The Compound Poisson Risk Model with a Threshold Dividend Strategy

Kristina P. Pavlova
University of Western Ontario

kpavlova@utstat.utoronto.ca

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• $\{N(t); \ t \geq 0\}$ - Poisson process with parameter $\lambda$ representing the number of claims at time $t$
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• \( \{V_1, V_2, \ldots\} \) - independent identically distributed (i.i.d.) interclaim times
• \( \{Y_1, Y_2, \ldots\} \) - i.i.d. claim amounts independent of \( \{V_1, V_2, \ldots\} \) and having common cumulative distribution function (c.d.f.) \( P(y) = 1 - \bar{P}(y) \)
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• $\{S(t) = \sum_{i=1}^{N(t)} Y_i; t \geq 0\}$ - aggregate-claim process
\[ dU(t) = c - dS(t) \]
- insurer’s surplus at time \( t \)

\[ u \]
- initial surplus
\[ dU(t) = c - dS(t) \]

- insurer’s surplus at time \( t \)

\( u \) – initial surplus
\[ U_b(t) \]

\[ b \]

\[ u \]

\[ T_b \]

\[ |U_b(T_b)| \]

\[ U_b(T_b^-) \]
\[ dU_b(t) = \begin{cases} 
 c_1 dt - dS(t), & U_b(t) \leq b \\
 c_2 dt - dS(t), & U_b(t) > b 
\end{cases} \]

- insurer’s surplus at time \( t \)
Problem description
Outline

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- Analytic solution
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  - Probability of ultimate ruin
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  - Surplus immediately before ruin and deficit at ruin
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  • Time of ruin
  • Surplus immediately before ruin and deficit at ruin
  • Exponential examples
Similarly to the classical compound Poisson model, we define the Gerber-Shiu discounted penalty function by

\[
m(u; b) = \mathbb{E}\left\{ e^{-\delta T_b} w(U_b(T_b^-), |U_b(T_b)|) I(T_b < \infty) | U_b(0) = u \right\}
\]

where \( \delta \geq 0 \) is the force of interest and \( I \) is the indicator function.
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where \( \delta \geq 0 \) is the force of interest and \( I \) is the indicator function. Then, for notational clarity, we set

\[ m(u; b) = \begin{cases} 
    m_1(u), & 0 \leq u \leq b \\
    m_2(u), & u > b.
\end{cases} \]
We demonstrate that the discounted penalty function satisfies

\[
m'(u; b) = \frac{\lambda + \delta}{c_i} m(u; b) - \frac{\lambda}{c_i} \int_0^u m(u - y; b) dP(y) - \frac{\lambda}{c_i} \zeta(u),
\]

where \( i = 1 \) or \( 2 \) depending on the interval for \( u \) and

\[
\zeta(u) = \int_u^\infty w(u, y - u) dP(y).
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\[ \int_0^{u-b} m_2(u - y) dP(y) + \int_{u-b}^u m_1(u - y) dP(y) \]

when \( u > b. \)
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where \( i = 1 \) or \( 2 \) depending on the interval for \( u \) and
\[ \zeta(u) = \int_u^{\infty} w(u, y - u) dP(y). \]

We are interested in finding an analytic solution to the above equation.
To do so, we first need to secure an initial condition. As such may serve the continuity of $m$. More specifically, it may be shown that $m_1(b) = m_2(b) := \lim_{u \to b^+} m_2(u)$. 
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Second, we assume that $m_1$ is known and proceed with finding $m_2$. 
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Second, we assume that \( m_1 \) is known and proceed with finding \( m_2 \).

The integro-differential equation for \( m_2 \) yields

\[
m_2(u) = \pi_2 \left[ \int_0^{u-b} m_2(u - y) dA_2(y) + \int_{u-b}^u m_1(u - y) dA_2(y) \right] + \frac{\lambda}{c_2} T_{\rho_2} \zeta(u), \quad u > b,
\]

where

\[
T_{\rho_2} \zeta(u) = \int_0^\infty e^{-\rho_2 y} \zeta(u + y) dy.
\]

X. Sheldon Lin and Kristina P. Pavlova – p. 7/?
Therefore, Theorem 2.1 in Lin and Willmot (1999) yields

\[ m_2(u) = \frac{1}{1 - \pi_2} \int_0^{u-b} h(u - y) dK_2(y) + h(u), \quad u > b, \]

where \( h \) is a function depending on \( m_1 \) and \( K_2 \) is a compound geometric c.d.f.
Third, we return to $m_1$ and consider the integro-differential equation with the relaxed condition $u \geq 0$. 
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\[ v'(u) = \frac{\lambda + \delta}{c_1} v(u) - \frac{\lambda}{c_1} \int_0^u v(u - y) dP(y), \]
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$$v'(u) = \frac{\lambda + \delta}{c_1} v(u) - \frac{\lambda}{c_1} \int_0^u v(u - y) dP(y),$$

