

**Differentiation of some
functionals of
multidimensional risk
processes and determination
of optimal reserve allocation**

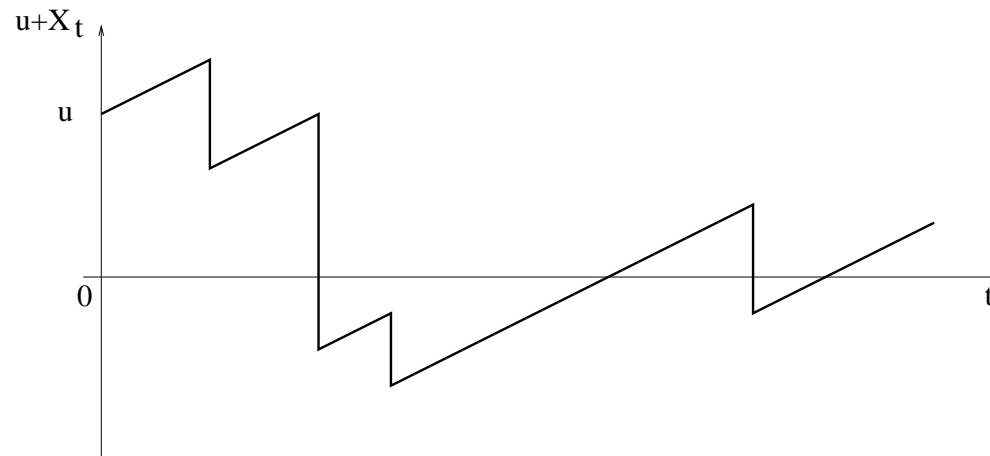
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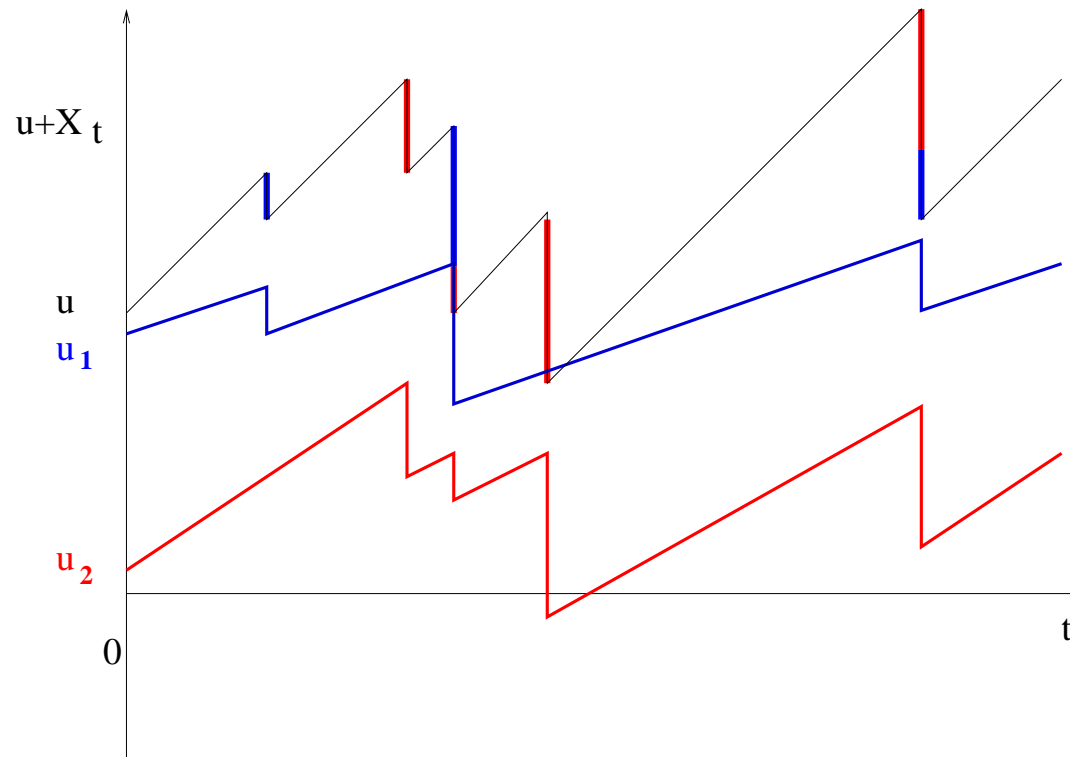




Historical model: for unidimensional risk processes $R_t = u + X_t$,

- with initial reserve u
- and with $X_t = ct - S_t$, where
 - $c > 0$ is the premium income rate,
 - $S_t = \sum_{i=1}^{N(t)} W_i$,
 - the W_i are i.i.d. nonnegative random variables, independent from $(N(t))_{t \geq 0}$,
 - with the convention that the sum is zero if $N(t) = 0$.

Probability of ruin: $\psi(u) = \mathbb{P}(\exists t \geq 0, R_t < 0)$.



