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DIFFERENTIATION OF SOME FUNCTIONALS OF RISK PROCESSES AND OPTIMAL RESERVE ALLOCATION.

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Abstract

For general risk processes, the expected time-integrated negative part of the process on a fixed time interval is introduced and studied. Differentiation theorems are stated and proved. They make it possible to derive the expected value of this risk measure, and to link it with the average total time below zero studied by Dos Reis (1993), and the probability of ruin. Differentiation of other functionals of unidimendional and multidimensional risk processes with respect to the initial reserve level are carried out. Applications to ruin theory, and to the determination of the optimal allocation of the global initial reserve which minimizes one of these risk measures, illustrate the variety of application fields and the benefits deriving from an efficient and effective use of such tools.

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Introduction

For unidimensional risk processes $R_t = u + X_t$ (representing the surplus of an insurance company at time t, with initial reserve u and with $X_t = ct - S_t$, where $c > 0$ is the premium income rate, and S_t is in the most classical case a compound Poisson process (here we do not limit ourselves to the Poisson case)), many risk measures have been considered (see for example Gerber (1988), Dufresne and Gerber (1988)

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and Picard (1994)): the time to ruin $T_u = \inf\{t > 0, u + X_t < 0\}$, the severity of ruin $u + X_{T_u}$, the couple $(T_u, u + X_{T_u})$, the time in the red (below 0) from the first ruin to the first time of recovery $T'_u - T_u$ where $T'_u = \inf\{t > T_u, u + X_t = 0\}$, the maximal ruin severity $(\inf_{t>0} u + X_t)$, the aggregate severity of ruin until recovery $J(u) = \int_{T_u}^{T'_u} |u + X_t| dt$,... Dos Reis (1993) studied the total time in the red $\tau(u) = \int_0^{+\infty} 1_{\{u+X_t<0\}} dt$ using results of Gerber (1988).

All these random variables are drawn from the infinite time ruin theory, or involve the behavior of the risk process between ruin times and recovery times. It seems interesting to consider risk measures based on some fixed time interval $[0, T]$ (T may be infinite). One of the simplest penalty functions may be the expected value of the time-aggregated negative part of the risk process:

$$E(I_T) = E\left(\int_0^T 1_{\{R_t<0\}} |R_t| dt\right).$$

Note that the probability $P(I_T = 0)$ is the probability of non ruin within finite time T. I_T may be seen as the penalty the company will have to pay due to its insolvency until the time horizon T. From an economic point of view, it seems more consistent to consider

$$I_{g,h}(u) = \left(\int_0^T \left(1_{\{u+X_t\geq 0\}} g(|u + X_t|) - 1_{\{u+X_t\leq 0\}} h(|u + X_t|)\right) dt\right)$$

with $0 \leq g \leq h$, where g corresponds to a reward function for positive reserves, and h is a penalty function in case of insolvency. To be consistent with the theory of utility functions, g should be increasing and concave, and h should be increasing and convex in the classical case. Besides, $g \leq h$ because usually the cost of ruin is higher than the reward of the opposite wealth level.

These risk measures may be differentiated with respect to the initial reserve u, which makes it possible to compute them quite easily as integrals of other functions of u such as the probability of ruin or the total time in the red. Moreover, they have the advantage that the integral over t and the mathematical expectation may be permuted thanks to Fubini's theorem.

Statements and proofs of differentiation theorems can be found in Sections 1 and 2.

Section 3 presents examples of applications to unidimensional risk measures, in particular a closed-form formula is derived for $E(I_\infty(u))$ in the Poisson-exponential case. One can also use these concepts to construct risk measures for multidimensional risk processes, modelling different lines of business of an insurance company (car insurance, health insurance, ...). In this framework, determining the global initial reserve needed for the global expected penalty to be small enough requires finding the optimal allocation of this reserve. Differentiation of unidimensional risk measures are useful for this purpose. All this is illustrated in Section 4.

1. Differentiation theorems

In the sequel, we will denote for $T \in [0, +\infty]$ the time in the red until time T by

$$\tau(u,T) = \int_0^T 1_{\{u+X_t<0\}} dt.$$

In most cases, T will be fixed, and we will use the notation $\tau(u)$ instead of $\tau(u,T)$. In Section 1, we assume that $T < +\infty$.

Theorem 1. *Assume $T \in \mathbb{R}^+$. Let $(X_t)_{t\in[0,T)}$ be a stochastic process with almost surely time-integrable sample paths. For $u \in \mathbb{R}$, denote by $\tau(u)$ the random variable corresponding to the time spent under zero by the process $u+X_t$ between the fixed times 0 and T:*

$$\tau(u) = \int_0^T 1_{\{u+X_t<0\}} dt,$$

Let $\tau_0(u)$ correspond to the time spent in zero by the process $u + X_t$:

$$\tau_0(u) = \int_0^T 1_{\{u+X_t=0\}} dt.$$

Let $I_T(u)$ represent the time-integrated negative part of the process $u + X_t$ between 0 and T:

$$I_T(u) = \left(\int_0^T 1_{\{u+X_t<0\}} |u + X_t| dt \right)$$

and $f(u) = E(I_T(u))$.

