

# *Decision principles derived from risk measures*

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## Back to the future

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- "WORST CASE RISK MEASUREMENT"

(1982) F. De Vylder

$$v(c) = \sup_{\mu \in M} (\int f d\mu \mid \int g_i d\mu = c_i, i = 1, \dots, n)$$

- "COHERENT RISK MEASURES"

(1981) P. Huber (Robust Statistics)

$$X \leq Y \Rightarrow \rho(X) < \rho(Y)$$

$$\rho(aX + b) = a\rho(X) + b$$

$$\rho(X + Y) \leq \rho(X) + \rho(Y)$$

- "CONVEX RISK MEASURES"

(1985) H. Gerber and O. Deprez

(Convex principles of premium calculation)

- "DISTORTION RISK MEASURES"

(1987) Yaari, The Dual theory of choice under risk (Econometrica)

# Risk Measures $\Leftrightarrow$ Decision Principles

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- *Risk Measure*: functional that assigns a real number to a random variable, by means of a set of axioms (top level)

Axioms are fixed by economic actors or agents

- *Decision Principle*: idem but at a lower level

! A "DERIVED" FUNCTIONAL !

- "Markowitz": In one context one basic set of axioms is appropriate while in another context a different set is appropriate.

- E.g. Premium Principles, Capital Allocation, Solvency Capital Principle (regulatory, economic or rating capital)

# Premium Principle

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- Insurer  $E[u(\omega + \rho(X) - X)] \geq u(\omega)$
- $\tilde{u}(\tilde{\omega} - \rho(X)) \geq E[\tilde{u}(\tilde{\omega} - X)]$
- $\tilde{\rho}(X) < \rho(X) \leq \rho^+(X)$
- Hierarchy between measures and principles:
  - Expected Utility = Risk Measure (units: arbitrary) (axioms)
  - Decision Principle = derived functional ( $\neq$  properties) (units: money) (properties, eventually derived axioms)

## Solvency capital principle

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- Risk:  $(X - \rho(X))_+ + i\rho(X)$
- Risk Measure:  $\Pi$  AXIOMS?
- Total Risk:  $\Pi[(X - \rho(X))_+ + i\rho(X)]$
- Example:  $\Pi(\cdot) = E(\cdot)$

Total Risk:  $E[(X - \rho(X))_+ + i\rho(X)]$

Derived Measure:  $\rho(X) = F_X^{-1}(1 - i)$

# Reinsurance Principle

- Insurer  $(X - d)_+ \rightarrow$  Reinsurer and he retains  $X - (X - d)_+$
- If reinsurance price =  $\rho(X, d) \rightarrow \inf_d \Pi[X - (X - d)_+ + \rho(X, d)]$
- For a utility risk measure:

$$E[u(r(X) - X)] \leq E[u((X - d)_+ - X)] \text{ and } E[r(X)] = E[(X - d)_+]$$

$$\text{But: } E[\tilde{u}(\omega + \rho - r(X))] \geq E[\tilde{u}(\omega + \rho - (X - d)_+)]$$

## Bridge actuarial-financial pricing

- Axiom: expected utility  $u(x) = -\alpha \exp(-\alpha x)$   $\alpha > 0$
- Derived pricing principle

$$\sup_{\phi} -\alpha E[\exp(-\alpha E(\phi(X)X) - X)]$$

$$\text{Solution: } \phi(X) = \frac{e^{\alpha X}}{E(e^{\alpha X})}$$

$$\text{Remark: } \alpha = \frac{|\ln(\epsilon)|}{u}$$

## Incomplete information

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- $V(\rho, i, m) = \sup_{F \in G} \left( \int_a^b (x - \rho)_+ dF(x) + i\rho \right)$

whereby  $\int_a^b x dF(x) = c$  and  $\int_a^b dF(x) = 1$ ,

$$= \left( c - \frac{1}{2}(a + m) \right) \frac{(b - \rho)^2}{(b - m)(b - a)} + i\rho$$

then  $\rho^* = b - i \frac{(b - m)(b - a)}{2c - (a + m)}$ .

