Decision principles derived from risk measures

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Back to the future

- "WORST CASE RISK MEASUREMENT"  
  (1982) F. De Vylder  
  \[ v(c) = \sup_{\mu \in M} (\int f d\mu | \int g_i d\mu = c_i, i = 1, \ldots, n) \]

- "COHERENT RISK MEASURES"  
  (1981) P. Huber (Robust Statistics)  
  \[ X \leq Y \Rightarrow \rho(X) < \rho(Y) \]  
  \[ \rho(aX + b) = a\rho(X) + b \]  
  \[ \rho(X + Y) \leq \rho(X) + \rho(Y) \]

- "CONVEX RISK MEASURES"  
  (1985) H. Gerber and O. Deprez  
  (Convex principles of premium calculation)

- "DISTORTION RISK MEASURES"  
Risk Measures $\leftrightarrow$ Decision Principles

- **Risk Measure**: functional that assigns a real number to a random variable, by means of a set of axioms (top level)

  Axioms are fixed by economic actors or agents

- **Decision Principle**: idem but at a lower level

  ! A "DERIVED" FUNCTIONAL !

- "Markowitz": In one context one basic set of axioms is appropriate while in another context a different set is appropriate.

- E.g. Premium Principles, Capital Allocation, Solvency Capital Principle (regulatory, economic or rating capital)
Premium Principle

- Insurer \( E[u(\omega + \rho(X) - X)] \geq u(\omega) \)

- \( \tilde{u}(\tilde{\omega} - \rho(X)) \geq E[\tilde{u}(\tilde{\omega} - X)] \)

- \( \tilde{\rho}(X) < \rho(X) \leq \rho^+(X) \)

- Hierarchy between measures and principles:
  - Expected Utility = Risk Measure (units: arbitrary) (axioms)
  - Decision Principle = derived functional (≠ properties) (units: money) (properties, eventually derived axioms)
Solvency capital principle

- Risk: \((X - \rho(X))_+ + i\rho(X)\)

- Risk Measure: \(\Pi\) AXIOMS?

- Total Risk: \(\Pi[(X - \rho(X))_+ + i\rho(X)]\)

- Example: \(\Pi(.) = E(.)\)

Total Risk: \(E[(X - \rho(X))]_+ + i\rho(X)\)

Derived Measure: \(\rho(X) = F_X^{-1}(1 - i)\)
Reinsurance Principle

- Insurer \((X - d)_+ \rightarrow\) Reinsurer and he retains \(X - (X - d)_+\)

- If reinsurance price = \(\rho(X, d) \rightarrow \inf_d \Pi[X - (X - d)_+ + \rho(X, d)]\)

- For a utility risk measure:

\[
E[u(r(X) - X)] \leq E[u((X - d)_+ - X)] \text{ and } E[r(X)] = E[(X - d)_+]
\]

But:
\[
E[\tilde{u}(\omega + \rho - r(X))] \geq E[\tilde{u}(\omega + \rho - (X - d)_+)]
\]
Bridge actuarial-financial pricing

- Axiom: expected utility \( u(x) = -\alpha \exp(-\alpha x) \quad \alpha > 0 \)

- Derived pricing principle

\[
\sup_{\phi} -\alpha E[\exp(-\alpha E(\phi(X)X) - X)]
\]

Solution: \( \phi(X) = \frac{e^{\alpha X}}{E(e^{\alpha X})} \)

Remark: \( \alpha = \frac{|\ln(\epsilon)|}{u} \).
Incomplete information

\[ V(\rho, i, m) = \sup_{F \in G} \left( \int_a^b (x - \rho) + dF(x) + i\rho \right) \]

whereby

\[ \int_a^b x dF(x) = c \text{ and } \int_a^b dF(x) = 1, \]

\[ = \left( c - \frac{1}{2}(a + m) \right) \frac{(b-\rho)^2}{(b-m)(b-a)} + i\rho \]

then

\[ \rho^* = b - i \frac{(b-m)(b-a)}{2c-(a+m)}. \]
Financial pricing

- Ordered Esscher-Girsanov transforms implies ordered prices. If the price measure is applied to normally distributed random variables, this axiom is equivalent to "respect for second-order stochastic dominance".

- The price measure is appropriately normalized such that the price of \( c \) non-random units is equal to \( c \) non-random units.

- Additivity for sums of Esscher-Girsanov transforms. If the price measure is applied to normally distributed random variables, the axiom is equivalent to "superadditivity and comonotonic additivity of the price measure".

- Topological conditions, which are necessary to establish the mathematical proofs.
Esscher transform

- \( dF_X^{(h)}(x) = \frac{e^{hx}dF_X(x)}{E[e^{hX}]}, \quad h \in \mathbb{R}. \)

- \( \psi_X(h) = \int_{-\infty}^{+\infty} xdF_X^{(h)}(x) = \frac{E[Xe^{hX}]}{E[e^{hX}]} \).

- A1. If \( \frac{E[Xe^{hX}]}{E[e^{hX}]} \leq \frac{E[Ye^{hY}]}{E[e^{hY}]} \) for all \( h \leq 0 \), then \( \pi[X] \leq \pi[Y] \);

- A2. \( \pi[c] = c \), for all \( c \);

- A3. \( \pi[X + Y] = \pi[X] + \pi[Y] \) when \( X \) and \( Y \) are independent;

- A4. If \( X_n \) converges weakly to \( X \), with \( \min[X_n] \to \min[X] \), then \( \lim_{n \to +\infty} \pi[X_n] = \pi[X] \).
Esscher transform

• A price measure $\pi[.]$ satisfies the set of axioms A1-A4 if and only if there exists some non-decreasing function

\[ H : [-\infty, 0] \rightarrow [0, 1] \]

such that

\[ \pi[X] = \int_{[-\infty,0]} \psi_X(h) dH(h). \]
The Esscher-Girsanov theorem

- \( \phi_X(x) = \Phi^{-1}(F_X(x)) \)
- For the cdf \( F_X(.) \) with differential \( dF_X(.) \) corresponding to a given r.v. \( X \), and a given real number \( v \), we define by

\[
dF_{(h,v)}^{(h,v)}(x) = \frac{e^{hv\phi_X(x)}}{E[e^{hv\phi_X(x)}]} dF_X(x) = e^{hv\phi_X(x)} - \frac{1}{2} h^2 v^2 dF_X(x), \ h \in \mathbb{R},
\]

its Esscher-Girsanov transform with parameters \( h \) and \( v \) (absolute risk aversion and penalty parameter, respectively).

