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# **DELTA METHOD and RESERVING**

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# Introduction

Presentation of methods based on reserve's moments allowing:

- Estimating:
  - *predictive distribution,*
  - *moments, percentiles (VaR) and their function*
- With estimation risk measure , confidence interval

Within the GLM approach (including Log-Poisson)

# I. Notations

- For a given line of business, claims are assumed to be closed in  $(n + 1)$  years.
- GLM approach is based on *incremental payments r.v.*:

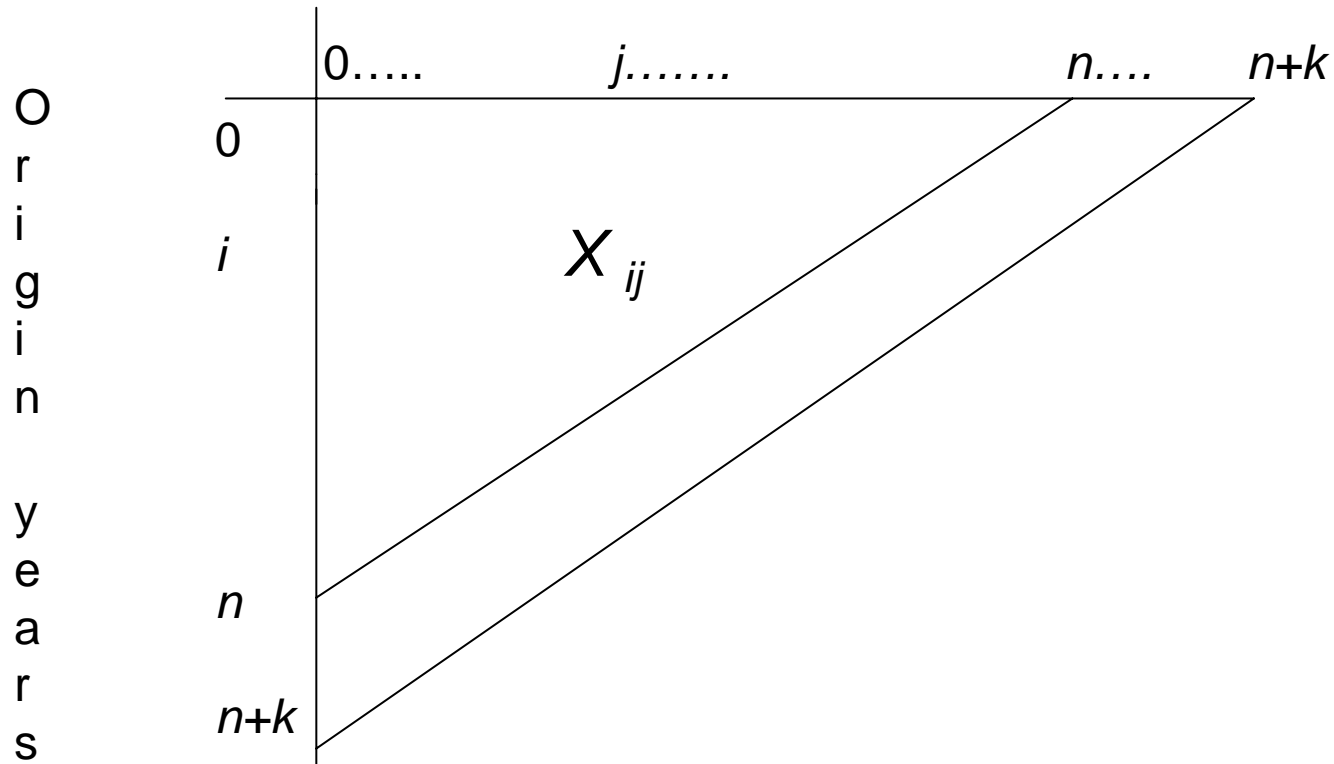
$$X_{ij} , i, j = 0, \dots, n$$

assumed to be **independent**

- Among the  $(n+1)^2$   $X_{ij}$  r. v. inside the run-off triangle those located in the upper triangle have been observed

# I. Notations

## Payments delays



# I. Notations

- Reserve for the  $i^{\text{th}}$  origin year

$$R_i = \sum_{h=n-i+1}^n X_{ih}$$

- Total reserve

$$R = \sum_{i=1}^n R_i$$

- Remark:* To analyze future annual cash flows by integrating new business, cash flows for the accident year  $(n+k)$ :

$$\sum_{i+j=n+k}^n X_{ij}$$

# II. Parameters

## 2.1. Interest Parameters

Interest parameters are linked to the d.f.  $F_R : \Pi(F_R)$

➤ Indicators depending on moments of  $R$  :

mean, dispersion (variance, standard deviation), margins such :

$$E(R) + \gamma \sigma(R)$$

➤ Other indicators: (tail) VaR (percentiles), TailVaR, ...  
probability of insufficiency, ...

