRAM OF THE LOGARITHM OF CLAIMS. THE SMOOTH CURVE SUPERIMPOSED ON THE HISTOGRAM IS THE FITTED PARETO DISTRIBUTION.

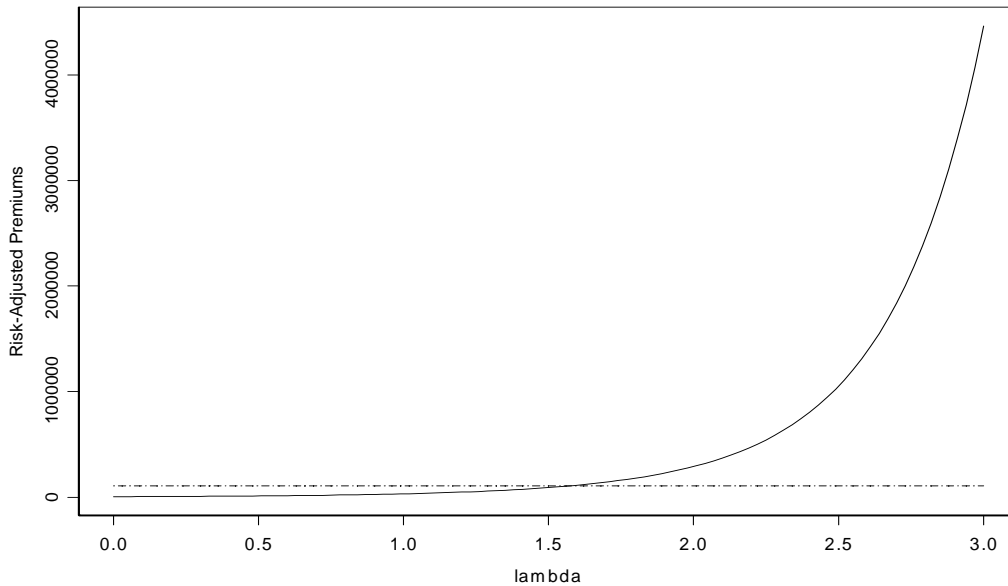


FIGURE 6: THE RISK-ADJUSTED PREMIUM AS A FUNCTION OF THE DISTRIBUTION SHIFT PARAMETER λ . THE BROKEN HORIZONTAL LINE GIVES THE UNADJUSTED NET PREMIUM, THAT IS, THE EXPECTED VALUE OF THE UN-TRANSFORMED DISTRIBUTION. A POSITIVE LOADING THEREFORE RESULTS ONLY WHEN λ IS CHOSEN SO THAT THE SMOOTH CURVE IS ABOVE THE HORIZONTAL LINE.

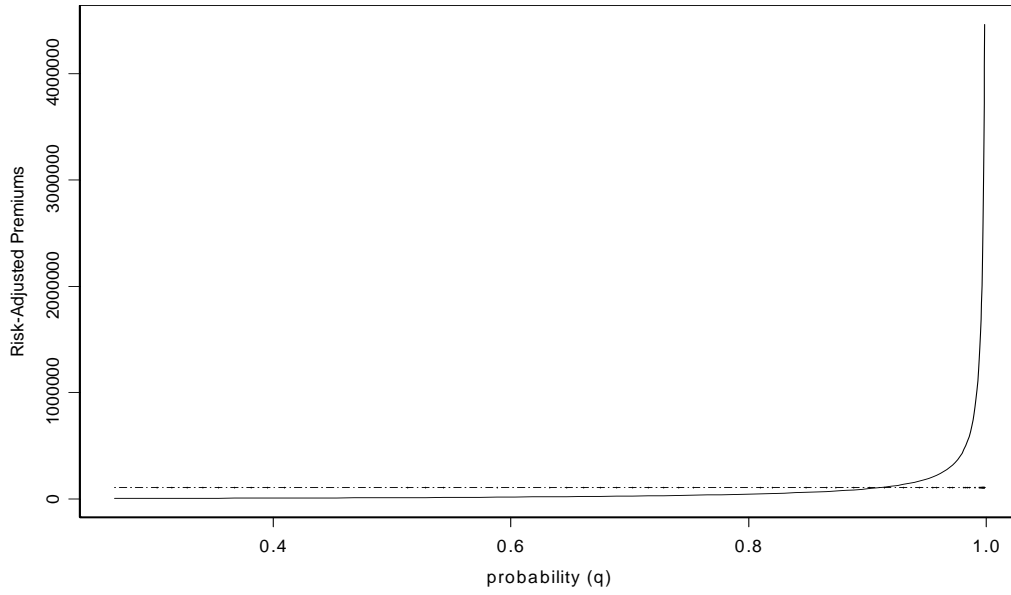


FIGURE 7: THE RISK-ADJUSTED PREMIUM AS A FUNCTION OF THE LEVEL OF PROBABILITY q . THE BROKEN HORIZONTAL LINE GIVES THE UNADJUSTED NET PREMIUM, THAT IS, THE EXPECTED VALUE OF THE UN-TRANSFORMED DISTRIBUTION.