Multivariate Chain–Ladder

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Multivariate Chain–Ladder

1. The Univariate Chain–Ladder Method

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2. The Univariate Model of Schnaus

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3. The Multivariate Model

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4. The Multivariate Chain–Ladder Method
5. Conclusion

1. The Univariate Chain–Ladder Method

An Example

Accident Year i	**Development Year k**					
	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	
2	1265	2433	3233	3977		
3	1490	2873	3880			
4	1725	3261				
5	1889					

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					3335	3483
						1.044

An Example

Accident Year i	Development Year k 0	1	2	3	4	5
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				6410	7179	
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An Example

Accident Year i	Development Year k 0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	
3	1490	2873	3880			
4	1725	3261				
5	1889					
				6410	7179	
					1,120	1.044

An Example

Accident Year i	Development Year k 0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	4650
3	1490	2873	3880			
4	1725	3261				
5	1889					
				6410	7179	
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2	1265	2433	3233	3977	4454	4650
3	1490	2873	3880	4780	5354	5590
4	1725	3261				
5	1889					
			8430	10387		
				1,232	1,120	1.044

An Example

Accident	**Development Year k**					
Year i	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
1	1113	2103	2774	3422	3844	4013
2	1265	2433	3233	3977	4454	4650
3	1490	2873	3880	4780	5354	5590
4	1725	3261	4334	5339	5980	6243
5	1889					
		9264	12310			
			1.329	1,232	1,120	1.044

An Example

Accident Year i	**Development Year** k 0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3483
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4	1725	3261	4334	5339	5980	6243
5	1889	3587	4767	5873	6578	6867
	6594	12525				
		1.899	1.329	1,232	1,120	1.044

An Example

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4	1725	3261	4334	5339	5980	6243
5	1889	3587	4767	5873	6578	6867

Abstract Run–Off Triangle

Run–off triangle of observable aggregate claims:

Accident Year	**Development Year**								
	0	1	$\dots$	k	$\dots$	$n-i$	$\dots$	$n-1$	n
0	$S_{0,0}$	$S_{0,1}$	$\dots$	$S_{0,k}$	$\dots$	$S_{0,n-i}$	$\dots$	$S_{0,n-1}$	$S_{0,n}$
1	$S_{1,0}$	$S_{1,1}$	$\dots$	$S_{1,k}$	$\dots$	$S_{1,n-i}$	$\dots$	$S_{1,n-1}$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			
i	$S_{i,0}$	$S_{i,1}$	$\dots$	$S_{i,k}$	$\dots$	$S_{i,n-i}$			
$\vdots$	$\vdots$	$\vdots$		$\vdots$					
$n-k$	$S_{n-k,0}$	$S_{n-k,1}$	$\dots$	$S_{n-k,k}$					
$\vdots$	$\vdots$	$\vdots$							
$n-1$	$S_{n-1,0}$	$S_{n-1,1}$							
n	$S_{n,0}$								

Chain–Ladder Estimation

- Individual Development Factors:

$$F_{i,k} := \frac{S_{i,k}}{S_{i,k-1}}$$

Chain–Ladder Estimation

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$$F_{i,k} \;:=\; \frac{S_{i,k}}{S_{i,k-1}}$$

- Chain–Ladder Factors:

$$\widehat{F}_k^{\mathrm{CL}} \;:=\; \frac{\sum_{j=0}^{n-k} S_{j,k}}{\sum_{j=0}^{n-k} S_{j,k-1}} \;=\; \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

(for $k \geq 1$)

Chain–Ladder Estimation

- Identity for aggregate claims:

$$S_{i,k} = S_{i,n-i} \prod_{l=n-i+1}^{k} F_{i,l}$$

Chain–Ladder Estimation

- Identity for aggregate claims:

$$S_{i,k} \;=\; S_{i,n-i} \prod_{l=n-i+1}^{k} F_{i,l}$$

- Chain–Ladder Estimators:

$$\widehat{S}_{i,k}^{\mathrm{CL}} \;:=\; S_{i,n-i} \prod_{k=n-i+1}^{k} \widehat{F}_{l}^{\mathrm{CL}}$$

(for $k \geq n-i$)

Chain–Ladder Estimation:

Chain–Ladder estimation is recursive since

$$\widehat{S}^{\text{CL}}_{i,n-i} = S_{i,n-i}$$

and

$$\widehat{S}^{\text{CL}}_{i,k} = \widehat{S}^{\text{CL}}_{i,k-1}\,\widehat{F}^{\text{CL}}_{k}$$

for $k = n - i + 1, \ldots, n$

Modified Chain–Ladder Estimation

- Chain–Ladder Factors:

$$\widehat{F}_k^{\mathrm{CL}} = \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

Modified Chain–Ladder Estimation

- Chain–Ladder Factors:

$$\widehat{F}_k^{\text{CL}} = \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

- Modified Chain–Ladder Factors:

$$\widehat{F}_k := \sum_{j=0}^{n-k} W_{j,k} \, F_{j,k}$$

with $\sum_{j=0}^{n-k} W_{j,k} = 1$

(for $k \geq 1$)

Modified Chain–Ladder Estimation

- Chain–Ladder Estimators:

$$\widehat{S}_{i,k}^{\mathrm{CL}} \; := \; S_{i,n-i} \prod_{k=n-i+1}^{k} \widehat{F}_{l}^{\mathrm{CL}}$$

Modified Chain–Ladder Estimation

- Chain–Ladder Estimators:

$$\widehat{S}_{i,k}^{\text{CL}} \;:=\; S_{i,n-i} \prod_{k=n-i+1}^{k} \widehat{F}_l^{\text{CL}}$$

- Modified Chain–Ladder Estimators:

$$\widehat{S}_{i,k} \;:=\; S_{i,n-i} \prod_{k=n-i+1}^{k} \widehat{F}_l$$

$\Longrightarrow$ Competition !!!

(for $k \geq n-i$)

2. The Univariate Model of Schnaus

Truncated Run–Off Square

Accident Year	**Development Year** 0	1	$\dots$	$k-1$	k	$\dots$
0	$S_{0,0}$	$S_{0,1}$	$\dots$	$S_{0,k-1}$	$S_{0,k}$	$\dots$
1	$S_{1,0}$	$S_{1,1}$	$\dots$	$S_{1,k-1}$	$S_{1,k}$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
$n-k$	$S_{n-k,0}$	$S_{n-k,1}$	$\dots$	$S_{n-k,k-1}$	$S_{n-k,k}$	$\dots$
$n-k+1$	$S_{n-k+1,0}$	$S_{n-k+1,1}$	$\dots$	$S_{n-k+1,k-1}$	$S_{n-k+1,k}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
$n-1$	$S_{n-1,0}$	$S_{n-1,1}$	$\dots$	$S_{n-1,k-1}$	$S_{n-1,k}$	$\dots$
n	$S_{n,0}$	$S_{n-1,1}$	$\dots$	$S_{n,k-1}$	$S_{n,k}$	$\dots$

We denote by $\mathcal{G}_{k-1}$ the information provided by development years $0, 1, \dots, k-1$.

The Model of Schnaus

Assumption:

For each $k = 1, \ldots, n$
there exist *random variables* F_k and V_k such that

$$E(S_{i,k}|\mathcal{G}_{k-1}) \;=\; S_{i,k-1}\, F_k$$

holds for all $i = 0, 1, \ldots, n$.

The Model of Schnaus

Assumption:

For each $k = 1, \ldots, n$
there exist *random variables* F_k and V_k such that

$$\begin{aligned} E(S_{i,k} | \mathcal{G}_{k-1}) &= S_{i,k-1} \, F_k \\ \operatorname{var}(S_{i,k} | \mathcal{G}_{k-1}) &= S_{i,k-1} \, V_k \end{aligned}$$

holds for all $i = 0, 1, \ldots, n$.

The Model of Schnaus

Assumption:

For each $k = 1, \dots, n$
there exist *random variables* F_k and V_k such that

$$
\begin{aligned}
E(S_{i,k}|\mathcal{G}_{k-1}) &= S_{i,k-1}\,F_k \\
\mathrm{var}(S_{i,k}|\mathcal{G}_{k-1}) &= S_{i,k-1}\,V_k \\
\mathrm{cov}(S_{i,k}, S_{j,k}|\mathcal{G}_{k-1}) &= 0
\end{aligned}
$$

holds for all $i, j = 0, 1, \dots, n$ such that $j \neq i$.