For $u \in \mathbb{R}$, if $E\tau_0(u) = 0$, then f is differentiable at u, and $f'(u) = -E\tau(u)$.

$I_T(u)$ is illustrated by Figure 1.

Proof. Fix $u \in \mathbb{R}$. For $\varepsilon \geq 0$, set

$$\tau_\varepsilon(u) = \int_0^T 1_{\{|u+X_t|<\varepsilon\}} dt.$$

Here, $\tau_\varepsilon(u)$ represents the time spent by the process $u + X_t$ in the interval $]-\varepsilon, \varepsilon[$ between dates 0 and T.
For each sample path (considered as a function of time t),

$$t \to 1_{\{|u+X_t|<\varepsilon\}}$$

pointwise converges, decreasingly to

$$t \to 1_{\{u+X_t=0\}}.$$

Besides, each of the integrals of the indicator functions is bounded by T. From the monotone convergence theorem, τ_ε is decreasing with respect to ε and converges surely to τ_0.
From the monotone convergence theorem (this time for mathematical expectation), $E\tau_\varepsilon \downarrow E\tau_0$ as $\varepsilon \downarrow 0$, because for all $\varepsilon \geq 0$, $E\tau_\varepsilon \leq T$.

Lemma 1. *For $\varepsilon \in \mathbb{R}$,*

$$|I_T(u+\varepsilon) - I_T(u) + \varepsilon\tau(u)| \leq |\varepsilon|\tau_\varepsilon(u).$$

Proof of the lemma. For $\varepsilon > 0$, $\{u + \varepsilon + X_t < 0\} \subset \{u + X_t < 0\}$, whence $I_T(u + \varepsilon) - I_T(u) =$

$$\int_0^T (|u+\varepsilon+X_t| - |u+X_t|)\, 1_{\{u+X_t<0\}} dt - \int_0^T |u+\varepsilon+X_t| 1_{\{-\varepsilon<u+X_t<0\}} dt.$$

$$I_T(u+\varepsilon) - I_T(u) = -\varepsilon \int_0^T 1_{\{u+X_t<0\}} dt - \int_0^T |u+\varepsilon+X_t| 1_{\{-\varepsilon<u+X_t<0\}} dt. \quad (1)$$

On the right side of (1), the left term corresponds to $-\varepsilon\tau(u)$. The absolute value under the integral of the second term is less than ε on the support of the indicator function.
Hence

$$|I_T(u+\varepsilon) - I_T(u) + \varepsilon\tau(u)| < \int_0^T \varepsilon 1_{\{-\varepsilon<u+X_t<0\}} dt,$$

which proves the lemma for $\varepsilon > 0$. A symmetrical procedure solves the case $\varepsilon \leq 0$.

From Lemma 1,

$$|EI_T(u+\varepsilon) - EI_T(u) + \varepsilon E\tau(u)| \leq |\varepsilon| E\tau_\varepsilon(u)$$

and

$$EI_T(u+\varepsilon) = EI_T(u) - \varepsilon E\tau(u) + \varepsilon v(u,\varepsilon)$$

where

$$|v(u,\varepsilon)| \leq E\tau_\varepsilon(u) \to E\tau_0(u) = 0$$

as $\varepsilon \to 0$, which proves that f is differentiable with respect to u and that for $u \in \mathbb{R}$, $f'(u) = -E\tau(u)$.

Corollary 1. *Using the notation of Theorem 1, let $X_t = ct - S_t$, where S_t is a jump process such that, almost surely, S_t has a finite number of nonnegative jumps in every finite interval, and that X_t has a positive drift ($X_t \to +\infty$ a.s.). Then f defined by $f(u) = E(I_T(u))$ for $u \in \mathbb{R}$ is differentiable on $\mathbb{R}$, and for $u \in \mathbb{R}$, $f'(u) = -E\tau(u)$.*

Proof. Only

$$E\tau_0(u) = \int_0^T 1_{\{u+ct-S_t=0\}} dt = 0$$

has to be shown. Note that $R_t = u + ct - S_t$ is a process whose sample paths are almost surely increasing between two consecutive jump instants. The number of jumps is almost surely finite on the time interval $[0,T]$. Between two times when the process is 0, there must be at least one jump instant.
This implies that the number of visits of 0 is almost surely finite as it is less than $N_T + 1$, where N_T is the number of jumps between 0 and T.
So $E\tau_0 = 0$ and the result comes from Theorem 1.