Two lines of business: classical, 1-dimensional surplus process (black),
2-dimensional process (1 for each line of business, blue and red).

$$u = u_1 + u_2$$

3 main directions:

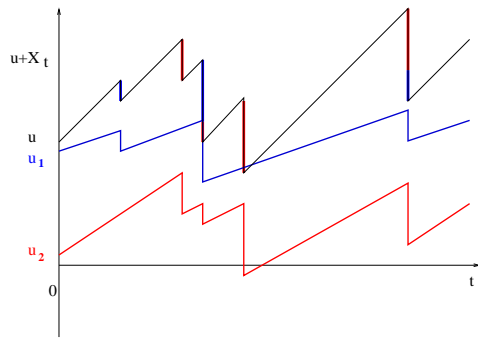


Figure 1: K -dimensional process

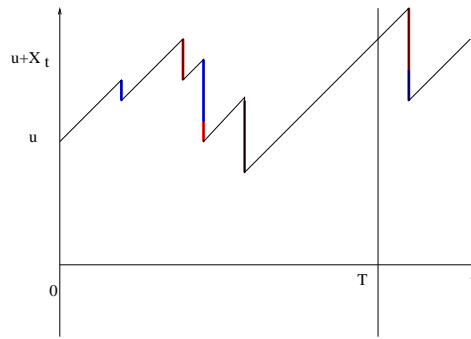


Figure 2: Finite-time.

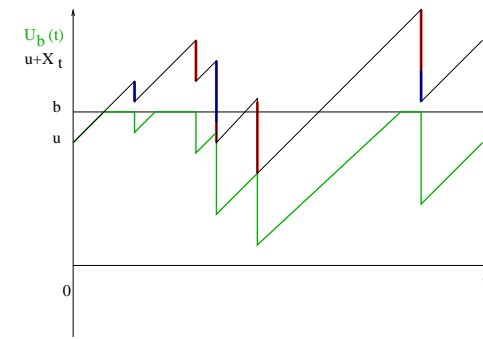


Figure 3: Dividends.

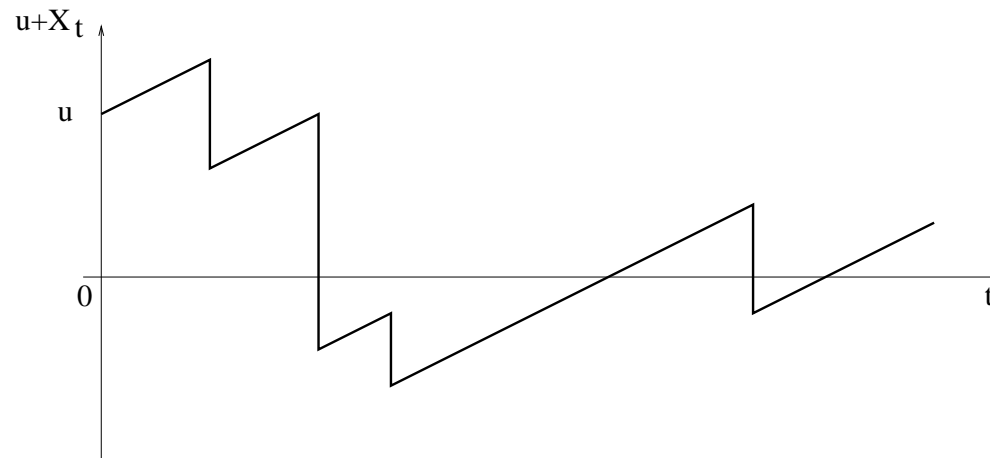
- How to model the stochastic dependence between the K lines of business ?
- Which ruin concept ?
- Within finite or infinite time ?
- Ruin or severity of ruin ?
- Optimal initial reserve allocation ?
- How to measure risk and profit (dividends)?

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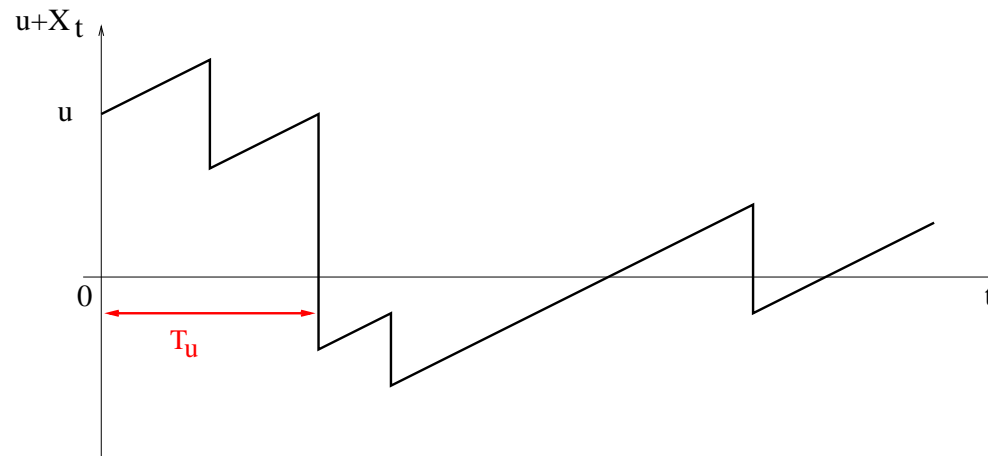
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- the time to ruin $T_u = \inf\{t > 0, u + X_t < 0\}$,
- the severity of ruin $|u + X_{T_u}|$, or the couple $(T_u, |u + X_{T_u}|)$,
- the time in the red (below 0) from the first ruin to the first time of recovery $T'_u - T_u$, where