The Model of Schnaus

Assumption:

For each $k = 1, \ldots, n$
there exist *random variables* F_k and V_k such that

$$\begin{aligned} E(S_{i,k} | \mathcal{G}_{k-1}) &= S_{i,k-1}\, F_k \\ \mathrm{var}(S_{i,k} | \mathcal{G}_{k-1}) &= S_{i,k-1}\, V_k \\ \mathrm{cov}(S_{i,k}, S_{j,k} | \mathcal{G}_{k-1}) &= 0 \end{aligned}$$

holds for all $i, j = 0, 1, \ldots, n$ such that $j \neq i$.

A slightly less general model is the Model of Mack

The Model of Schnaus

Equivalent formulation of the assumption:

For each $k = 1, \ldots, n$
there exist *random variables* F_k and V_k such that

$$\begin{aligned} E(F_{i,k}|\mathcal{G}_{k-1}) &= F_k \\ \mathrm{var}(F_{i,k}|\mathcal{G}_{k-1}) &= V_k/S_{i,k-1} \\ \mathrm{cov}(F_{i,k}, F_{j,k}|\mathcal{G}_{k-1}) &= 0 \end{aligned}$$

holds for all $i, j = 0, 1, \ldots, n$ such that $j \neq i$.

Estimation of a non–observable $F_{i,k}$ $(i+k > n)$:

- Admissible estimators:

$$\widehat{F}_k := \sum_{j=0}^{n-k} W_{j,k} F_{j,k}$$

with *random variables* $W_{0,k}, W_{1,k}, \ldots, W_{n-k,k} \in \mathcal{G}_{k-1}$ satisfying $\sum_{j=0}^{n-k} W_{j,k} = 1$.

Estimation of a non–observable $F_{i,k}$ $(i+k > n)$:

- The Chain–Ladder Factor

$$\widehat{F}_k^{\mathrm{CL}} := \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

is an admissible estimator.

Estimation of a non–observable $F_{i,k}$ $(i+k > n)$:

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  is an admissible estimator.

- Every admissible estimator is a Modified Chain–Ladder Factor.

Estimation of a non–observable $F_{i,k}$ $(i+k > n)$:

- The Chain–Ladder Factor

$$\widehat{F}_k^{\mathrm{CL}} := \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

  is an admissible estimator.

- Every admissible estimator is a Modified Chain–Ladder Factor.

- *Every* admissible estimator is unbiased in the sense that

$$E\Big(\widehat{F}_k \,\Big|\, \mathcal{G}_{k-1}\Big) = E\Big(F_{i,k} \,\Big|\, \mathcal{G}_{k-1}\Big)$$

Estimation of a non–observable $F_{i,k}$ $(i+k > n)$:

- Optimization problem:
  Minimize

$$E\left(\left(\widehat{F}_k - F_{i,k}\right)^2 \,\middle|\, \mathcal{G}_{k-1}\right)$$

  over all admissible estimators.

Estimation of a non–observable $F_{i,k}$ $(i+k > n)$:

- Optimization problem:
  Minimize

$$E\left(\left(\widehat{F}_k - F_{i,k}\right)^2 \,\middle|\, \mathcal{G}_{k-1}\right)$$

  over all admissible estimators.

- Solution: The optimization problem has the unique solution

$$\widehat{F}_k^{\mathrm{CL}} \;:=\; \sum_{j=0}^{n-k} \frac{S_{j,k-1}}{\sum_{h=0}^{n-k} S_{h,k-1}} F_{j,k}$$

Estimation of $S_{i,n-i+1}$:

- Admissible estimators:

$$\widehat{S}_{i,n-i+1} \;:=\; S_{i,n-i}\,\widehat{F}_{n-i+1}$$

with an admissible estimator $\widehat{F}_{n-i+1}$ of $F_{i,n-i+1}$.

Estimation of $S_{i,n-i+1}$:

- The Chain–Ladder Estimator

$$\widehat{S}^{\mathrm{CL}}_{i,n-i+1} \;:=\; S_{i,n-i}\,\widehat{F}^{\mathrm{CL}}_{n-i+1}$$

is an admissible estimator.

Estimation of $S_{i,n-i+1}$:

- The Chain–Ladder Estimator

$$\widehat{S}^{\mathrm{CL}}_{i,n-i+1} \quad := \quad S_{i,n-i} \, \widehat{F}^{\mathrm{CL}}_{n-i+1}$$

  is an admissible estimator.
- Every admissible estimator is a Modified Chain–Ladder Estimator.