Proposition 1. *More generally, all processes for which the distribution of R_t is diffuse for all $t \in \mathbb{R}^+ - N$ satisfy the condition $E\tau_0 = 0$, if N is a null subset of $\mathbb{R}^+$ for the Lebesgue measure.*
Theorem 1 is also satisfied for this wide class of processes.

Proof. For $T \in \bar{\mathbb{R}}$, from Fubini's theorem,

$$E\tau_0(T) \leq E\left(\int_0^{+\infty} 1_{\{R_t=0\}} dt\right) = \int_0^{+\infty} P(R_t = 0) dt$$

which provides the required result.

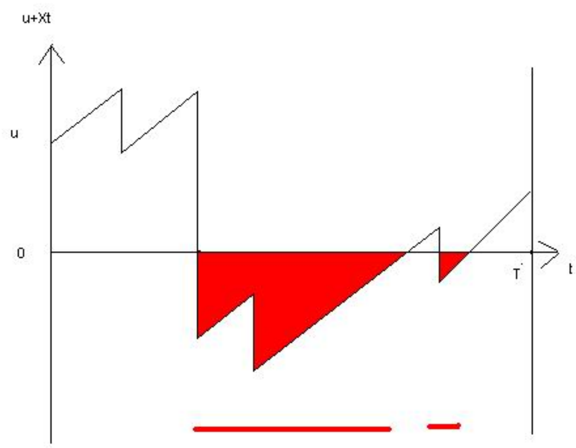

FIGURE 1: The area in red represents $I_T(u) = \int_0^T 1_{\{u+X_t<0\}}|u+X_t|dt$

Theorem 2. *Let $g \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+)$ be a convex or concave function, such that $g(0) = 0$. Let X_t be a stochastic process such that, for $u \in \mathbb{R}$, $t \to g(-(u+X_t))1_{\{u+X_t<0\}}$ is almost surely integrable with respect to t. Let I_g be the function from $\mathbb{R}$ into the space of nonnegative random variables, and defined by*

$$I_g(u) = \left(\int_0^T 1_{\{u+X_t<0\}} g(-(u+X_t))dt \right)$$

for $u \in \mathbb{R}$ and let $f(.) = EI_g(.)$.
For $u \in \mathbb{R}$, if $f(u) < +\infty$, $EI_{g'}(u) < +\infty$ and $E\tau_0(u) = 0$, then f is differentiable at point u, and

$$f'(u) = -E \left(\int_0^T 1_{\{u+X_t<0\}} g'(|u+X_t|)dt \right).$$

Proof. Fix $u \in \mathbb{R}$.

$$\frac{I_g(u+\varepsilon) - I_g(u)}{\varepsilon} = \int_0^T \frac{g(|u+\varepsilon+X_t|) - g(|u+X_t|)}{\varepsilon} 1_{\{u+X_t<0\}} dt - \int_0^T \frac{g(|u+\varepsilon+X_t|)}{\varepsilon} 1_{\{-\varepsilon<u+X_t<0\}} dt.$$

For $t \in [0, T]$,

$$\frac{g(-(u+\varepsilon+X_t)) - g(-(u+X_t))}{-\varepsilon} 1_{\{u+X_t<0\}} \uparrow \text{ (resp. } \downarrow \text{) } g'(-(u+X_t))1_{\{u+X_t<0\}}$$

almost surely as $\varepsilon \downarrow 0$, from the increase (resp. decrease) of the rates of increase of convex (resp. concave) functions.

From the monotone convergence theorem, for $t \in [0,T]$,

$$E\left(\frac{g(-(u+\varepsilon+X_t))-g(-(u+X_t))}{\varepsilon}1_{\{u+X_t<0\}}\right) \to -E\left(g'(-(u+X_t))1_{\{u+X_t<0\}}\right).$$

From Fubini's theorem,

$$E\left(\int_0^T \frac{g(-(u+\varepsilon+X_t))-g(-(u+X_t))}{\varepsilon}1_{\{u+X_t<0\}}dt\right) \to -EI_{g'}(u)$$

as $\varepsilon \downarrow 0$, where

$$I_{g'}(u) = \int_0^T g'(-(u+X_t))1_{\{u+X_t<0\}}dt.$$

Hence

$$|f(u+\varepsilon)-f(u)+\varepsilon EI_{g'}(u)+\varepsilon w(u,\varepsilon)| \leq E\left(\int_0^T g(-(u+\varepsilon+X_t))1_{\{-\varepsilon<u+X_t<0\}}dt\right)$$

with $w(u,\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, and

$$|f(u+\varepsilon)-f(u)+\varepsilon EI_{g'}(u)+\varepsilon w(u,\varepsilon)| \leq \varepsilon E\tau_\varepsilon(u)E\left(\sup_{t\in[0,\varepsilon]} g'(t)\right).$$

$$EI_g(u+\varepsilon) = EI_g(u) - \varepsilon EI_{g'}(u) + \varepsilon(v(u,\varepsilon)-w(u,\varepsilon))$$

where

$$|v(u,\varepsilon)| \leq KE\tau_\varepsilon(u) \to KE\tau_0(u) = 0$$

as $\varepsilon \downarrow 0$, which proves that f is right-differentiable at point u and that

$$f'_r(u) = -E\left(\int_0^T g'(-(u+X_t))1_{\{u+X_t<0\}}dt\right).$$

With similar reasoning, f is left-differentiable and $f'_l = f'_r$, which ends the proof.