$$T'_u = \inf\{t > T_u, u + X_t = 0\},$$

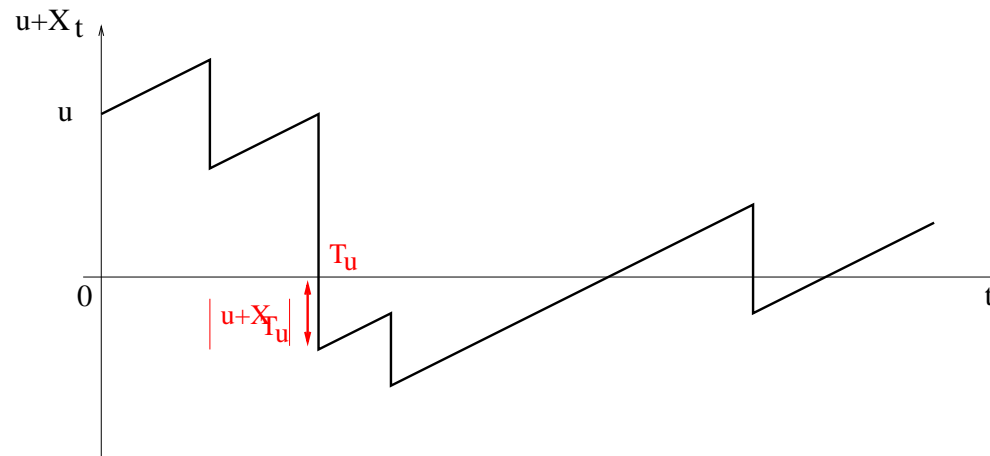
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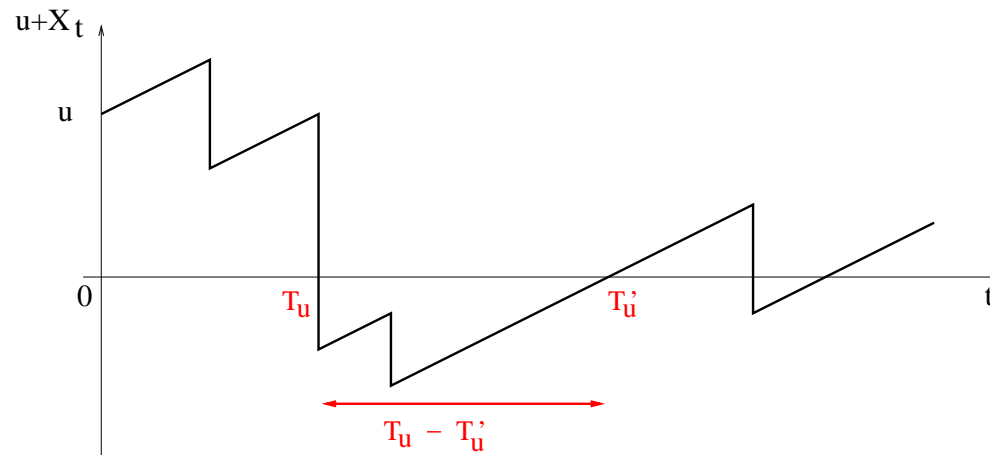
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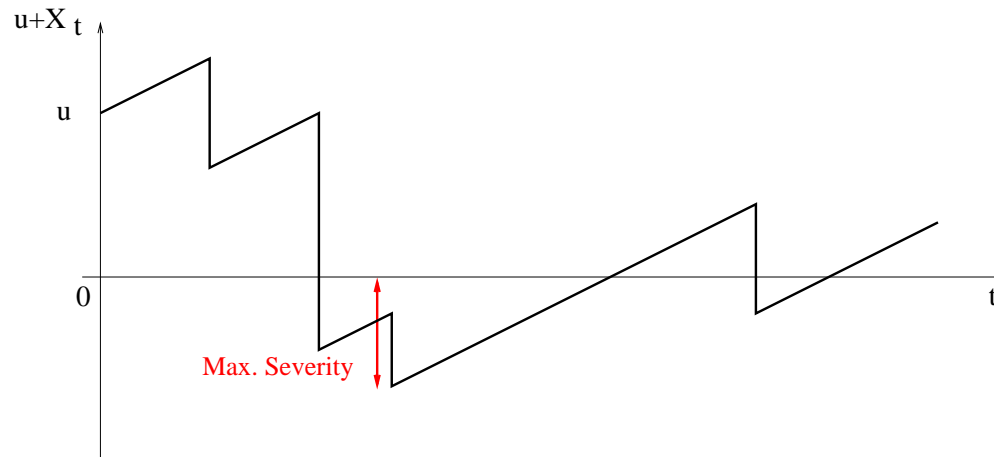
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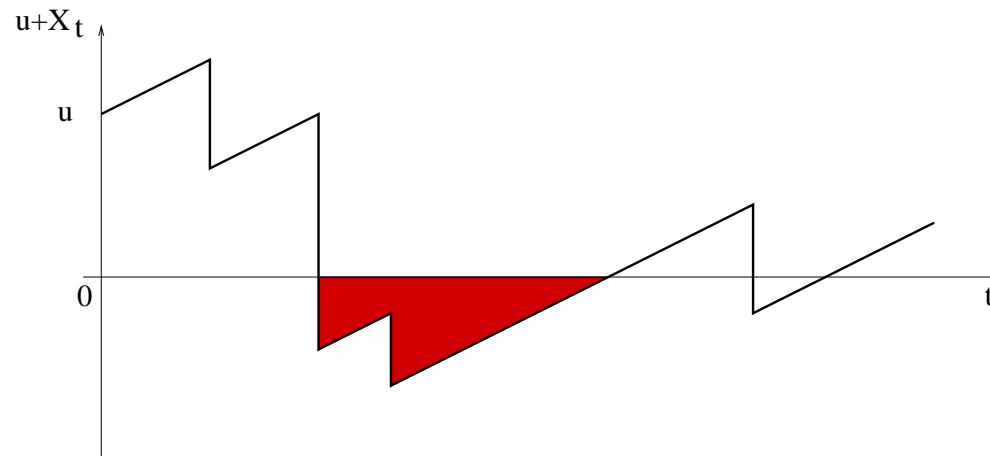