Estimation of $S_{i,n-i+1}$:

- The Chain–Ladder Estimator

$$\widehat{S}^{\mathrm{CL}}_{i,n-i+1} \;:=\; S_{i,n-i}\,\widehat{F}^{\mathrm{CL}}_{n-i+1}$$

  is an admissible estimator.

- Every admissible estimator is a Modified Chain–Ladder Estimator.

- *Every* admissible estimator is unbiased in the sense that

$$E\Big(\widehat{S}_{i,n-i+1}\,\Big|\,\mathcal{G}_{n-i}\Big) \;=\; E\Big(S_{i,n-i+1}\,\Big|\,\mathcal{G}_{n-i}\Big)$$

Estimation of $S_{i,n-i+1}$:

- Optimization problem:
  Minimize

$$E\left(\left(\widehat{S}_{i,n-i+1} - S_{i,n-i+1}\right)^2 \,\middle|\, \mathcal{G}_{n-i}\right)$$

  over all admissible estimators.

Estimation of $S_{i,n-i+1}$:

- Optimization problem:
  Minimize

$$E\left(\left(\widehat{S}_{i,n-i+1} - S_{i,n-i+1}\right)^2 \,\middle|\, \mathcal{G}_{n-i}\right)$$

  over all admissible estimators.

- Solution: The optimization problem has the unique solution

$$\widehat{S}^{\mathrm{CL}}_{i,n-i+1} \quad := \quad S_{i,n-i}\,\widehat{F}^{\mathrm{CL}}_{n-i+1}$$

$\Longrightarrow$ Chain–Ladder is the Winner !!!

Estimation of $S_{i,n-i+1}$:

- Proof:

$$E\left(\left(\widehat{S}_{i,n-i+1} - S_{i,n-i+1}\right)^2 \,\middle|\, \mathcal{G}_{n-i}\right)$$

$$= E\left(\left(S_{i,n-i}\widehat{F}_{n-i+1} - S_{i,n-i}F_{i,n-i+1}\right)^2 \,\middle|\, \mathcal{G}_{n-i}\right)$$

$$= S_{i,n-i}^2 \cdot E\left(\left(\widehat{F}_{n-i+1} - F_{i,n-i+1}\right)^2 \,\middle|\, \mathcal{G}_{n-i}\right)$$

Sequential Optimality:

As a consequence of the previous result,
the chain–ladder estimators are optimal in a sequential sense.

3. The Multivariate Model

Correlated Lines Of Business

- In the present section we consider the loss–development for $m = 2$ correlated lines of business.

Correlated Lines Of Business

- In the present section we consider the loss–development for $m = 2$ correlated lines of business.
- All definitions and results can be extended to arbitrary m.

Run–Off Triangle for Portfolio (1)

Accident Year	**Development Year**								
	0	1	$\dots$	k	$\dots$	$n-i$	$\dots$	$n-1$	n
0	$S_{0,0}^{(1)}$	$S_{0,1}^{(1)}$	$\dots$	$S_{0,k}^{(1)}$	$\dots$	$S_{0,n-i}^{(1)}$	$\dots$	$S_{0,n-1}^{(1)}$	$S_{0,n}^{(1)}$
1	$S_{1,0}^{(1)}$	$S_{1,1}^{(1)}$	$\dots$	$S_{1,k}^{(1)}$	$\dots$	$S_{1,n-i}^{(1)}$	$\dots$	$S_{1,n-1}^{(1)}$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			
i	$S_{i,0}^{(1)}$	$S_{i,1}^{(1)}$	$\dots$	$S_{i,k}^{(1)}$	$\dots$	$S_{i,n-i}^{(1)}$			
$\vdots$	$\vdots$	$\vdots$		$\vdots$					
$n-k$	$S_{n-k,0}^{(1)}$	$S_{n-k,1}^{(1)}$	$\dots$	$S_{n-k,k}^{(1)}$					
$\vdots$	$\vdots$	$\vdots$							
$n-1$	$S_{n-1,0}^{(1)}$	$S_{n-1,1}^{(1)}$							
n	$S_{n,0}^{(1)}$								

Run–Off Triangle for Portfolio (2)