**Need** : estimation of d.f.  $F_R$  directly or by inversion of m.g.f.

$$M_R(s) = E(s^R)$$

# II. Parameters

## 2.2. Estimation

For an estimator  $\hat{\Pi} = \hat{\Pi} \left[ (X_{ij})_{i+j \leq n} \right]$  of  $\Pi(F_R)$  uncertainty related to this estimation will be measured by:

▪ asymptotic variance:  $V_{as}(\hat{\Pi})$

▪ standard deviation :  $s.e._{as}(\hat{\Pi}) = \sqrt{V_{as}(\hat{\Pi})} \quad \frac{s.e._{as}(\hat{\Pi})}{\hat{\Pi}}$

In addition, a level 95% asymptotic confidence interval for  $\Pi(F_R)$  is

$$P \left\{ A \left[ (X_{ij})_{i+j \leq n} \right] \leq \Pi(F_R) \leq B \left[ (X_{ij})_{i+j \leq n} \right] \right\} \rightarrow 0,95$$

# III. GLM models

## 3.1. Random component

- independent «responses» :  $X_{ij}$  ( $i, j = 0, \dots, n$ )
- with “exponential” type dist.

$$f(x_{ij}; \theta_{ij}, \phi) = \exp \left\{ \left[ \theta_{ij} x_{ij} - b(\theta_{ij}) \right] / \phi + c(x_{ij}, \phi) \right\}$$

- $\theta_{ij}$  : *natural parameters*
- $\phi > 0$  : *dispersion parameter.*
- $b, c$  *specific functions, b being ‘regular’*

*Ex. : Poisson, Normal, Gamma, IG, Tweedie dist...*

- Moments:  $\mu_{ij} = E(X_{ij}) = b'(\theta_{ij})$   
 $V(X_{ij}) = \phi b''(\theta_{ij}) = \phi V(\mu_{ij})$  **With  $V$  variance function**

# III. GLM models

## 3.1. Random component

- Third moment:  $\mu_3(X_{ij}) = \phi^2 b'''(\theta_{ij}) = V(\mu_{ij})V'(\mu_{ij})$

- Skewness :  $\gamma_1(X_{ij}) = \frac{\mu_3(X_{ij})}{[V(X_{ij})]^{3/2}} = \sqrt{\phi} \frac{b'''(\theta_{ij})}{[b''(\theta_{ij})]^{3/2}} = \sqrt{\phi} \frac{V'(\mu_{ij})}{\sqrt{V(\mu_{ij})}}$

- m.g.f. and cumulant g.f:

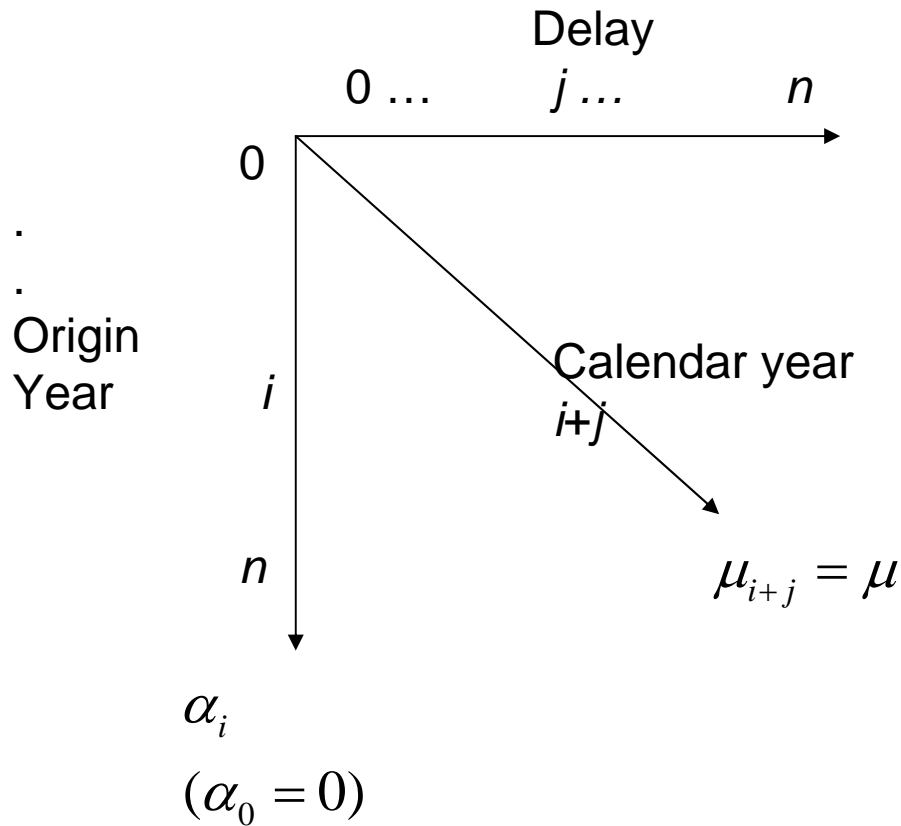
$$M_{X_{ij}}(s) = \exp\left\{\frac{1}{\phi}[b(\theta_{ij} + s\phi) - b(\theta_{ij})]\right\}, C_{X_{ij}}(s) = \log M_{X_{ij}}(s) = \frac{1}{\phi}[b(\theta_{ij} + s\phi) - b(\theta_{ij})]$$

- moments of  $X_{ij}$  functions of  $(\theta_{ij}, \phi)$ , then of  $(\mu_{ij}, \phi)$  :

$$\kappa_r(X_{ij}) = \phi^{r-1} b^{(r)}(\theta_{ij})$$

# III. GLM models

## 3.2. Systematic Component, link function



regression parameters :

$$\xi = \left[ \mu, (\alpha_i)_{i=1, \dots, n}, (\beta_j)_{j=1, \dots, n} \right]$$

# III. GLM models

## 3.2. Systematic Component, link function

- systematic component :  $\eta_{ij} = \mu + \alpha_i + \beta_j$  ( $i, j = 0, \dots, n$ )
- link function : *monotone and derivable real function*  $g$  :

$$\eta_{ij} = g(\mu_{ij}) \Leftrightarrow \mu_{ij} = g^{-1}(\eta_{ij})$$

- Identity link :  $\eta_{ij} = \mu_{ij} = \mu + \alpha_i + \beta_j$
- Log link :  $\eta_{ij} = \log \mu_{ij} \Leftrightarrow \mu_{ij} = e^{\mu + \alpha_i + \beta_j}$

# IV. Estimation


▪ based on:

- upper triangle likelihood

$$L \left[ \left( x_{ij} \right)_{i+j \leq n} ; \mu, (\alpha_i), (\beta_j), \phi \right]$$

- and Wedderburn equations

$$\frac{\delta \log L}{\delta \xi} = 0$$

m.l.e of	
$\hat{\xi} = \left[ \hat{\mu}, (\hat{\alpha}_i), (\hat{\beta}_j) \right]$	$\xi = \left[ \mu, (\alpha_i), (\beta_j) \right]$
$\hat{\eta}_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j, \hat{\mu}_{ij} = g^{-1}(\hat{\eta}_{ij})$	$\eta_{ij}, \mu_{ij} = E(X_{ij})$
As: $R = \sum_{i+j > n} X_{ij} \Rightarrow E(R) = \sum_{i+j > n} \mu_{ij}$	$E(R)$
$E(R) = \sum_{i+j > n} \hat{\mu}_{ij}$	

# IV. Estimation

## 4.1. Estimation risk (Delta method)

- m.l.e.: -  $\hat{\xi}$        $AN\left[\xi, \Sigma_{as}(\hat{\xi})\right]$        $\Sigma_{as}(\hat{\xi}) = I^{-1}(\xi)$
- $\hat{\eta} = (\hat{\eta}_{ij})$        $AN\left[\eta, \Sigma_{as}(\hat{\eta})\right]$        $\Sigma_{as}(\hat{\eta}) = J_{\eta} \Sigma_{as}(\hat{\xi}) J'_{\eta}$