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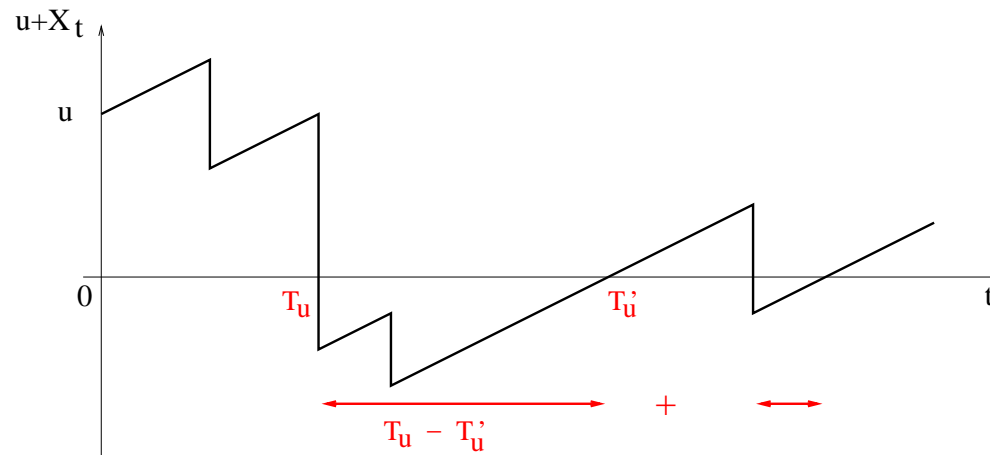
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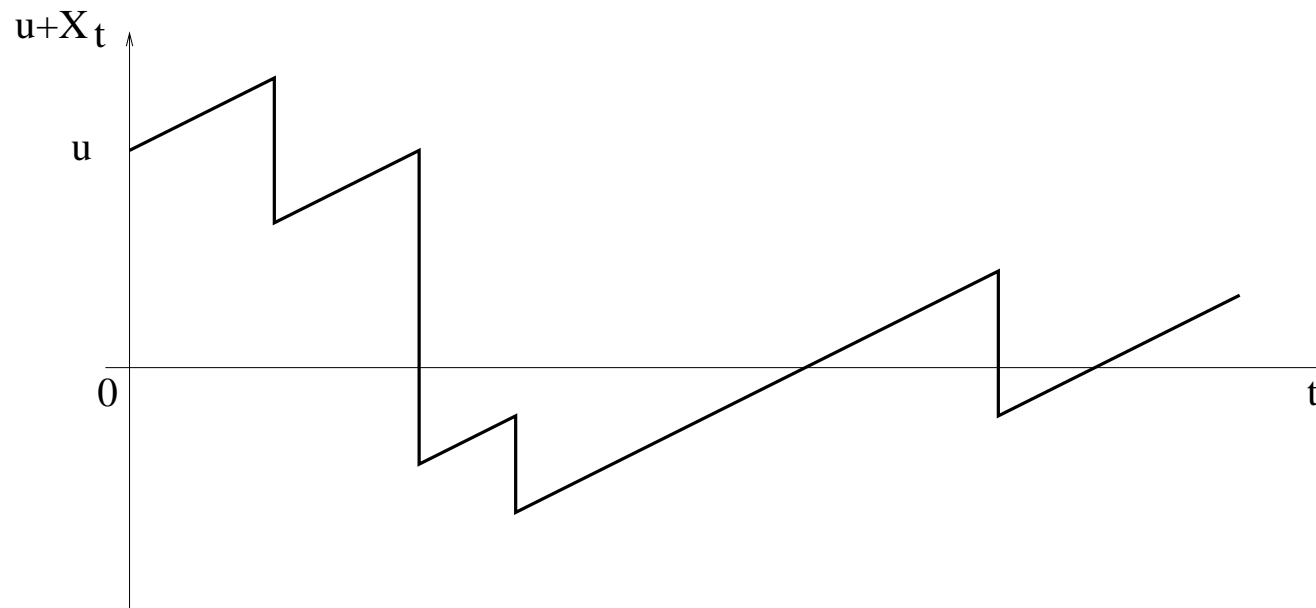
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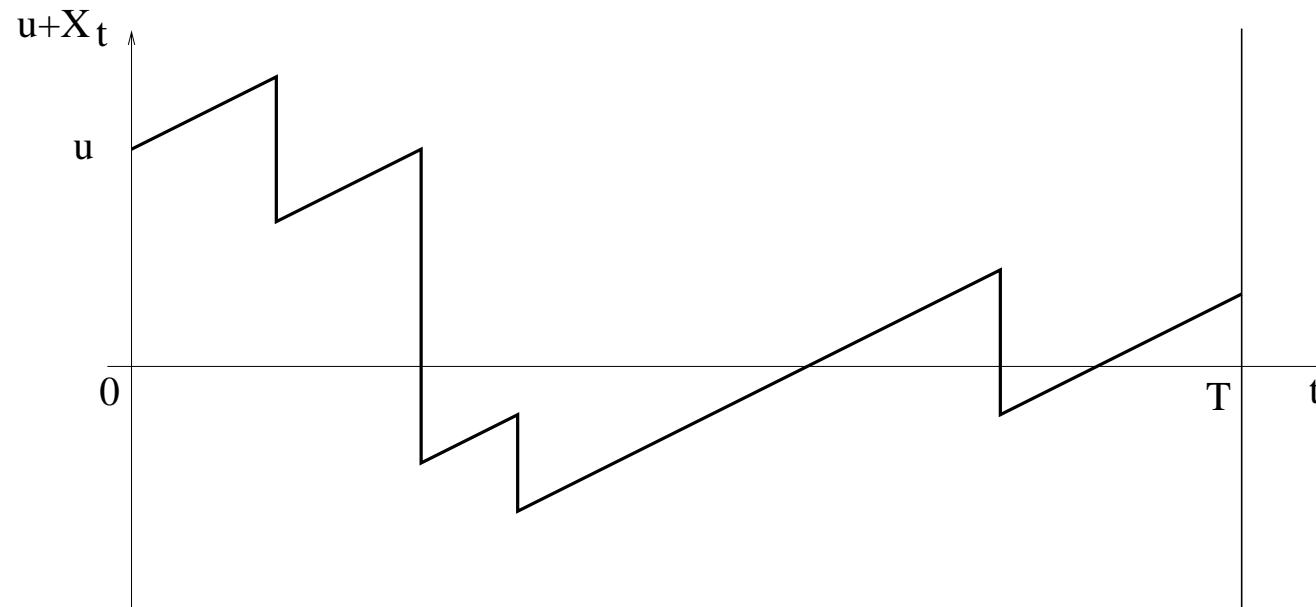