Accident	**Development Year**								
Year	0	1	$\dots$	k	$\dots$	$n-i$	$\dots$	$n-1$	n
0	$S^{(2)}_{0,0}$	$S^{(2)}_{0,1}$	$\dots$	$S^{(2)}_{0,k}$	$\dots$	$S^{(2)}_{0,n-i}$	$\dots$	$S^{(2)}_{0,n-1}$	$S^{(2)}_{0,n}$
1	$S^{(2)}_{1,0}$	$S^{(2)}_{1,1}$	$\dots$	$S^{(2)}_{1,k}$	$\dots$	$S^{(2)}_{1,n-i}$	$\dots$	$S^{(2)}_{1,n-1}$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			
i	$S^{(2)}_{i,0}$	$S^{(2)}_{i,1}$	$\dots$	$S^{(2)}_{i,k}$	$\dots$	$S^{(2)}_{i,n-i}$			
$\vdots$	$\vdots$	$\vdots$		$\vdots$					
$n-k$	$S^{(2)}_{n-k,0}$	$S^{(2)}_{n-k,1}$	$\dots$	$S^{(2)}_{n-k,k}$					
$\vdots$	$\vdots$	$\vdots$							
$n-1$	$S^{(2)}_{n-1,0}$	$S^{(2)}_{n-1,1}$							
n	$S^{(2)}_{n,0}$								

Run–Off Triangle for Both Portfolios

Accident Year	Development Year 0	1	$\ldots$	k	$\ldots$	$n-i$	$\ldots$	$n-1$	n
0	$\mathbf{S}_{0,0}$	$\mathbf{S}_{0,1}$	$\ldots$	$\mathbf{S}_{0,k}$	$\ldots$	$\mathbf{S}_{0,n-i}$	$\ldots$	$\mathbf{S}_{0,n-1}$	$\mathbf{S}_{0,n}$
1	$\mathbf{S}_{1,0}$	$\mathbf{S}_{1,1}$	$\ldots$	$\mathbf{S}_{1,k}$	$\ldots$	$\mathbf{S}_{1,n-i}$	$\ldots$	$\mathbf{S}_{1,n-1}$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			
i	$\mathbf{S}_{i,0}$	$\mathbf{S}_{i,1}$	$\ldots$	$\mathbf{S}_{i,k}$	$\ldots$	$\mathbf{S}_{i,n-i}$			
$\vdots$	$\vdots$	$\vdots$		$\vdots$					
$n-k$	$\mathbf{S}_{n-k,0}$	$\mathbf{S}_{n-k,1}$	$\ldots$	$\mathbf{S}_{n-k,k}$					
$\vdots$	$\vdots$	$\vdots$							
$n-1$	$\mathbf{S}_{n-1,0}$	$\mathbf{S}_{n-1,1}$							
n	$\mathbf{S}_{n,0}$								

Notation

$$\mathbf{S}_{i,k} := \begin{pmatrix} S_{i,k}^{(1)} \\ S_{i,k}^{(2)} \end{pmatrix}$$

$$\mathbf{T}_{i,k} := \begin{pmatrix} S_{i,k}^{(1)} & 0 \\ 0 & S_{i,k}^{(2)} \end{pmatrix}$$

$$\mathbf{F}_{i,k} := \begin{pmatrix} F_{i,k}^{(1)} \\ F_{i,k}^{(2)} \end{pmatrix}$$

Truncated Run–Off Square

Accident Year	**Development Year** 0	1	$\dots$	$k-1$	k	$\dots$
0	$\mathbf{S}_{0,0}$	$\mathbf{S}_{0,1}$	$\dots$	$\mathbf{S}_{0,k-1}$	$\mathbf{S}_{0,k}$	$\dots$
1	$\mathbf{S}_{1,0}$	$\mathbf{S}_{1,1}$	$\dots$	$\mathbf{S}_{1,k-1}$	$\mathbf{S}_{1,k}$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
$n-k$	$\mathbf{S}_{n-k,0}$	$\mathbf{S}_{n-k,1}$	$\dots$	$\mathbf{S}_{n-k,k-1}$	$\mathbf{S}_{n-k,k}$	$\dots$
$n-k+1$	$\mathbf{S}_{n-k+1,0}$	$\mathbf{S}_{n-k+1,1}$	$\dots$	$\mathbf{S}_{n-k+1,k-1}$	$\mathbf{S}_{n-k+1,k}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	
$n-1$	$\mathbf{S}_{n-1,0}$	$\mathbf{S}_{n-1,1}$	$\dots$	$\mathbf{S}_{n-1,k-1}$	$\mathbf{S}_{n-1,k}$	$\dots$
n	$\mathbf{S}_{n,0}$	$\mathbf{S}_{n-1,1}$	$\dots$	$\mathbf{S}_{n,k-1}$	$\mathbf{S}_{n,k}$	$\dots$