## Financial pricing

- Ordered Esscher-Girsanov transforms implies ordered prices. If the price measure is applied to normally distributed random variables, this axiom is equivalent to "respect for second-order stochastic dominance".
- The price measure is appropriately normalized such that the price of  $c$  non-random units is equal to  $c$  non-random units.
- Additivity for sums of Esscher-Girsanov transforms. If the price measure is applied to normally distributed random variables, the axiom is equivalent to "superadditivity and comonotonic additivity of the price measure".
- Topological conditions, which are necessary to establish the mathematical proofs.

## Esscher transform

- $dF_X^{(h)}(x) = \frac{e^{hx} dF_X(x)}{E[e^{hX}]}, \quad h \in \mathbb{R}.$
- $\psi_X(h) = \int_{-\infty}^{+\infty} x dF_X^{(h)}(x) = \frac{E[Xe^{hX}]}{E[e^{hX}]}.$
- **A1.** If  $\frac{E[Xe^{hX}]}{E[e^{hX}]} \leq \frac{E[Ye^{hY}]}{E[e^{hY}]}$  for all  $h \leq 0$ , then  $\pi[X] \leq \pi[Y]$ ;
- **A2.**  $\pi[c] = c$ , for all  $c$ ;
- **A3.**  $\pi[X + Y] = \pi[X] + \pi[Y]$  when  $X$  and  $Y$  are independent;
- **A4.** If  $X_n$  converges weakly to  $X$ , with  $\min[X_n] \rightarrow \min[X]$ , then  $\lim_{n \rightarrow +\infty} \pi[X_n] = \pi[X]$ .

## Esscher transform

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- A price measure  $\pi[\cdot]$  satisfies the set of axioms A1-A4 if and only if there exists some non-decreasing function

$$H : [-\infty, 0] \rightarrow [0, 1]$$

such that

$$\pi[X] = \int_{[-\infty, 0]} \psi_X(h) dH(h).$$

# The Esscher-Girsanov theorem

- $\phi_X(x) = \Phi^{-1}(F_X(x))$
- For the cdf  $F_X(\cdot)$  with differential  $dF_X(\cdot)$  corresponding to a given r.v.  $X$ , and a given real number  $v$ , we define by

$$dF_X^{(h,v)}(x) = \frac{e^{hv\phi_X(x)}}{E[e^{hv\phi_X(x)}]} dF_X(x) = e^{hv\phi_X(x) - \frac{1}{2}h^2v^2} dF_X(x), \quad h \in \mathbb{R},$$

its Esscher-Girsanov transform with parameters  $h$  and  $v$  (absolute risk aversion and penalty parameter, respectively).

- $\psi_X^v(h) = \int_{-\infty}^{+\infty} x dF_X^{(h,v)}(x) = E[X e^{hv\phi_X(x) - \frac{1}{2}h^2v^2}], \quad h \in \mathbb{R}.$

## Esscher-Girsanov price

- $\pi^v[X] = \rho^v[\psi_X^v]$ .
- **B1.** If  $\psi_X^v(h) \leq \psi_Y^v(h)$  for all  $h \leq 0$ , then  $\rho^v[\psi_X^v] \leq \rho^v[\psi_Y^v]$ ;
- **B2.**  $\rho^v[c] = c$ , for all  $c$ ;
- **B3.**  $\rho^v[\psi_X^v + \psi_Y^v] = \rho^v[\psi_X^v] + \rho^v[\psi_Y^v]$ ;
- **B4.** If  $\psi_{X_n}^v(h)$  converges to  $\psi_X^v(h)$  for all  $h \in [-\infty, 0]$ , then  $\lim_{n \rightarrow +\infty} \rho^v[\psi_{X_n}^v] = \rho^v[\psi_X^v]$ .
- If  $X$  and  $Y$  are two normally distributed r.v.'s with linear correlation coefficient  $\rho_{XY}$ , then

$$\psi_{X+Y}^v(h) = \mu_X + \mu_Y + hv\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}.$$

