

A Bonus-Malus System as a Markov Set-Chain



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A Bonus-Malus System as a Markov Set-Chain

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A Bonus-Malus System as a Markov Set-Chain

1. Markov set-chain

Definition 1*

Let N^1 be a compact set of $r \times r$ stochastic matrices. Let consider Markov chains with the state space $S = \{1, 2, \dots, r\}$, having all their transition matrices in N^1 . A *Markov set-chain* is the sequence

$$N^1, N^2, N^3, \dots,$$

where $N^k = \{\mathbf{P}: \mathbf{P} = \mathbf{P}_1 \mathbf{P}_2 \cdots \mathbf{P}_k, \text{ where } \mathbf{P}_i \in N^1 \text{ for all } i = 1, 2, \dots, k\}$ for each k .

*Hartfiel D. J. [1998], *Markov Set-Chains*, Springer Verlag, New York

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1. Markov set-chain

Definition 2*

A *matrix interval* is the interval

$$[\mathbf{K}, \mathbf{Q}] = \{\mathbf{P}: \mathbf{K} \leq \mathbf{P} \leq \mathbf{Q}\},$$

where $\mathbf{P}=[p_{ij}]$ denotes a $r \times r$ stochastic matrix and $\mathbf{K}=[k_{ij}]$ and $\mathbf{Q}=[q_{ij}]$ are nonnegative $r \times r$ matrices such that $\mathbf{K} \leq \mathbf{Q}$.

Definition 3*

A matrix interval $[\mathbf{K}, \mathbf{Q}]$ is *tight* if $k_{ij} = \min_{\mathbf{P} \in [\mathbf{K}, \mathbf{Q}]} p_{ij}$ and $q_{ij} = \max_{\mathbf{P} \in [\mathbf{K}, \mathbf{Q}]} p_{ij}$ for all i and j .

*Hartfiel D. J. [1998], *Markov Set-Chains*, Springer Verlag, New York

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2. Model of bonus-malus system

A system employed in automobile insurance is called a *bonus-malus system* when*:

- in each period (usually a year) all policyholders of a given tariff group are divided into a finite number of classes C_i ($i=1, 2, \dots, r$) and their premium depends only on the class they belong to;
- the class of a policyholder for a given period is determined uniquely by the class in the preceding period and the number of claims reported in that period.

*Lemaire J. [1995], *Bonus-malus systems in automobile insurance*, Kluwer Academic Publishers, Boston

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2. Model of bonus-malus system

The classical model of a bonus-malus system – a finite homogeneous Markov chain with the state space $S = \{1, 2, \dots, r\}$ and transition matrix*

$$\mathbf{P}(\lambda) = [p_{ij}(\lambda)] = \sum_{k=0}^{\infty} p_k(\lambda) \mathbf{T}_k,$$

where $p_k(\lambda)$ is the probability that the policyholder with claim frequency λ reports k claims in one period and \mathbf{T}_k - the matrix of transition rules.

*Lemaire J. [1995], *Bonus-malus systems in automobile insurance*, Kluwer Academic Publishers, Boston

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2. Model of bonus-malus system

Assumptions:

1. the number of claims of a policyholder characterised by λ conforms to a Poisson distribution;

2. $\lambda^{(1)}$ and $\lambda^{(2)}$ such that

$$0 < \lambda^{(1)} < \lambda^{(2)} < 1 \quad (1)$$

determine the interval $[\lambda^{(1)}, \lambda^{(2)}]$ of the claim frequency variability.

Under the above assumptions:

$$p_k(\lambda) \in [\min\{p_k(\lambda^{(1)}); p_k(\lambda^{(2)})\}, \max\{p_k(\lambda^{(1)}); p_k(\lambda^{(2)})\}], \quad (2)$$

for $k = 0, 1, 2, \dots$ and $\lambda \in [\lambda^{(1)}, \lambda^{(2)}]$.