2. Differentiation of the average time in the red and other generalizations

Recall that the time in the red is the time spent by the wealth process below 0, between time 0 and some fixed time horizon T:

$$\tau(u) = \int_0^T 1_{\{u+X_t<0\}}dt.$$

T is first supposed to be finite.

Theorem 3. *Let $X_t = ct - S_t$, where S_t is a jump process satisfying hypothesis (H1): S_t has a finite expected number of nonnegative jumps in every finite interval, and for each t, the distribution of S_t is absolutely continuous.*
For example, S_t might be a compound Poisson process with a continuous jump size distribution. Consider $T < +\infty$ and define h by $h(u) = E(\tau(u))$ for $u \in \mathbb{R}$. h is differentiable on $\mathbb{R}_*^+$, and for $u > 0$,

$$h'(u) = -\frac{1}{c} EN^0(u,T),$$

where $N^0(u,T) = Card(\{t \in [0,T], \quad u + ct - S_t = 0\})$.

Proof. Almost surely in ω, the number of jumps $N(T)$, and so $N^0(u,T)$, is finite. Consider a sample path $(X_t(\omega))_{0\le t\le T}$. Let $R_t = u + X_t$ and denote by T_i the i^{th} jump instant. Define

$$\varepsilon_0(\omega) = \inf_{n \le N(T), R_{T_n} > 0} R_{T_n}.$$

If $N^0(u,T) = 0$, then define

$$\varepsilon^+ = \inf\left(\{u + X_t, \quad 0 \le t \le T\} \cap \mathbb{R}^+\right)$$

and

$$\varepsilon^- = -\sup\left(\{u + X_t, \quad 0 \le t \le T\} \cap \mathbb{R}^-\right).$$

Here, ε^- and ε^+ are almost surely positive. If $|\varepsilon| < \inf(\varepsilon^+, \quad \varepsilon^-)$, then $\tau(u-\varepsilon) - \tau(u) = 0$, and the following reasoning remains valid.

Otherwise, for $1 \le i \le N^0(u,T)$, denote by t_i the instant of the i^{th} visit of R_t in 0, and by t'_i the instant of the first jump of R_t after t_i. The sample paths of the process R_t are almost surely right-continuous, and the probability that $R_T = 0$ is zero. So one may consider

$$\varepsilon_1(\omega) = \min\left(\min_{1 \le i \le N^0(u,T)} c(t'_i - t_i), c(T - t_{N^0(u,T)})\right).$$

Then, for $0 < \varepsilon < \min(\varepsilon_0(\omega), \varepsilon_1(\omega))$,

$$\{0 < u + ct - S_t < \varepsilon\} = \bigcup_{i=1}^{N^0(u,T)} \{]t_i, t_i + \varepsilon/c[\}$$

and so

$$\tau(u-\varepsilon) - \tau(u) = \int_0^T \left(1_{\{u-\varepsilon+ct-S_t<0\}} - 1_{\{u+ct-S_t<0\}}\right) dt$$

$$= \int_0^T 1_{\{0 \leq u+ct-S_t<\varepsilon\}} dt = \sum_{k=1}^{N^0(u,T)} \frac{\varepsilon}{c}.$$

Hence

$$\frac{\tau(u-\varepsilon)-\tau(u)}{\varepsilon} \to \frac{1}{c} N^0(u,T)$$

almost surely as $\varepsilon \to 0$. Moreover, between two consecutive jumps of R_t, the difference between the two integrals is less than $\frac{\varepsilon}{c}$ in absolute value, whence

$$\int_{T_i}^{T_{i+1}} 1_{\{0 \leq u+ct-S_t<\varepsilon\}} dt \leq \frac{\varepsilon}{c}.$$

So for $\varepsilon > 0$ small enough, with notations $T_{N(T)+1} = T$ and $T_0 = 0$,

$$\left(\frac{\tau(u-\varepsilon)-\tau(u)}{\varepsilon} \right) = \left(\sum_{i=0}^{N(T)} \frac{1}{\varepsilon} \int_{T_i}^{T_{i+1}} 1_{\{0 \leq u+ct-S_t<\varepsilon\}} dt \right).$$

$$\left(\frac{\tau(u-\varepsilon)-\tau(u)}{\varepsilon} \right) \leq \left(\sum_{i=0}^{N(T)} \frac{1}{\varepsilon} \frac{\varepsilon}{c} \right) \leq \frac{1}{c}(N(T)+1).$$