$J_{\eta}$  jacobian matrix of       $\eta : \xi \rightarrow \eta(\xi) = (\eta_{ij})$

$$\frac{\partial \eta_{ij}}{\partial \mu} = 1, \quad \frac{\partial \eta_{ij}}{\partial \alpha_k} = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}, \quad \frac{\partial \eta_{ij}}{\partial \beta_l} = \begin{cases} 1 & \text{if } l = j \\ 0 & \text{if } l \neq j \end{cases}$$

- $\hat{\mu} = (\hat{\mu}_{ij})$        $AN\left[\mu, \Sigma_{as}(\hat{\mu})\right]$        $\Sigma_{as}(\hat{\mu}) = D \Sigma_{as}(\hat{\eta}) D$

$D$  Jacobian matrix (diagonal) of       $(\eta_{ij}) \rightarrow [g^{-1}(\eta_{ij})] = [\mu_{ij}]$

# IV. Estimation

## 4.1. Estimation risk (Delta method)

- Then  $\left[ \hat{E}(R_i) \right]_{i=1, \dots, n} \sim AN \left\{ \left[ E(R_i) \right]_{i=1, \dots, n}, \Sigma_{as} \left\{ \left[ \hat{E}(R_i) \right] \right\} \right\}$

$$\text{With - } \Sigma_{as} \left\{ \left[ \hat{E}(R_i) \right] \right\} = J_{\mu} \Sigma_{as}(\mu) J_{\mu}'$$

$$\text{- } J_{\mu} \text{ Jacobian matrix of } (\mu_{ij}) \rightarrow \left[ E(R_i) \right]$$

- And  $\hat{E}(R) \sim AN \left\{ E(R), \sigma_{as}^2 \left[ \hat{E}(R) \right] \right\}$

$$\text{With: - } \sigma_{as}^2 \left[ \hat{E}(R) \right] = J_R \Sigma_{as} \left\{ \left[ \hat{E}(R_i) \right] \right\} J_R'$$

$$\text{- } J_R = (1, 1, \dots, 1) \text{ Jacobian matrix of } \left[ E(R_i) \right]_{i=1, \dots, n} \rightarrow E(R)$$

⇒ Asymptotic s.e. and confidence interval for  $E(R)$

using only products of matrix (spreadsheet)

# IV. Estimation

## 4.2. Extensions

➤ Same approach could be applied to:

- variance: 
$$V(R) = \sum_{i=1}^n \sum_{j>n-i} V(X_{ij}) = \phi \sum_{i=1}^n \sum_{j>n-i} V(\mu_{ij})$$

- more generally, to cumulants:

$$\kappa(R) = \sum_{i=1}^n \sum_{j>n-i} \kappa(X_{ij})$$

- Then to any regular function of moments of  $R$

➤ Giving only a variance function and dispersion parameter by quasi-likelihood  $V(\mu) > 0$ ,  $\phi$

Ex. : over-dispersed Poisson

# V. Predictive distribution

## 5.1. Inversion of the m.g.f

- m.g.f:

$$M_R(s) = \prod_{i=1}^n \prod_{i+j \geq n} M_{X_{ij}}(s) = \exp \left\{ \frac{1}{\phi} \sum_{i=1}^n \sum_{j=n-i+1}^n \left\{ b \left[ b'^{-1}(g^{-1}(\eta_{ij})) + s\phi \right] - b \left[ b'^{-1}(g^{-1}(\eta_{ij})) \right] \right\} \right\}$$

with  $(\hat{\eta}_{ij})$  for m.l.e.

- Inversion by F.F.T. if no standard m.g.f.

# V. Predictive distribution

## 5.2. Approximated distributions using moments

From  $R = \sum_{i=1}^n \sum_{j=n-i+1}^n X_{ij}$  and independence of  $X_{ij}$ , moments of  $R$  are functions of  $(\mu_{ij})$ :

$$\mu = E(R) = \sum_{i=1}^n \sum_{j=n-i+1}^n \mu_{ij}$$

$$\sigma^2 = V(R) = \sum_{i=1}^n \sum_{j=n-i+1}^n V(X_{ij}) = \phi \sum_{i=1}^n \sum_{j=n-i+1}^n V(\mu_{ij})$$

$$\mu_3 = \mu_3(R) = \sum_{i=1}^n \sum_{j=n-i+1}^n \mu_3(X_{ij}) = \phi^2 \sum_{i=1}^n \sum_{j=n-i+1}^n V'(\mu_{ij})V(\mu_{ij})$$

$$\gamma_1 = \gamma_1(R) = \frac{\mu_3}{\sigma^3}$$

(m.l.e. with  $\mu_{ij}$ )