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2. Model of bonus-malus system

Since $\lambda \in [\lambda^{(1)}, \lambda^{(2)}]$, the lower and upper bounds on the transition matrix interval are:

$$\mathbf{K} = \sum_{k=0}^{\infty} \min\{p_k^{(1)}, p_k^{(2)}\} \mathbf{T}_k = [\min\{p_{ij}^{(1)}, p_{ij}^{(2)}\}], \quad (3)$$

$$\mathbf{Q} = \sum_{k=0}^{\infty} \max\{p_k^{(1)}, p_k^{(2)}\} \mathbf{T}_k = [\max\{p_{ij}^{(1)}, p_{ij}^{(2)}\}], \quad (4)$$

where upper indices (1) and (2) indicate that a given probability is calculated for $\lambda^{(1)}$ and $\lambda^{(2)}$ respectively.

$$\mathbf{P}(\lambda) \in [\mathbf{K}, \mathbf{Q}] \quad \text{for } \lambda \in [\lambda^{(1)}, \lambda^{(2)}]$$

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2. Model of bonus-malus system

Theorem 1

Let a Markov set-chain be a model of a bonus-malus system under assumptions (1)-(2). Let $[\mathbf{K}, \mathbf{Q}]$ be its transition matrix interval, where \mathbf{K} and \mathbf{Q} are given by formulas (3) and (4). Then *the Markov set-chain is ergodic.*

Theorem 2

Let a Markov set-chain be a model of a bonus-malus system under assumptions (1)-(2). Let $[\mathbf{K}, \mathbf{Q}]$ be its transition matrix interval, where \mathbf{K} and \mathbf{Q} are given by formulas (3) and (4). Then *the interval $[\mathbf{K}, \mathbf{Q}]$ is tight.*

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3. Example

Bonus-malus system of PZU SA (April 2003)

| Class number | Premium level (in percentage) | Class number after | | | | | | |
|--------------|----------------------------------|--------------------|----|---|---|---|---|-----------|
| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 or more |
| 1 | 200 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 150 | 3 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 130 | 4 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 115 | 5 | 2 | 1 | 1 | 1 | 1 | 1 |
| <u>5</u> | <u>100</u> | 6 | 3 | 1 | 1 | 1 | 1 | 1 |
| 6 | 90 | 7 | 4 | 2 | 1 | 1 | 1 | 1 |
| 7 | 80 | 8 | 5 | 3 | 1 | 1 | 1 | 1 |
| 8 | 80 | 9 | 6 | 4 | 2 | 1 | 1 | 1 |
| 9 | 70 | 10 | 7 | 5 | 3 | 1 | 1 | 1 |
| 10 | 60 | 11 | 8 | 6 | 4 | 2 | 1 | 1 |
| 11 | 50 | 12 | 9 | 7 | 5 | 3 | 1 | 1 |
| 12 | 50 | 13 | 10 | 8 | 6 | 4 | 2 | 1 |
| 13 | 40 | 13 | 11 | 9 | 7 | 5 | 3 | 1 |

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3. Example

Claim frequency interval: $[0.1; 0.2]$

Bounds on the interval of stationary probability distributions:

$$\mathbf{l}^{\infty} = \begin{bmatrix} 0.00002 \\ 0.00004 \\ 0.00011 \\ 0.00022 \\ 0.00056 \\ 0.00108 \\ 0.00297 \\ 0.00506 \\ 0.01621 \\ 0.02180 \\ 0.08555 \\ 0.06722 \\ 0.51409 \end{bmatrix}'$$

$$\mathbf{h}^{\infty} = \begin{bmatrix} 0.00246 \\ 0.00358 \\ 0.00534 \\ 0.00771 \\ 0.01174 \\ 0.01652 \\ 0.02633 \\ 0.03497 \\ 0.06154 \\ 0.07160 \\ 0.15314 \\ 0.13363 \\ 0.77898 \end{bmatrix}'$$

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3. Example

Lower bound on the interval of mean first passage times:

$$\bar{\mathbf{M}}^l = \begin{bmatrix} 407.33 & 1.11 & 2.33 & 3.68 & 5.06 & 6.46 & 7.87 & 9.29 & 10.71 & 12.13 & 13.54 & 14.96 & 16.38 \\ 496.29 & 279.46 & 1.22 & 2.57 & 3.95 & 5.36 & 6.77 & 8.18 & 9.60 & 11.02 & 12.44 & 13.86 & 15.28 \\ 604.95 & 339.87 & 187.12 & 1.35 & 2.73 & 4.14 & 5.55 & 6.96 & 8.38 & 9.80 & 11.22 & 12.64 & 14.05 \\ 737.67 & 413.65 & 226.82 & 129.76 & 1.38 & 2.79 & 4.20 & 5.61 & 7.03 & 8.45 & 9.87 & 11.29 & 12.70 \\ 800.51 & 503.98 & 275.52 & 156.67 & 85.15 & 1.40 & 2.82 & 4.23 & 5.65 & 7.07 & 8.49 & 9.90 & 11.32 \\ 855.54 & 546.35 & 335.25 & 189.79 & 102.13 & 60.53 & 1.41 & 2.83 & 4.24 & 5.66 & 7.08 & 8.50 & 9.92 \\ 886.27 & 583.36 & 362.86 & 230.53 & 123.16 & 72.04 & 37.98 & 1.42 & 2.83 & 4.25 & 5.67 & 7.09 & 8.51 \\ 909.07 & 603.70 & 386.87 & 248.98 & 149.14 & 86.39 & 44.48 & 28.59 & 1.42 & 2.84 & 4.25 & 5.67 & 7.09 \\ 922.60 & 618.60 & 399.72 & 264.93 & 160.48 & 104.23 & 52.73 & 33.01 & 16.25 & 1.42 & 2.84 & 4.26 & 5.67 \\ 931.58 & 627.13 & 408.91 & 273.12 & 170.15 & 111.65 & 63.12 & 38.72 & 17.93 & 13.97 & 1.42 & 2.84 & 4.26 \\ 936.66 & 632.54 & 413.84 & 278.77 & 174.73 & 117.86 & 66.95 & 46.01 & 20.30 & 15.15 & 6.53 & 1.42 & 2.84 \\ 939.47 & 635.28 & 416.68 & 281.47 & 177.64 & 120.45 & 70.00 & 48.34 & 23.51 & 16.90 & 6.06 & 7.48 & 1.42 \\ 940.56 & 636.46 & 417.73 & 282.71 & 178.60 & 121.84 & 70.76 & 50.07 & 23.88 & 19.35 & 5.81 & 7.23 & 1.28 \end{bmatrix}$$

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3. Example

Upper bound on the interval of mean first passage times:

$$\overline{\mathbf{M}}^h = \begin{bmatrix} 48039.25 & 1.22 & 2.71 & 4.54 & 6.52 & 8.64 & 10.84 & 13.10 & 15.41 & 17.74 & 21.39 & 23.99 & 24.82 \\ 53090.48 & 22425.35 & 1.49 & 3.31 & 5.30 & 7.42 & 9.62 & 11.88 & 14.19 & 16.52 & 20.17 & 22.77 & 23.60 \\ 58672.94 & 24782.61 & 9314.63 & 1.82 & 3.80 & 5.92 & 8.13 & 10.39 & 12.69 & 15.02 & 18.68 & 21.28 & 22.11 \\ 64842.53 & 27387.79 & 10292.87 & 4518.94 & 1.98 & 4.10 & 6.31 & 8.57 & 10.87 & 13.20 & 16.85 & 19.46 & 20.28 \\ 66351.92 & 30267.07 & 11374.11 & 4992.74 & 1785.76 & 2.12 & 4.32 & 6.59 & 8.89 & 11.22 & 14.87 & 17.47 & 18.30 \\ 67461.81 & 30970.92 & 12569.22 & 5516.52 & 1972.05 & 927.84 & 2.20 & 4.47 & 6.77 & 9.10 & 12.75 & 15.35 & 16.18 \\ 67806.03 & 31488.27 & 12860.73 & 6095.58 & 2178.12 & 1023.87 & 336.15 & 2.26 & 4.57 & 6.90 & 10.55 & 13.15 & 13.98 \\ 68007.59 & 31648.20 & 13074.79 & 6236.26 & 2406.07 & 1130.20 & 369.92 & 197.78 & 2.30 & 4.63 & 8.29 & 10.89 & 11.71 \\ 68079.66 & 31741.53 & 13140.39 & 6339.38 & 2460.80 & 1247.94 & 407.47 & 216.98 & 61.71 & 2.33 & 5.98 & 8.58 & 9.41 \\ 68116.41 & 31774.42 & 13178.33 & 6370.47 & 2500.68 & 1275.68 & 449.19 & 238.43 & 66.59 & 45.87 & 3.65 & 6.25 & 7.08 \\ 68130.08 & 31790.82 & 13191.16 & 6388.15 & 2512.11 & 1295.72 & 458.33 & 262.38 & 72.22 & 49.08 & 11.69 & 3.90 & 4.73 \\ 68135.98 & 31796.44 & 13197.14 & 6393.64 & 2518.24 & 1300.98 & 464.68 & 267.16 & 78.69 & 52.87 & 11.16 & 14.88 & 2.37 \\ 68137.60 & 31798.44 & 13198.65 & 6395.81 & 2519.55 & 1303.48 & 465.67 & 270.31 & 79.19 & 57.30 & 10.81 & 14.65 & 1.95 \end{bmatrix}$$