Hence, from the dominated convergence theorem,

$$E\left(\frac{\tau(u-\varepsilon)-\tau(u)}{\varepsilon} \right) \to \frac{1}{c} EN^0(u,T)$$

as $\varepsilon \to 0$. This proves that h is left-differentiable on $\mathbb{R}_*^+$, and that for $u > 0$,

$$h_l'(u) = -\frac{1}{c} EN^0(u,T).$$

With similar reasoning, h is right-differentiable on $\mathbb{R}_*^+$, and $h_l' = h_r'$. Hence h is differentiable on $\mathbb{R}_*^+$, and for $u > 0$, $h'(u) = -\frac{1}{c} EN^0(u,T)$.

Remark 1. This provides the second-order derivative of $EI_T(.)$, which appears to be positive. Thus, $EI_T(.)$ is strictly convex, which will be very important for the minimization in Section 4.

Remark 2. This second-order differentiate corresponds in the general case to the expectation of the local time $L_T(0)$ in 0 of the process $u + X_t$ up to time T:

$$L_T(0) = \lim_{\varepsilon \downarrow 0} \left(\frac{1}{2\varepsilon} \int_0^T P(|u+X_t| < \varepsilon) dt \right).$$

Theorem 4. *Let g, h be two convex or concave functions in $\mathcal{C}^1(\mathbb{R}^+,\mathbb{R}^+)$, such that for $x \geq 0$, $g(x) \geq g(0)$ and $h(x) \geq h(0)$. Let X_t be a stochastic process such that $t \to g(u+X_t)$ and $t \to h(u+X_t)$ are almost surely integrable on $[0,T]$. Let I_g^+ be the function from $\mathbb{R}$ into the space of nonnegative random variables, and defined by*

$$I_g^+(u) = \int_0^T 1_{\{u+X_t \geq 0\}} g(u+X_t)dt$$

for $u \geq 0$ and let $f(.) = EI_g^+(.) - EI_h(.)$.
If, for $u \in \mathbb{R}$,

$$EI_g^+(u), \quad EI_h(u), \quad EI_{g'}^+(u), \quad EI_{h'}(u) \quad < +\infty,$$

and if $E\tau_0(u) = 0$, then f is differentiable on $\mathbb{R}_*^+$, and for $u > 0$,

$$f'(u) = EI_{g'}^+(u) - EI_{h'}(u) - (g(0)+h(0))EL_T(0).$$

Corollary 2. *With the hypotheses of Theorem 4, if besides $X_t = ct - S_t$, where S_t satisfies hypothesis (H1) of Theorem 3, then the differentiate may be rewritten for $u > 0$ as:*

$$f'(u) = EI_{g'}^+(u) - EI_{h'}(u) + \frac{(g(0)+h(0))EN^0(u,T)}{c} \tag{2}$$

where $N^0(u,T) = Card(\{t \in [0,T], \quad u + ct - S_t = 0\})$.

Proof of Corollary 2. Immediate from Theorem 4, after replacing the last term in (2) following the proof of Theorem 3.

Proof of Theorem 4. Decompose

$$I_g^+(u) - I_h(u) = -\tilde{I}_{(g-g(0))}(-u) - I_{(h-h(0))}(u) - h(0)\tau(u) + g(0)(T - \tau(u)),$$

where $\tilde{I}_g$ is obtained from I_g by changing X_t into $-X_t$. From linearity of expectation and of differentiation, applying theorem 2 to $g - g(0)$ with $-X_t$ and to $h - h(0)$ with X_t, and using Theorem 3 end the proof of Theorem 4.

Theorem 5. *If besides the process X_t converges almost surely to $+\infty$ as $t \to +\infty$, and if for $u \geq 0$, $EI_\infty < +\infty$ and $E\tau(u,\infty) < +\infty$, then Theorem 1 remains valid with $T = +\infty$.*

Proof. Same kind of reasoning as previously.

Remark 3. These conditions of integrability are fulfilled if the time spent below 0 for a single ruin is integrable.

Denote by $\psi(u)$ the probability of ruin in infinite time with initial reserve u.

Theorem 6. *Theorem 3 remains valid with $T = +\infty$ if besides X_t has a positive drift and if $\tau(u)$ is integrable for all $u > 0$. Besides, in the compound Poisson case, for $u > 0$,*

$$h'(u) = -\frac{1}{c}\frac{1}{1-\psi(0)}\psi(u).$$

Proof. For $T \in [0, +\infty]$, recall the notation

$$\tau(u,T) = \int_0^T 1_{\{u+X_t<0\}} dt.$$

Note that $(N^0(u,n))_{n\geq 0}$ is a nondecreasing sequence of random variables which converges surely to $N^0(u,+\infty)$, possibly infinite.

We shall show that $EN^0(u,+\infty) < +\infty$.