We denote by $\mathcal{G}_{k-1}$ the information provided by development years $0, 1, \dots, k-1$.

The Multivariate Model

Assumption:

For each $k = 1, \ldots, n$ there exists a *random vector* $\mathbf{F}_k$ and a *random matrix* $\mathbf{V}_k$ such that

$$E(\mathbf{S}_{i,k} | \mathcal{G}_{k-1}) = \mathbf{S}_{i,k-1} \, \mathbf{F}_k$$

holds for all $i = 0, 1, \ldots, n$.

The Multivariate Model

Assumption:

For each $k = 1, \ldots, n$ there exists a *random vector* $\mathbf{F}_k$ and a *random matrix* $\mathbf{V}_k$ such that

$$\begin{aligned} E(\mathbf{S}_{i,k}|\mathcal{G}_{k-1}) &= \mathbf{T}_{i,k-1}\,\mathbf{F}_k \\ \text{var}(\mathbf{S}_{i,k}|\mathcal{G}_{k-1}) &= \mathbf{T}_{i,k-1}^{1/2}\,\mathbf{V}_k \mathbf{T}_{i,k-1}^{1/2} \end{aligned}$$

holds for all $i = 0, 1, \ldots, n$.

The Multivariate Model

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holds for all $i, j = 0, 1, \dots, n$ such that $j \neq i$.

The Multivariate Model

Assumption:

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holds for all $i, j = 0, 1, \ldots, n$ such that $j \neq i$.

A slightly less general model is the Model of Braun [2004].

The Multivariate Model

Equivalent formulation of the assumption:

For each $k = 1, \ldots, n$ there exists
a *random vector* $\mathbf{F}_k$ and a *random matrix* $\mathbf{V}_k$ such that

$$\begin{aligned} E(\mathbf{F}_{i,k} | \mathcal{G}_{k-1}) &= F_k \\ \mathrm{var}(\mathbf{F}_{i,k} | \mathcal{G}_{k-1}) &= \mathbf{T}_{i,k-1}^{-1/2} \mathbf{V}_k \mathbf{T}_{i,k-1}^{-1/2} \\ \mathrm{cov}(\mathbf{F}_{i,k}, \mathbf{F}_{j,k} | \mathcal{G}_{k-1}) &= \mathbf{O} \end{aligned}$$

holds for all $i, j = 0, 1, \ldots, n$ such that $j \neq i$.

4. The Multivariate Chain–Ladder Method

Estimation of a non–observable $\mathbf{F}_{i,k}$ $(i+k > n)$:

- Admissible estimators:

$$\widehat{\mathbf{F}}_k := \sum_{j=0}^{n-k} \mathbf{W}_{j,k} \mathbf{F}_{j,k}$$

with *random matrices* $\mathbf{W}_{0,k}, \mathbf{W}_{1,k}, \ldots, \mathbf{W}_{n-k,k} \in \mathcal{G}_{k-1}$ satisfying $\sum_{j=0}^{n-k} \mathbf{W}_{j,k} = \mathbf{I}$.

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- *Every* admissible estimator is unbiased in the sense that

$$E\left(\widehat{\mathbf{F}}_k \,\middle|\, \mathcal{G}_{n-i}\right) = E\left(\mathbf{F}_{i,k} \,\middle|\, \mathcal{G}_{n-i}\right)$$

Estimation of a non–observable $\mathbf{F}_{i,k}$ $(i+k > n)$:

Optimization problem:
Minimize

$$E\left(\left(\widehat{\mathbf{F}}_k - \mathbf{F}_{i,k}\right)'\left(\widehat{\mathbf{F}}_k - \mathbf{F}_{i,k}\right) \,\middle|\, \mathcal{G}_{k-1}\right)$$

over all admissible estimators.