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3. Example

The difference between the upper and lower bounds on the interval of mean first passage times:

| | | | | | | | | | | | | |
|-----------------|-------------|----------|---------|---------|---------|--------|--------|-------|-------|------|------|------|
| 47631.92 | <u>0.12</u> | 0.39 | 0.86 | 1.46 | 2.18 | 2.97 | 3.81 | 4.70 | 5.61 | 7.85 | 9.03 | 8.44 |
| 52594.18 | 22145.89 | 0.27 | 0.74 | 1.34 | 2.06 | 2.85 | 3.70 | 4.58 | 5.50 | 7.73 | 8.91 | 8.32 |
| 58067.99 | 24442.74 | 9127.51 | 0.47 | 1.07 | 1.79 | 2.58 | 3.43 | 4.31 | 5.23 | 7.46 | 8.64 | 8.05 |
| 64104.86 | 26974.14 | 10066.05 | 4389.18 | 0.60 | 1.32 | 2.11 | 2.96 | 3.84 | 4.75 | 6.99 | 8.17 | 7.58 |
| 65551.41 | 29763.09 | 11098.59 | 4836.07 | 1700.61 | 0.72 | 1.51 | 2.36 | 3.24 | 4.15 | 6.39 | 7.57 | 6.98 |
| 66606.28 | 30424.57 | 12233.97 | 5326.73 | 1869.92 | 867.31 | 0.79 | 1.64 | 2.52 | 3.44 | 5.67 | 6.85 | 6.26 |
| 66919.75 | 30904.91 | 12497.87 | 5865.04 | 2054.96 | 951.83 | 298.17 | 0.85 | 1.73 | 2.65 | 4.88 | 6.06 | 5.47 |
| 67098.51 | 31044.50 | 12687.92 | 5987.28 | 2256.93 | 1043.81 | 325.44 | 169.18 | 0.89 | 1.80 | 4.03 | 5.21 | 4.62 |
| 67157.06 | 31122.93 | 12740.67 | 6074.45 | 2300.32 | 1143.71 | 354.74 | 183.97 | 45.46 | 0.91 | 3.15 | 4.33 | 3.74 |
| 67184.84 | 31147.29 | 12769.41 | 6097.35 | 2330.53 | 1164.04 | 386.07 | 199.71 | 48.66 | 31.90 | 2.23 | 3.42 | 2.82 |
| 67193.41 | 31158.28 | 12777.31 | 6109.37 | 2337.38 | 1177.86 | 391.38 | 216.37 | 51.92 | 33.93 | 5.16 | 2.49 | 1.89 |
| 67196.50 | 31161.16 | 12780.46 | 6112.17 | 2340.60 | 1180.53 | 394.68 | 218.82 | 55.17 | 35.97 | 5.09 | 7.39 | 0.95 |
| <u>67197.03</u> | 31161.98 | 12780.91 | 6113.10 | 2340.95 | 1181.64 | 394.92 | 220.23 | 55.31 | 37.94 | 5.00 | 7.42 | 0.66 |

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3. Example

Lower bound on the interval of mean first passage times:

$$\bar{\mathbf{M}}^l = \begin{bmatrix} 407.33 & 1.11 & 2.33 & 3.68 & 5.06 & 6.46 & 7.87 & 9.29 & 10.71 & 12.13 & 13.54 & 14.96 & 16.38 \\ 496.29 & 279.46 & 1.22 & 2.57 & 3.95 & 5.36 & 6.77 & 8.18 & 9.60 & 11.02 & 12.44 & 13.86 & 15.28 \\ 604.95 & 339.87 & 187.12 & 1.35 & 2.73 & 4.14 & 5.55 & 6.96 & 8.38 & 9.80 & 11.22 & 12.64 & 14.05 \\ 737.67 & 413.65 & 226.82 & 129.76 & 1.38 & 2.79 & 4.20 & 5.61 & 7.03 & 8.45 & 9.87 & 11.29 & 12.70 \\ 800.51 & 503.98 & 275.52 & 156.67 & 85.15 & 1.40 & 2.82 & 4.23 & 5.65 & 7.07 & 8.49 & 9.90 & \underline{11.32} \\ 855.54 & 546.35 & 335.25 & 189.79 & 102.13 & 60.53 & 1.41 & 2.83 & 4.24 & 5.66 & 7.08 & 8.50 & 9.92 \\ 886.27 & 583.36 & 362.86 & 230.53 & 123.16 & 72.04 & 37.98 & 1.42 & 2.83 & 4.25 & 5.67 & 7.09 & 8.51 \\ 909.07 & 603.70 & 386.87 & 248.98 & 149.14 & 86.39 & 44.48 & 28.59 & 1.42 & 2.84 & 4.25 & 5.67 & 7.09 \\ 922.60 & 618.60 & 399.72 & 264.93 & 160.48 & 104.23 & 52.73 & 33.01 & 16.25 & 1.42 & 2.84 & 4.26 & 5.67 \\ 931.58 & 627.13 & 408.91 & 273.12 & 170.15 & 111.65 & 63.12 & 38.72 & 17.93 & 13.97 & 1.42 & 2.84 & 4.26 \\ 936.66 & 632.54 & 413.84 & 278.77 & 174.73 & 117.86 & 66.95 & 46.01 & 20.30 & 15.15 & 6.53 & 1.42 & 2.84 \\ 939.47 & 635.28 & 416.68 & 281.47 & 177.64 & 120.45 & 70.00 & 48.34 & 23.51 & 16.90 & 6.06 & 7.48 & 1.42 \\ 940.56 & 636.46 & 417.73 & 282.71 & 178.60 & 121.84 & 70.76 & 50.07 & 23.88 & 19.35 & 5.81 & 7.23 & 1.28 \end{bmatrix}$$