## Esscher-Girsanov price

- A functional  $\rho^v[\cdot]$  satisfies the set of axioms B1-B4 if and only if there exists some non-decreasing function  $H : [-\infty, 0] \rightarrow [0, 1]$  such that

$$\rho^v[\psi_X^v] = \int_{[-\infty, 0]} \psi^v(X)(h) dH(h).$$

- For the cdf  $F_{X_n}(\cdot)$  with differential  $dF_{X_n}(\cdot)$  corresponding to a given r.v.  $X_n$  and a given real-valued function  $v(\cdot)$ , we define by

$$\begin{aligned} dF_{X_n|X_0}^{(h,v(\cdot))}(x_n|x_0) &= \int_{x_{n-1}} \dots \int_{x_1} e^{h \sum_{j=0}^{n-1} v(x_j) \phi_{X_{j+1}|X_j}(x_{j+1}|x_j) - \frac{1}{2} h^2 v(x_j)^2} \\ &\times dF_{X_n|X_{n-1}}(x_n|x_{n-1}) \dots dF_{X_1|X_0}(x_1|x_0) \end{aligned}$$

its discrete Esscher-Girsanov transform with parameter  $h$  and penalty function  $v(\cdot)$ .

## Financial Derivative Pricing by Esscher-Girsanov Transforms

- $S_0 = s_0, \quad dS_t = \mu(S_t)dt + \sigma(S_t)dB_t$
- $Z_t^T(S, h, v(\cdot)) = h \int_t^T v(S_\tau)dB_\tau - \frac{1}{2}h^2 \int_t^T v(S_\tau)^2 d\tau,$
- $E_t[e^{Z_t^T(S, h, v(\cdot))}] = 1,$
- $\pi_t^{v(\cdot)}[g(S_T)] = \int_{[-\infty, 0]} E_t[e^{-\int_t^T r(S_\tau)d\tau} \phi^{(h)}(S_T, T) e^{Z_t^T(S, h, v(\cdot))}] dH(h) \quad (1).$
- The function  $\phi^{(h)}(\cdot, \cdot)$  will be chosen such that the calculation of the Feynman-Kac path integral on the right-hand side of (1) becomes feasible. Whatever function  $\phi^{(h)}(\cdot, \cdot)$  introduced, the right-hand side of (1) only depends on the terminal values  $\phi^{(h)}(S_T, T) = g(S_T).$

## Financial Derivative Pricing by Esscher-Girsanov Transforms

- Let  $\phi^{(h)}(S_T, T) = g(S_T)$ . Then

$$\begin{aligned}\pi_t^{v(\cdot)}[g(S_T)] &= \int_{[-\infty, 0]} E_t[e^{-\int_t^T r(S_\tau) d\tau} \phi^{(h)}(S_T, T) e^{Z_t^T(S, h, v(\cdot))}] dH(h) \\ &= \int_{[-\infty, 0]} \phi^{(h)}(S_t, t) dH(h),\end{aligned}$$

whenever  $\phi^{(h)}(\cdot, \cdot)$  is the solution of the PDE

$$\begin{aligned}&= \frac{\partial \phi^{(h)}(x, \tau)}{\partial \tau} + (\mu(x) + hv(x)\sigma(x)) \frac{\partial \phi^{(h)}(x, \tau)}{\partial x} \\ &= \frac{1}{2} \sigma(x)^2 \frac{\partial^2 \phi^{(h)}(x, \tau)}{\partial x^2} = r(x) \phi^{(h)}(x, \tau), \quad \tau \in [t, T].\end{aligned}$$

## Financial Derivative Pricing by Esscher-Girsanov Transforms

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- Suppose that  $S$  is a tradable asset. If  $v(x) = \frac{\mu(x) - r(x)x}{\sigma(x)}$  and

$$H(h) = \begin{cases} 1, & h \geq -1 \\ 0, & \text{otherwise,} \end{cases}$$

then  $\pi_t^{v(\cdot)}[g(S_T)]$  coincides with the approximate arbitrage-free price of the financial derivative  $g(S_T)$ .