- Consider risk measures based on some fixed time interval $[0, T]$ (T may be infinite).

- Simple penalty function (expected penalty to pay due to insolvency until time horizon T) :

$$\mathbb{E} (I_T(u)) = \mathbb{E} \left(\int_0^T 1_{\{u+X_t < 0\}} |u + X_t| dt \right).$$

- Note that the probability $\mathbb{P} (I_T(u) = 0)$ is the probability of non ruin within finite time T.

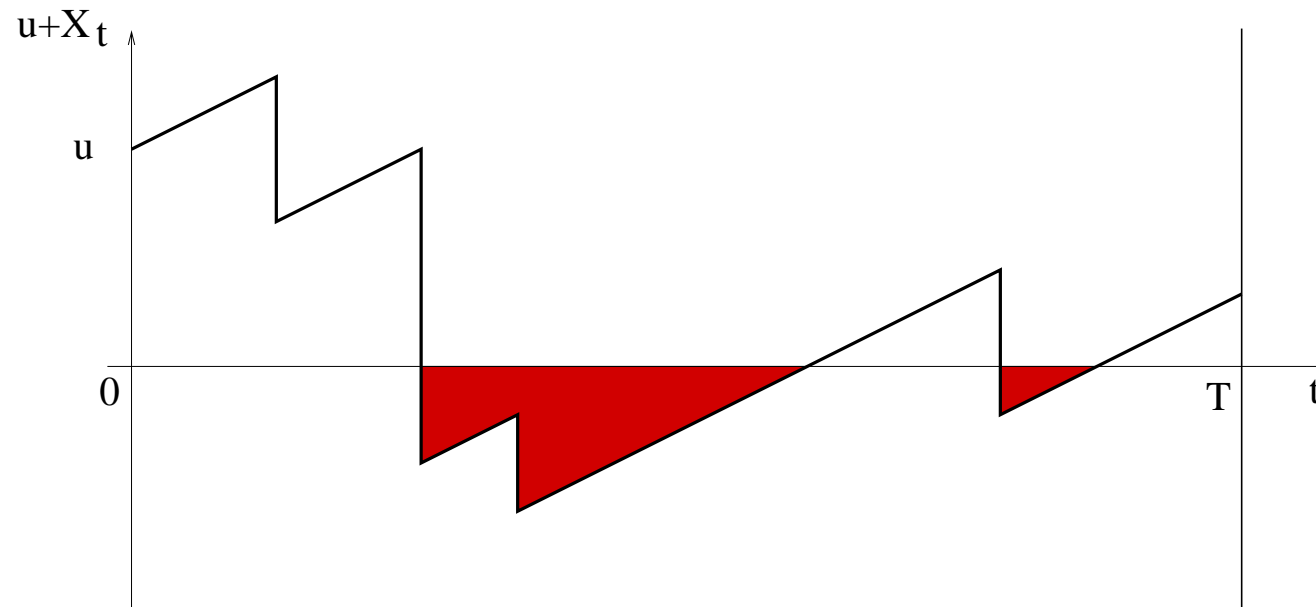


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► From an economical point of view, it seems more consistent to consider

$$\mathbb{E}I_{g,h}(u) = \mathbb{E} \left(\int_0^T (1_{\{u+X_t \geq 0\}} g(|u + X_t|) - 1_{\{u+X_t \leq 0\}} h(|u + X_t|)) dt \right)$$