Almost surely, $u + X_t \to +\infty$ as $t \to +\infty$. Hence, almost surely, $N^0(u,+\infty) < +\infty$ and is equal to the number of ruins:

$$N^0(u,\infty) = \text{Card}\left(\{t > 0, \quad u + ct - S_t < 0 \text{ and } u + ct^- - S_{t^-} > 0\}\right).$$

Indeed, after each ruin, there is a recovery because X_t converges almost surely to $+\infty$ as t goes to $+\infty$, and the number of jumps which lead exactly to the value 0 is finite almost surely. Besides, in the compound Poisson case, the number of ruins has the following distribution:

$$P(N^0(u,\infty) = n) = \psi(u)\psi(0)^{n-1}(1-\psi(0))$$

for $n \geq 1$ and $P(N^0(u,\infty) = 0) = 1 - \psi(u)$. So $N^0(u,\infty)$ follows a zero-modified geometric distribution : $P(N^0(u,\infty) = 0) = 1 - \psi(u)$ and for $n > 0$,

$$P(N^0(u,\infty) = n | N^0(u,\infty) > 0) = \psi(0)^{n-1}(1-\psi(0)).$$

Hence $N^0(u,\infty)$ is integrable and

$$EN^0(u,\infty) = \psi(u)\frac{1}{1-\psi(0)}.$$

For all ω and for $\varepsilon > 0$, the function

$$(T,\omega) \to \frac{\tau(u+\varepsilon,T)-\tau(u,T)}{\varepsilon}(\omega)$$

is increasing with respect to T, and its limit expectation is equal to $-\frac{1}{c}EN^0(u,T)$ as $\varepsilon \downarrow 0$. From the monotone convergence theorem,

$$E\left[\lim_{\varepsilon\downarrow 0}\left(\frac{\tau(u+\varepsilon,\infty)-\tau(u,\infty)}{\varepsilon}\right)\right] = -\frac{1}{c}EN^0(u,\infty).$$

Remark 4. In infinite time, the probability of ruin may be regarded as the expectation of the local time in 0 of the process (up to multiplication by a constant number).

3. Applications to the unidimensional case

Theorem 7. *In the Poisson(λ)-Exponential($1/\mu$) case, with positive safety loading* $\rho = \frac{c-\lambda\mu}{\lambda\mu}$,

$$\psi(u) = (1-\mu R)e^{-Ru},$$

with $R = \frac{1}{\mu}\left(1-\frac{\lambda\mu}{c}\right)$. *Hence, for* $T = +\infty$,

$$E\tau(u) = \frac{(1-\mu R)}{c\mu R^2}e^{-Ru}$$

and

$$EI_\infty(u) = \frac{(1-\mu R)}{c\mu R^3}e^{-Ru}.$$

Proof. This comes simply from integration of the well-known formula for $\psi(u)$, as the functions considered tend to 0 as $u \to +\infty$.

This method provides a way to get back the average total time in the red from the integration of the probability of ruin. Dos Reis (1993) derived this result for $E\tau(u,\infty)$ by considering the number of ruins, and using the distributions of the length of the first period in the red (until recovery), and of those of the following periods in the red, which had been derived by Gerber (1988).

Remark 5. It is possible to derive $EI_\infty(u)$ for Gamma-distributed or phase-type-distributed claim amounts, as we know the probability of ruin in these cases. The results are not reported here in the interest of conciseness.

The parallel with the Brownian case is also interesting. The local time of a standard Brownian motion W_t in x is defined by

$$L_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \int_0^t 1_{\{|W_s - x| < \varepsilon\}} ds.$$

This provides a density for the occupation time $\Gamma_t(B)$ of a Borelian set B between 0 and t:

$$\Gamma_t(B) = \int_B 2L_t(x)dx.$$

Paul Lévy's Brownian local time representation theorem with downcrossings states that

Theorem 8. *(Paul Lévy)*

$$2L_t(0) = \lim_{\varepsilon \downarrow 0} \varepsilon D_t(\varepsilon)$$

where $D_t(\varepsilon)$ is the number of downcrossings of the interval $[0, \varepsilon]$ by the process W_s between 0 and t .

This well-known theorem might be viewed as a limit case of Theorem 3.