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3. Example

Upper bound on the interval of mean first passage times:

$$\overline{\mathbf{M}}^h = \begin{bmatrix} 48039.25 & 1.22 & 2.71 & 4.54 & 6.52 & 8.64 & 10.84 & 13.10 & 15.41 & 17.74 & 21.39 & 23.99 & 24.82 \\ 53090.48 & 22425.35 & 1.49 & 3.31 & 5.30 & 7.42 & 9.62 & 11.88 & 14.19 & 16.52 & 20.17 & 22.77 & 23.60 \\ 58672.94 & 24782.61 & 9314.63 & 1.82 & 3.80 & 5.92 & 8.13 & 10.39 & 12.69 & 15.02 & 18.68 & 21.28 & 22.11 \\ 64842.53 & 27387.79 & 10292.87 & 4518.94 & 1.98 & 4.10 & 6.31 & 8.57 & 10.87 & 13.20 & 16.85 & 19.46 & 20.28 \\ 66351.92 & 30267.07 & 11374.11 & 4992.74 & 1785.76 & 2.12 & 4.32 & 6.59 & 8.89 & 11.22 & 14.87 & 17.47 & \mathbf{18.30} \\ 67461.81 & 30970.92 & 12569.22 & 5516.52 & 1972.05 & 927.84 & 2.20 & 4.47 & 6.77 & 9.10 & 12.75 & 15.35 & 16.18 \\ 67806.03 & 31488.27 & 12860.73 & 6095.58 & 2178.12 & 1023.87 & 336.15 & 2.26 & 4.57 & 6.90 & 10.55 & 13.15 & 13.98 \\ 68007.59 & 31648.20 & 13074.79 & 6236.26 & 2406.07 & 1130.20 & 369.92 & 197.78 & 2.30 & 4.63 & 8.29 & 10.89 & 11.71 \\ 68079.66 & 31741.53 & 13140.39 & 6339.38 & 2460.80 & 1247.94 & 407.47 & 216.98 & 61.71 & 2.33 & 5.98 & 8.58 & 9.41 \\ 68116.41 & 31774.42 & 13178.33 & 6370.47 & 2500.68 & 1275.68 & 449.19 & 238.43 & 66.59 & 45.87 & 3.65 & 6.25 & 7.08 \\ 68130.08 & 31790.82 & 13191.16 & 6388.15 & 2512.11 & 1295.72 & 458.33 & 262.38 & 72.22 & 49.08 & 11.69 & 3.90 & 4.73 \\ 68135.98 & 31796.44 & 13197.14 & 6393.64 & 2518.24 & 1300.98 & 464.68 & 267.16 & 78.69 & 52.87 & 11.16 & 14.88 & 2.37 \\ 68137.60 & 31798.44 & 13198.65 & 6395.81 & 2519.55 & 1303.48 & 465.67 & 270.31 & 79.19 & 57.30 & 10.81 & 14.65 & 1.95 \end{bmatrix}$$

A Bonus-Malus System as a Markov Set-Chain

4. Conclusions

The application of a Markov set-chain to the analysis of a bonus-malus system enables us to:

- broaden the scope of research in comparison with classical Markov chains;
- relax the assumption of the constant claim frequency of a policyholder;
- examine the consequences of claim frequency changes within a given interval;
- determine the variability range of stationary distribution as well as of mean first passage times.

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Thank you

Małgorzata Niemiec