- \( \psi^v_X(h) = \int_{-\infty}^{+\infty} x dF_{(h,v)}^{(h,v)}(x) = E[X e^{hv\phi_X(x)} - \frac{1}{2} h^2 v^2], \ h \in \mathbb{R}. \)
Esscher-Girsanov price

- \( \pi^v[X] = \rho^v[\psi^v_X] \).
- B1. If \( \psi^v_X(h) \leq \psi^v_Y(h) \) for all \( h \leq 0 \), then \( \rho^v[\psi^v_X] \leq \rho^v[\psi^v_Y] \);
- B2. \( \rho^v[c] = c \), for all \( c \);
- B3. \( \rho^v[\psi^v_X + \psi^v_Y] = \rho^v[\psi^v_X] + \rho^v[\psi^v_Y] \);
- B4. If \( \psi^v_{X_n}(h) \) converges to \( \psi^v_X(h) \) for all \( h \in [-\infty, 0] \), then \( \lim_{n \to +\infty} \rho^v[\psi^v_{X_n}] = \rho^v[\psi^v_X] \).

- If \( X \) and \( Y \) are two normally distributed r.v.'s with linear correlation coefficient \( \rho_{XY} \), then

\[
\psi^v_{X+Y}(h) = \mu_X + \mu_Y + hv\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}.
\]
## Esscher-Girsanov price

- A functional $\rho^v[.]$ satisfies the set of axioms B1-B4 if and only if there exists some non-decreasing function $H : [\infty, 0] \rightarrow [0, 1]$ such that

$$\rho^v[\psi^v_X] = \int_{[\infty, 0]} \psi^v(X)(h)dH(h).$$

- For the cdf $F_{X_n}(.)$ with differential $dF_{X_n}(.)$ corresponding to a given r.v. $X_n$ and a given real-valued function $v(.)$, we define by

$$dF^{(h,v(.)|X_0|X_0)}_{X_n|X_0}(x_n|x_0) = \int_{x_{n-1}}^{x_n} \ldots \int_{x_1}^{x_n} e^{h \sum_{j=0}^{n-1} v(x_j)\phi_{X_{j+1}|X_{j}}(x_{j+1}|x_j)-\frac{1}{2}h^2v(x_j)^2} \times dF_{X_n|X_{n-1}}(x_n|x_{n-1}) \ldots dF_{X_1|X_0}(x_1|x_0)$$

its discrete Esscher-Girsanov transform with parameter $h$ and penalty function $v(.)$. 
Financial Derivative Pricing by Esscher-Girsanov Transforms

- \( S_0 = s_0, \quad dS_t = \mu(S_t)dt + \sigma(S_t)dB_t \)
- \( Z^T_t(S, h, v(\cdot)) = h \int_t^T v(S_\tau)dB_\tau - \frac{1}{2} h^2 \int_t^T v(S_\tau)^2d\tau, \)
- \( E_t[e^{Z^T_t(S, h, v(\cdot))}] = 1, \)
- \( \pi^v_t[\phi(S_T)] = \int_{-\infty}^0 E_t[e^{-\int_t^T r(S_\tau)d\tau} \phi(h)(S_T, T)e^{Z^T_t(S, h, v(\cdot))}]dH(h) \quad (1). \)
- The function \( \phi(h)(\cdot, \cdot) \) will be chosen such that the calculation of the Feynman-Kac path integral on the right-hand side of (1) becomes feasible. Whatever function \( \phi(h)(\cdot, \cdot) \) introduced, the right-hand side of (1) only depends on the terminal values \( \phi(h)(S_T, T) = g(S_T). \)
Financial Derivative Pricing by Esscher-Girsanov Transforms

Let $\phi_h(S_T, T) = g(S_T)$. Then

$$
\pi^v_t[g(S_T)] = \int_{[-\infty, 0]} E_t[e^{-\int_t^T r(S_{\tau}) d\tau} \phi_h(S_T, T) e^{Z^T_t(S, h, v(.))}] dH(h)
$$

$$
= \int_{[-\infty, 0]} \phi_h(S_t, t) dH(h),
$$

whenever $\phi_h(., .)$ is the solution of the PDE

$$
\begin{align*}
&= \frac{\partial \phi_h(x, \tau)}{\partial \tau} + (\mu(x) + hv(x)\sigma(x)) \frac{\partial \phi_h(x, \tau)}{\partial x} \\
&= \frac{1}{2} \sigma(x)^2 \frac{\partial^2 \phi_h(x, \tau)}{\partial x^2} = r(x) \phi_h(x, \tau), \quad \tau \in [t, T].
\end{align*}
$$
• Suppose that $S$ is a tradable asset. If $v(x) = \frac{\mu(x) - r(x)x}{\sigma(x)}$ and

$$H(h) = \begin{cases} 
1, & h \geq -1 \\
0, & \text{otherwise,}
\end{cases}$$

then $\pi^v_t[g(S_T)]$ coincides with the approximate arbitrage-free price of the financial derivative $g(S_T)$. 