4. Multidimensional risk measures and optimal allocation

For a unidimensional risk process, one classical goal is to determine the minimal initial reserve u_ε needed for the probability of ruin to be less than ε.
In a multidimensional framework, modelling the evolution of the different lines of business of an insurance company by a multirisk process $(u_1 + X_t^1, \ldots, u_K + X_t^K)$ ($u_k + X_t^k$ corresponds to the wealth of the k^{th} line of business at time t), one could look for the global initial reserve u which ensures that the probability of ruin ψ satisfies

$$\psi(u_1, \ldots, u_K) \leq \varepsilon$$

for the optimal allocation $(u_1, \ldots, u_K)$ such that

$$\psi(u_1, \ldots, u_K) = \inf_{v_1 + \cdots + v_K = u} \psi(v_1, \ldots, v_K)$$

with

$$\psi(u_1, \ldots, u_K) = P(\exists k \in [1, K], \exists t > 0, u_k + X_t^k < 0).$$

Instead of the probability of crossing some barriers, it may be more interesting to minimize the sum of the expected cost of the ruin for each line of business until time T, which may be represented by the expectation of the sum of integrals over time of the negative part of the process. In both cases, finding the global reserve needed requires determination of the optimal allocation. It has just been shown in the previous sections how to compute $E(I_T)$ for one line of business, and the linearity of the expectation makes it possible to compute the sum for K dependent lines of business just as in the independent case. The structure of dependence between lines of business has no impact on this risk measure. This may be considered as a problem of optimal allocation of resource under budget constraints as in economics, the goal being to maximize the utility function given by the opposite of the sum of the $E(I_T^i)$.

4.1. Minimizing the penalty function

Recall that what has to be minimized is

$$A(u_1, \dots, u_K) = \sum_{i=k}^{K} EI_T^k$$

where

$$EI_T^k = E\left[\int_0^T |R_t^k| 1_{\{R_t^k<0\}} dt\right]$$

with $R_t^k = u_k + X_t^k$ under the constraint $u_1 + \cdots + u_K = u$. This does not depend on the dependence structure between the lines of business because of the linearity of the expectation. Denote $v_k(u_k)$ the differentiate of EI_T^k with respect to u_k. Using Lagrange multipliers implies that if $(u_1, \dots, u_K)$ minimizes A, then $v_k(u_k) = v_1(u_1)$ for all $1 \leq k \leq K$. Compute $v_k(u_k)$:

$$v_k(u_k) = \left(E\left[\int_0^T |R_t^k| 1_{\{R_t^k<0\}} dt\right]\right)' = -E\tau^k = -\int_0^T P\left[\{R_t^k < 0\}\right] dt$$

where τ^k represents the time spent in the red between 0 and T for line of business k. The differentiation theorem of the previous section justifies the previous derivation. The sum of the average times spent under 0 is a decreasing function of the u_k. So A is strictly convex. On the compact space

$$\mathcal{S} = \{(u_1, \dots, u_K) \in (\mathbb{R}^+)^K, \quad u_1 + \cdots + u_K = u\},$$

A admits a unique minimum. The optimal allocation is thus the following: there is a subset $J \subset [1, K]$ such that for $k \notin J$, $u_k = 0$, and for $k, j \in J$, $E\tau_k = E\tau_j$. The interpretation is quite intuitive: the safest lines of business do not require any reserve, and the other ones share the global reserve in order to get equal average times in the red for those lines of business.

Relaxing nonnegativity, on $\{u_1 + \cdots + u_K = u\}$, if $(u_1, \ldots, u_K)$ is an extremum point for A, then for the K lines of business, the average times spent under 0 are equal to one another. If it is a minimum for the sum of the times spent below 0 for each line of business, then the average number of visits is proportional to the c_k, and in infinite time the ruin probabilities are in fixed proportions. However the existence of a minimum is not guaranteed, because $(u_1, \ldots, u_K)$ is no longer compact. The problem would be more tractable with the average time in the red or with minimization on the c_k, because some factors penalize very negative u_k in these cases.

4.2. Example

In the Poisson(λ)-Exponential($\frac{1}{\mu}$) case, recall that

$$EI_\infty(u) = \frac{(1 - \mu R)}{c\mu R^3} e^{-Ru}.$$

Consider a two-line-of-business model, with the following parameters:
$\mu_1 = \mu_2 = 1$, $c_1 = c_2 = 1$, $R_2 = 0.4$ and $u = 10$. We want to minimize $A(u_1, u_2)$ for $0 \leq u_1, u_2 \leq 10$ such that $u_1 + u_2 = 10$. A mere modification of the adjustment coefficient R_1 makes the optimal allocation vary strongly. When $R_1 = 0.5 > R_2$, (Figure 2), line of business 1 is safer than line 2 from the comparison between the adjustment coefficients, and line 2 should receive a greater initial reserve than line 1. The optimal allocation is about ($u_1 = 3.745990378, \quad u_2 = 6.254009622$). When $R_1 = 0.3 < R_2$ (Figure 3), line of business 1 is riskier, and this is the opposite. The optimal allocation is in this case ($u_1 = 6.756449750, \quad u_2 = 3.243550250$). When $R_1 = 0.08$ (Figure 4), the optimal allocation is ($u_1 = 10, \quad u_2 = 0$). In that case, line of business 1 is much more risky than line of business 2, which justifies the transfer of the global initial reserve u to line of business 1. For more properties or examples about optimal reserve allocation, the interested reader may consult Loisel (2004).