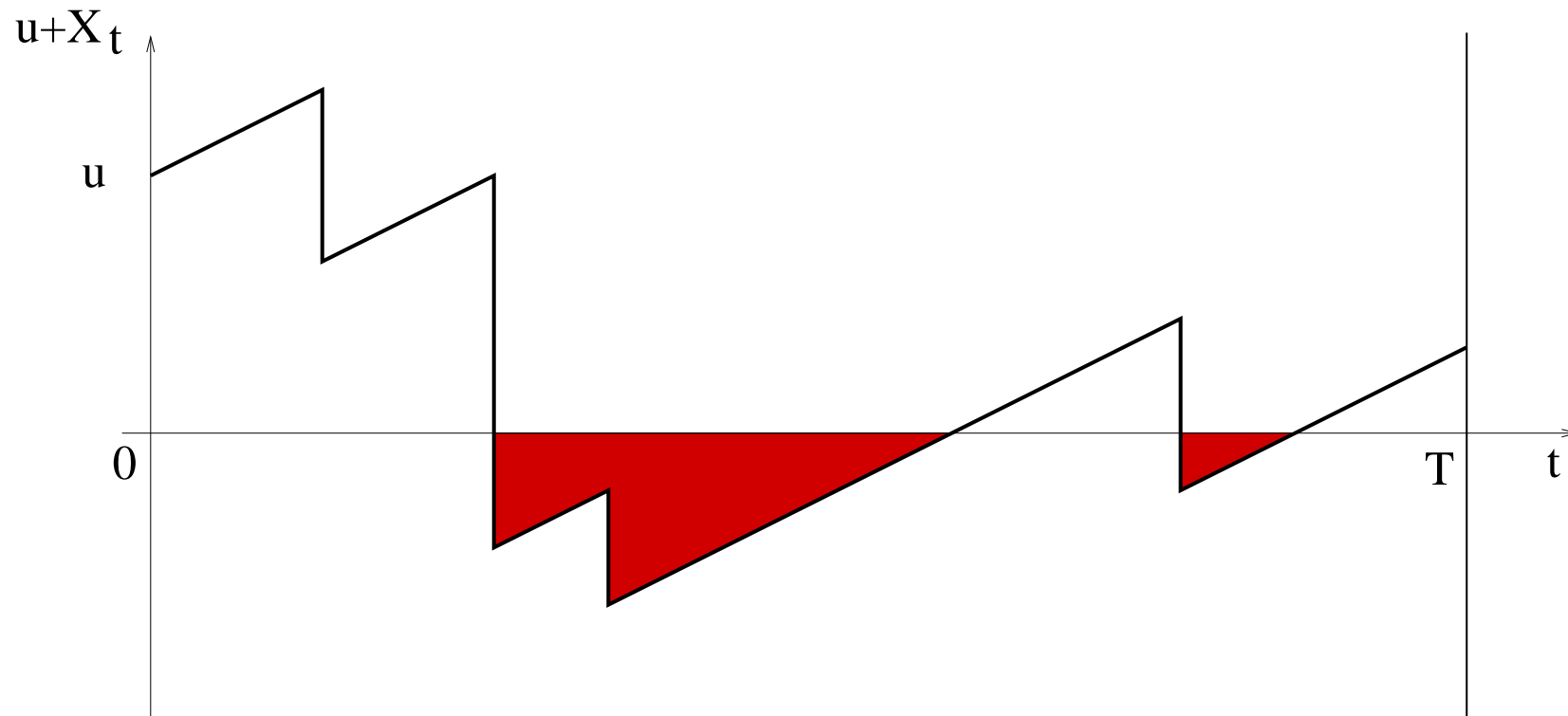
- $0 \leq g \leq h$
- g corresponds to a reward function for positive reserves,
- and h is a penalty function in case of insolvency.

► These risk measures may be differentiated with respect to the initial reserve u .

► Fubini's theorem.

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$$I(u) = \int_0^T 1_{\{u+X_t < 0\}} |u + X_t| dt$$

Th. (L., 2004b):

Let $(X_t)_{t \in [0, T]}$ be a stochastic process with almost surely time-integrable sample paths.

For $u \in \mathbb{R}$, denote by $\tau_0(u)$ the time spent in zero by the process $u + X_t$:

$$\tau_0(u) = \int_0^T 1_{\{u+X_t=0\}} dt.$$

Let f be defined by $f(u) = \mathbb{E}(I_T(u))$ for $u \in \mathbb{R}$, where

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► For $u \in \mathbb{R}$, if $\mathbb{E}\tau_0(u) = 0$, then f is differentiable at u , and $f'(u) = -\mathbb{E}\tau(u)$.