and a solution of the nonhomogeneous equation

$$m'_\infty(u) = \frac{\lambda + \delta}{c_1} m_\infty(u) - \frac{\lambda}{c_1} \int_0^u m_\infty(u - y) dP(y) - \frac{\lambda}{c_1} \zeta(u).$$
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For more detail about the approach, see Lin et al. (2003).
Bühlman (1970) demonstrates that

\[ v(u) = \frac{1 - \Psi(u)}{1 - \Psi(0)} e^{\rho_1 u}, \quad u \geq 0, \]

where \( \Psi \) is a compound geometric tale.
Bühlman (1970) demonstrates that

\[ v(u) = \frac{1 - \Psi(u)}{1 - \Psi(0)} e^{\rho_1 u}, \quad u \geq 0, \]

where \( \Psi \) is a compound geometric tale and Gerber and Shiu (1998) find that the solution in the classical compound Poisson case satisfies

\[ m_\infty(u) = \pi_1 \int_0^u m_\infty(u - y) dA_1(y) + \frac{\lambda}{c_1} T_{\rho_1} \zeta(u), \quad u \geq 0, \]
which by Theorem 2.1 in Lin and Willmot (1999) may be analytically expressed as

\[
m_\infty(u) = \frac{\lambda}{c_1(1 - \pi_1)} \int_0^u T_{\rho_1} \zeta(u - y) dK_1(y) + \frac{\lambda}{c_1} T_{\rho_1} \zeta(u),
\]

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where \( K_1 \) is a compound geometric c.d.f. Finally, the solution of the integro-differential equation for \( m_1 \) is of the form

\[ m_1(u) = m_\infty(u) + \kappa \nu(u), \quad u \geq 0, \]

where \( \kappa \) is a constant to be determined through the initial condition.
The probability of ultimate ruin $\psi(u; b)$ equals

$$\psi_1(u) = 1 - q(b) + q(b)\psi_{1,\infty}(u), \quad 0 \leq u \leq b$$

$$\psi_2(u) = -\frac{1 + \theta_2}{\theta_2} \int_0^{u-b} h(u - y) d\psi_{2,\infty}(y) + h(u), \quad u > b,$$

where $q(b) \in [0, 1]$ is a constant, $\psi_{i,\infty}, i = 1, 2,$ is the probability of ultimate ruin under the classical compound Poisson model with premium rate $c_i$, $h$ is a function depending on $\psi_1$, and $\theta_2$ is the relative security loading when the surplus is above the barrier $b$. 
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where $q(b) \in [0, 1]$ is a constant, $\psi_{i,\infty}, i = 1, 2$, is the probability of ultimate ruin under the classical compound Poisson model with premium rate $c_i$, $h$ is a function depending on $\psi_1$, and $\theta_2$ is the relative security loading when the surplus is above the barrier $b$. A related result is obtained in Asmussen (2000).
The Laplace transform of the time of ruin $\mathcal{L}(u; b)$ equals

$$
\mathcal{L}_1(u) = \mathcal{L}_\infty(u) + \kappa \frac{1 - \Psi(u)}{1 - \Psi(0)} e^{\rho_1 u}, \quad 0 \leq u \leq b
$$

$$
\mathcal{L}_2(u) = \frac{1}{1 - \pi_2} \int_0^{u-b} h(u - y) dK_2(y) + h(u), \quad u > b,
$$

where $\kappa$ is known and $h$ is a function depending on $\mathcal{L}_1$. 
Applications

The Laplace transform of the time of ruin $\mathcal{L}(u; b)$ equals

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$$

$$
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$$

where $\kappa$ is known and $h$ is a function depending on $\mathcal{L}_1$. Both the joint and the marginal defective distributions, along with the moments, of the surplus before ruin and the deficit at ruin may be expressed analytically.
If the claim amounts have exponential distribution
\[ P(y) = 1 - e^{-\mu y}, \; y \geq 0, \]
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- the probability of ultimate ruin \( \psi(u; b) \) becomes

\[
\psi_1(u) = 1 - q(b) + \frac{q(b)}{1 + \theta_1} e^{-\beta_1 u}, \quad 0 \leq u \leq b
\]

\[
\psi_2(u) = \frac{1}{1 + \theta_2} \left[ 1 - q(b) + q(b)e^{-\beta_1 b} \right] e^{-\beta_2(u-b)}, \quad u > b,
\]

where \( q(b) \in [0, 1] \) is a known constant and
\[ \beta_i = \frac{\theta_i}{1+\theta_i} \mu, \ i = 1, 2. \]
• the Laplace transform of the time of ruin $\mathcal{L}(u; b)$ becomes

$$\mathcal{L}_1(u) = [1 - r(b)]e^{\rho_1 u} + \pi_1 r(b)e^{-\tau_1 u}, \ 0 \leq u \leq b$$

$$\mathcal{L}_2(u) = \pi_2 \left\{ \frac{\rho_1}{\mu + \rho_1}[1 - r(b)]e^{-\mu b} + \frac{\mu}{\mu + \rho_1}[1 - r(b)]e^{\rho_1 b} + r(b)e^{-\tau_1 b} \right\} e^{-\tau_2(u-b)}, \ u > b,$$

where $r(b)$ is a known constant.
The End