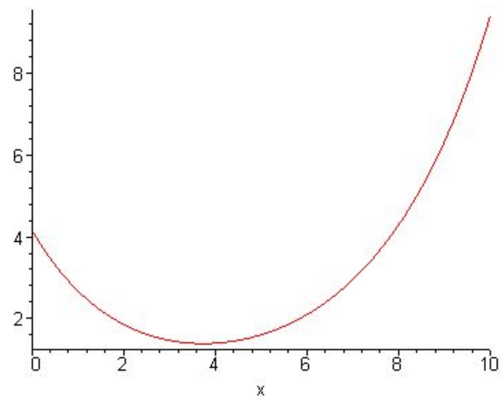

FIGURE 2: Graph of $A(x, 10-x)$ with $R_1 = 0.5$.: line of business 2 should have greater initial reserve than line 1.

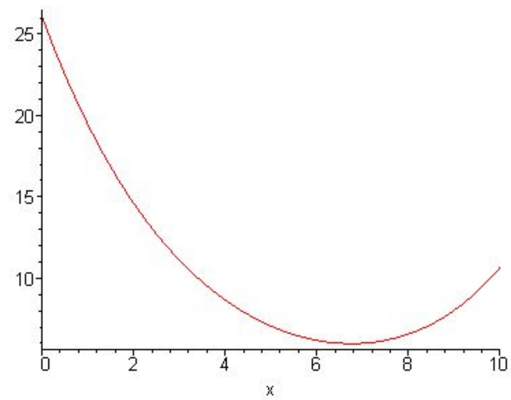

FIGURE 3: Graph of $A(x, 10-x)$ with $R_1 = 0.3$.: line of business 1 should have greater initial reserve than line 2.

4.3. Further applications

The multidimensional risk measure A, which does not depend on the structure of dependence between lines of business, is one example of what can be considered. Another possibility would be to minimize the sum

$$B = \sum_{k=1}^{K} E\tau'_k(u)$$

where

$$E\tau'_k(u) = E\left(\int_0^T 1_{\{R_t^k<0\}} 1_{\{\sum_{j=1}^K R_t^j>0\}} dt\right).$$

Here B takes dependence into account, and the following proposition shows what can be done:

Proposition 2. *Let $X_t = ct - S_t$, where S_t satisfies hypothesis (H1) of Theorem 3. Define B by $B(u_1, \dots, u_K) = \sum_{k=1}^K E(\tau'_k(u))$ for $u \in \mathbb{R}^K$. B is differentiable on $(\mathbb{R}_*^+)^K$, and for $u_1, \dots, u_K > 0$,*

$$\frac{\partial B}{\partial u_k} = -\frac{1}{c_k} E N_k^0(u, T),$$

where $N_k^0(u,T) = \mathit{Card}\left(\{t \in [0,T], \quad \left(R_t^k = 0\right) \cap \left(\sum_{j=1}^K R_t^j > 0\right)\}\right).$

It is also possible to differentiate with respect to c instead of u.

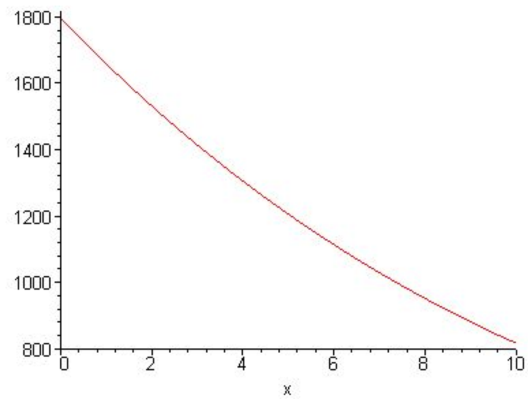

FIGURE 4: Graph of $A(x, 10-x)$ with $R_1 = 0.08$.: line of business 1 should receive the whole initial reserve.

Theorem 9. *With the notation of Theorem 1, consider the case* $X_t = ct - S_t$*, where* S_t *satisfies hypothesis (H1) of Theorem 3, and define* $\tilde{f}(c) = E(I_T(c))$.
If for all c, $E\tau_0(c) = 0$*, then* $\tilde{f}$ *is differentiable on* $\mathbb{R}$ *and for* $c \in \mathbb{R}$,

$$\tilde{f}'(u) = -\int_0^T tP(R_t < 0)dt.$$

It is interesting to look for the optimal allocation of the global premium $c = c_1 + \cdots + c_K$ because if c_k is small enough to make the safety loading negative, the process R_t^k tends to $-\infty$. Quite often, optimizing with the c_k will be easier than with the u_k for this reason. These examples illustrate how these differentiation results may be used.
The differentiation developed here is quite general and may be useful to solve many problems involving multirisk models. For a discussion about multidimensional risk measures, optimal allocation procedures, and impact of dependence between lines of business, the interested reader may consult Loisel (2004).

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