► Th. (L., 2004b): Let $X_t = ct - S_t$, where S_t is a jump process satisfying hypothesis (H1): S_t has a finite expected number of nonnegative jumps in every finite interval, and for each t , the distribution of S_t is absolutely continuous. For example, S_t might be a compound Poisson process with a continuous jump size distribution. Define h by $h(u) = \mathbb{E}(\tau(u))$ for $u \in \mathbb{R}$. h is differentiable on \mathbb{R}_+^* , and for $u > 0$,

$$h'(u) = -\frac{1}{c} \mathbb{E}N^0(u),$$

where $N^0(u) = \text{Card}(\{t \in [0, T], u + ct - S_t = 0\})$.

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- This remains valid with $T = +\infty$ if X_t has a positive drift and $\tau(u)$ is integrable. In the compound Poisson case, for $u \geq 0$,

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- $\mathbb{E}I_T(\cdot)$ is thus well strictly convex, which will be very important for minimization.

- Theorem: In the Poisson-Exponential($1/\mu$) case, $\psi(u) = \frac{1-\mu R}{\mu R} e^{-Ru}$, with $R = \frac{1}{\mu} \left(1 - \frac{\lambda\mu}{c}\right)$. Hence, for $T = +\infty$,

$$\mathbb{E}\tau(u) = \frac{1 - \mu R}{c\mu R^2} e^{-Ru} \quad \text{Gerber, Dos Reis (1993)}$$

and

$$\mathbb{E}I_\infty(u) = \frac{1 - \mu R}{c\mu R^3} e^{-Ru} \quad \text{L. (2004b)}$$

- Proof: Integration of the well-known formula for $\psi(u)$. The considered functions tend to 0 as $u \rightarrow +\infty$.
- It is possible to derive $\mathbb{E}I_\infty(u)$ explicitly for Γ and *phase-type*-distributed claim amounts.

► Th. (L., 2004b): Let g, h be two convex or concave functions in $\mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$, such that for $x \geq 0$, $g(x) \geq g(0)$ and $h(x) \geq h(0)$. Let X_t be a stochastic process such that $t \rightarrow g(u + X_t)$ and $t \rightarrow h(u + X_t)$ are almost surely integrable on $[0, T]$. Let I_g^+ be the function from \mathbb{R} into the space of nonnegative random variables, and defined by

$$I_g^+(u) = \int_0^T 1_{\{u+X_t \geq 0\}} g(u + X_t) dt$$

for $u \geq 0$ and let $f(\cdot) = \mathbb{E}I_g^+(\cdot) - \mathbb{E}I_h(\cdot)$. Define also

$$L_T(0) = \lim_{\varepsilon \downarrow 0} \left(\frac{1}{2\varepsilon} \int_0^T 1_{\{|u+X_t| < \varepsilon\}} dt \right).$$

If, for $u \in \mathbb{R}$, $\mathbb{E}I_g^+(u)$, $\mathbb{E}I_h(u)$, $\mathbb{E}I_{g'}^+(u)$, $\mathbb{E}I_{h'}(u) < +\infty$,

and if $\mathbb{E}\tau_0(u) = 0$, then f is differentiable on \mathbb{R}_*^+ , and for $u > 0$,

$$f'(u) = \mathbb{E}I_{g'}^+(u) - \mathbb{E}I_{h'}(u) - (g(0) + h(0))\mathbb{E}L_T(0)$$

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► What has to be minimized is

$$A(u_1, \dots, u_K) = \sum_{k=1}^K \mathbb{E} I_T^k(u_k)$$

under the constraint $u_1 + \dots + u_K = u$, where

$$\mathbb{E} I_T^k(u_k) = \mathbb{E} \left[\int_0^T |R_t^k| 1_{\{R_t^k < 0\}} dt \right]$$

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- This does **not** depend on the dependence structure.
- From previous differentiation theorems, A is strictly convex. On the compact space

$$\mathcal{S} = \{(u_1, \dots, u_K) \in (\mathbb{R}^+)^K, \quad u_1 + \dots + u_K = u\},$$

A admits a unique minimum.

► Lagrange multipliers → optimal allocation:

there is a subset $J \subset [1, K]$ such that

- for $j \notin J$, $u_j = 0$,
- and for $j, k \in J$, $\mathbb{E}\tau_j = \mathbb{E}\tau_k$.

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► In the Poisson-Exponential($\frac{1}{\mu}$) case, recall that

$$\mathbb{E}I_u = \frac{1 - \mu R}{c\mu R^3} e^{-Ru}.$$

Consider a two-line model, with the following parameters:

$\mu_1 = \mu_2 = 1$, $c_1 = c_2 = 1$, $R_2 = 0.4$ and $u = 10$.

Three values of R_1 → different optimal allocation strategies.