# V. Predictive distribution

## 5.2. Approximated distributions using moments

- Using :
  - NP-approximation,
  - Gamma approximations (Translated, Bowers),...

based on *m.l.e.* of  $(\mu, \sigma, \gamma_1)$

*For instance*

$$F_R(x) \approx F^{(NP)}(x) = \Phi \left[ \frac{-3}{\gamma_1} + \sqrt{\frac{9}{\gamma_1^2} + 1 + \frac{6}{\gamma_1} \left( \frac{x - \mu}{\sigma} \right)} \right]$$

$$q_{1-\eta}^{(NP)} = \mu + \sigma \left[ \frac{\gamma_1}{6} q_{1-\eta}^2 + q_{1-\eta} - 1 \right]$$

- We obtain *m.l.e.* of approximations of d.f. , VaR,...

enhanced by their asymptotic s.e. and confidence interval

# VI.Example

incremental claims amounts for some line of Marine business

<b>Years</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>
<b>0</b>	1 381	4 399	4 229	435	465	205	110	67
<b>1</b>	859	6 940	2 619	1 531	517	572	287	
<b>2</b>	6 482	6 463	3 995	1 420	547	723		
<b>3</b>	2 899	16 428	5 521	2 424	477			
<b>4</b>	3 964	15 872	8 178	3 214				
<b>5</b>	6 809	24 484	27 928					
<b>6</b>	11 155	38 229						
<b>7</b>	10 641							

chain ladder reserve : 133750

Comparing models by extended quasi-likelihood :

1. Log / Gamma

2/.Log / overdispersed Poisson

# VI.Example

	Overdispersed Poisson model				Bootstrap (1000 samples)	
	Estimation of $\Phi$ : Deviance		Estimation of $\Phi$ : Pearson residuals			
	Estimates of $E(R_i)$	$se(R_i)/R_i$	Estimates of $E(R_i)$	$se(R_i)/R_i$	Estimates of $E(R_i)$	$se(R_i)/R_i$
<b>1</b>	80	329%	80	348%	71	331%
<b>2</b>	442	134%	442	142%	427	123%
<b>3</b>	1 631	70%	1 631	74%	1 624	62%
<b>4</b>	2 811	53%	2 811	56%	2 777	46%
<b>5</b>	117 86	32%	11 786	34%	11 706	28%
<b>6</b>	41 864	20%	41 864	21%	41 799	18%
<b>7</b>	75 137	31%	75 137	33%	75 595	29%
<b>Total</b>	133 750	19%	133 750	20%	134 000	20%

# VI.Example

Estimates of the percentiles of  $R$  directly and using Normal Power approximation

Gamma approximations not available ( $\gamma_1 = 0.075$ )

	0.50	0.75	0.80	0.90	0.95	0.99
$\hat{q}_{1-\eta}(R)$	133750	140733	142463	147019	150781	157836
$\hat{q}_{1-\eta}^{(NP)}(R)$	123396	130441	132205	136885	140789	148205

# VI. Example

	Gamma Model					
	Estimation of $\Phi$ : maximisation of the likelihood		Estimation of $\Phi$ : Deviance		Estimation of $\Phi$ : Pearson	
	Estimates of $E(R_i)$	$se(R_i)/R_i$	Estimates of $E(R_i)$	$se(R_i)/R_i$	Estimates of $E(R_i)$	$se(R_i)/R_i$
<b>1</b>	101	37%	101	49%	101	50%
<b>2</b>	494	25%	494	33%	494	34%
<b>3</b>	1 286	22%	1 286	29%	1 286	30%
<b>4</b>	2 793	22%	2 793	28%	2 793	29%
<b>5</b>	11 262	23%	11 262	30%	11 262	31%
<b>6</b>	36 702	27%	36 702	35%	36 702	36%
<b>7</b>	69 563	36%	69 563	48%	69 563	49%
<b>Total</b>	122 200	22%	122 200	29%	122 200	30%

# VI.Example

Estimates of the percentiles of  $R$  by Normal Power and Gamma approx.

$1-\eta$	0.50	0.75	0.80	0.90	0.95	0.99
$q_{1-\eta}^{(NP)}(R)$	112621	127662	131632	142550	152071	171164
$q_{1-\eta}^{(GT)}(R)$	132 027	146 885	150 820	161 671	171 188	190 445

# Conclusion