Estimation of a non–observable $\mathbf{F}_{i,k}$ $(i+k > n)$:

Solution:
The optimization problem has the unique solution

$$\widehat{\mathbf{F}}_k^{\mathrm{CL}} := \left(\sum_{j=0}^{n-k} \mathbf{T}_{j,k-1}^{1/2} \mathbf{V}_k^{-1} \mathbf{T}_{j,k-1}^{1/2} \right)^{-1} \cdot \sum_{j=0}^{n-k} \mathbf{T}_{j,k-1}^{1/2} \mathbf{V}_k^{-1} \mathbf{T}_{j,k-1}^{1/2} \mathbf{F}_{j,k}$$

The unique solution $\widehat{\mathbf{F}}_k^{\mathrm{CL}}$ is said to be the Multivariate Chain–Ladder Factor of development year k.

Estimation of $\mathrm{S}_{i,n-i+1}$:

- Admissible estimators:

$$\widehat{\mathbf{S}}_{i,n-i+1} \quad := \quad \mathbf{T}_{i,n-i} \, \widehat{\mathbf{F}}_{n-i+1}$$

with an admissible estimator $\widehat{\mathbf{F}}_{n-i+1}$ of $\mathbf{F}_{i,n-i+1}$.

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- *Every* admissible estimator is unbiased in the sense that

$$E\left(\widehat{\mathbf{S}}_{i,n-i+1} \,\middle|\, \mathcal{G}_{n-i}\right) = E\left(\mathbf{S}_{i,n-i+1} \,\middle|\, \mathcal{G}_{n-i}\right)$$

Estimation of $\mathrm{S}_{i,n-i+1}$:

- Optimization problem:
  Minimize

$$E\left(\left(\widehat{\mathbf{S}}_{i,n-i+1}-\mathbf{S}_{i,n-i+1}\right)^2 \,\middle|\, \mathcal{G}_{n-i}\right)$$

  over all admissible estimators.

Estimation of $\mathrm{S}_{i,n-i+1}$:

- Optimization problem:
  Minimize

$$E\left(\left(\widehat{\mathbf{S}}_{i,n-i+1} - \mathbf{S}_{i,n-i+1} \right)^2 \,\middle|\, \mathcal{G}_{n-i} \right)$$

  over all admissible estimators.

- Solution:
  The optimization problem has the unique solution

$$\widehat{\mathbf{S}}^{\mathrm{CL}}_{i,n-i+1} \;:=\; \mathbf{T}_{i,n-i}\, \widehat{\mathbf{F}}^{\mathrm{CL}}_{n-i+1}$$

Estimation of $\mathbf{S}_{i,n-i+1}$:

The unique solution $\widehat{\mathbf{S}}_k^{\mathrm{CL}}$ is said to be the Multivariate Chain–Ladder Estimator of $\mathbf{S}_{i,n-i+1}$.

Recursive Chain–Ladder Estimation:

Define

$$\widehat{\mathbf{S}}^{\mathrm{CL}}_{i,n-i} := \mathbf{S}_{i,n-i}$$

and for $k = n-i+1, \ldots, n$

$$\begin{aligned} \widehat{\mathbf{T}}^{\mathrm{CL}}_{i,k-1} &:= \mathrm{diag}(\widehat{\mathbf{S}}^{\mathrm{CL}}_{i,k-1}) \\ \widehat{\mathbf{S}}^{\mathrm{CL}}_{i,k} &:= \widehat{\mathbf{T}}^{\mathrm{CL}}_{i,k-1}\,\widehat{\mathbf{F}}^{\mathrm{CL}}_{k} \end{aligned}$$

Then $\widehat{\mathbf{S}}^{\mathrm{CL}}_{i,k}$ is said to be the Multivariate Chain–Ladder Estimator of $\mathbf{S}_{i,k}$.

Sequential Optimality:

Sequential optimality obtains as in the univariate case.

5. Conclusion

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  the method is there and a stochastic model had to be found for its justification.

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- In the univariate case,
  the method is there and a stochastic model had to be found for its justification.
- In the multivariate case,
  an extension of the stochastic model of Schnaus leads to an extension of the method.
- The coordinates of the Multivariate Chain–Ladder Estimators usually differ from the Univariate Chain–Ladder Estimators of the Coordinates.
- The multivariate approach resolves the problem of additivity.