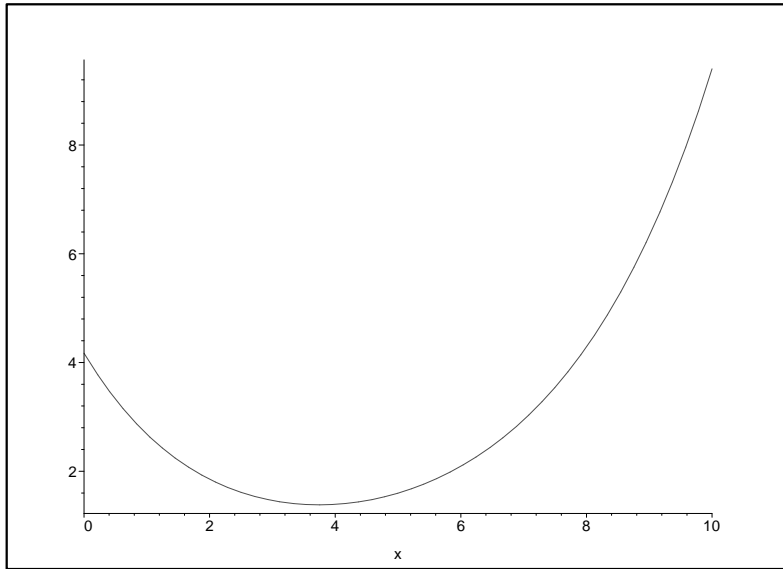


Figure 4: Graph of $A(x, 10 - x)$.

When $R_1 = 0.5 > R_2$,

line of business 1 is safer than line 2.

→ $u_1 < u_2$.

The optimal allocation is about

$(u_1 = 3.5, u_2 = 6.5)$

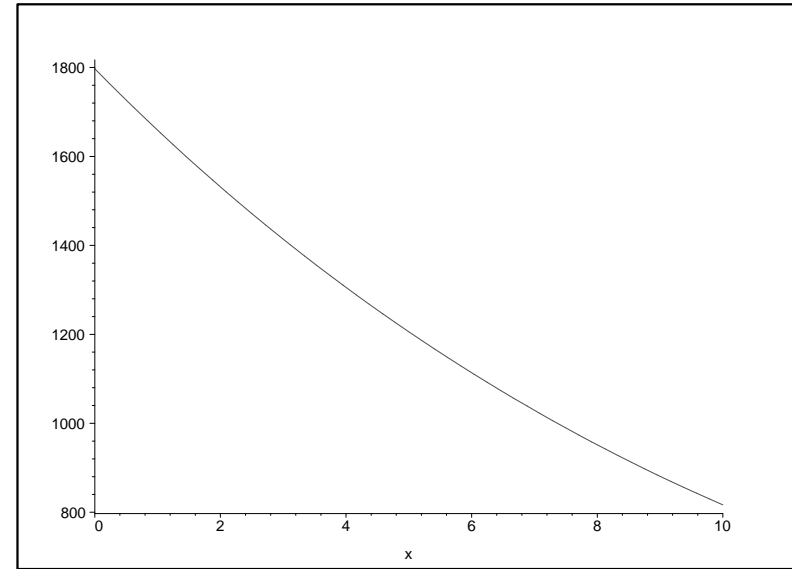


Figure 5: Graph of $A(x, 10 - x)$.

When $R_1 = 0.08 < R_2$,

line of business 1 is much riskier than line 2.

→ $u_1 = u = 10$ and $u_2 = 0$.

Transfer of the whole reserve to line 1.

- Consider the sum

$$B = \sum_{k=1}^K \mathbb{E} \tau'_k(u_1, \dots, u_K)$$

where

$$\mathbb{E} \tau'_k(u_1, \dots, u_K) = \mathbb{E} \left(\int_0^T 1_{\{R_t^k < 0\}} 1_{\{\sum_{j=1}^K R_t^j > 0\}} dt \right).$$

B takes dependence into account, and the following proposition prescribes to do the same kind of reasoning:

- Proposition: Let $X_t = ct - S_t$, where S_t satisfies hypothesis (H1). Define B by $B(u_1, \dots, u_K) = \sum_{k=1}^K \mathbb{E}(\tau'_k(u_1, \dots, u_K))$ for $u \in \mathbb{R}^K$. B is differentiable on $(\mathbb{R}_+^*)^K$, and for $u_1, \dots, u_K > 0$,

$$\frac{\partial B}{\partial u_k} = -\frac{1}{c_k} \mathbb{E} N_k^0(u_1, \dots, u_K),$$

where $N_k^0(u_1, \dots, u_K) = \text{Card} \left(\{t \in [0, T], (R_t^k = 0) \cap \left(\sum_{j=1}^K R_t^j > 0 \right)\} \right)$.

- It is also possible to differentiate with respect to c instead of u .
- Theorem: With the previous notation, consider the case $X_t = ct - S_t$, where S_t satisfies hypothesis (H1), and define $\tilde{f}(c) = \mathbb{E}(I(c))$.
If for all c , $\mathbb{E}\tau_0(c) = 0$, then \tilde{f} is differentiable on \mathbb{R} and for $c \in \mathbb{R}$,

$$\tilde{f}'(c) = - \int_0^T tP(R_t < 0)dt.$$

- It is interesting to look for the optimal allocation of the global premium $c = c_1 + \dots + c_n$ because if c_i is small enough to make the safety loading negative, the process R_t^i tends to $-\infty$. Quite often, optimizing with the c_i will be easier than with the u_i for this reason.
- These examples illustrate how these differentiation results may be used.
The differentiation developed here is quite general and may be useful to solve many problems involving multi-risks models.