Classification and Ordering of Portfolios and of New Insured Unities of Risks

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Introduction

The classical definitions of classification and ordering of risks.

- The **classification** of risks is used to group individual risks to which it must be applied the same premium.

The aim is the protection of the insurance system’s financial soundness.

- The **ordering** of risks is a comparison of risks belonging to two different classes.

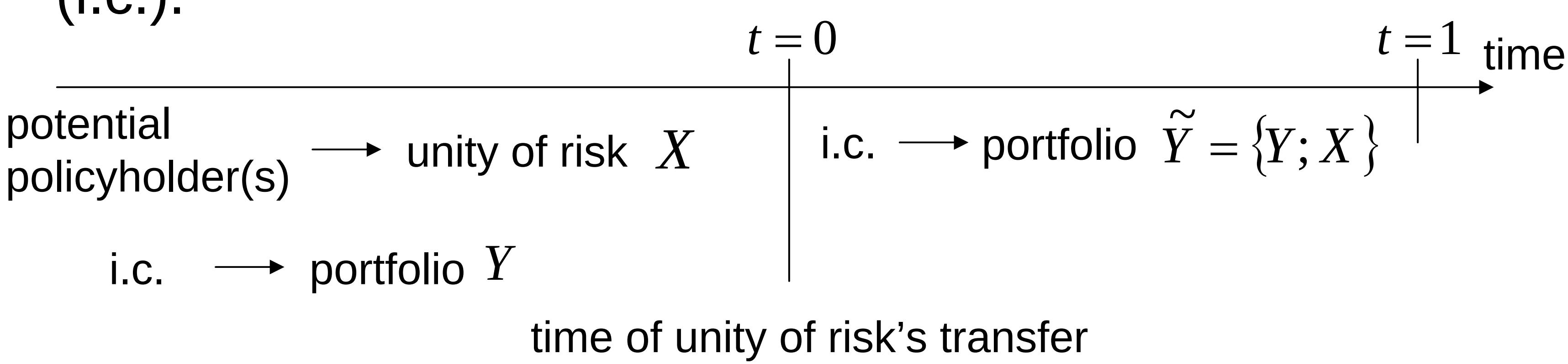
The aim is to establish to which risk it must be applied the greater premium.

- Both the classification and the ordering are based on the risks’ measures.

Introduction

The basic ideas of our model.

- The classification and the ordering are made after the risks are insured (the purpose is the outline of a reinsurance strategy).
- The classification and the ordering are based on the changes produced in the state of the business in the passage from a generic portfolio Y to the new portfolio $\tilde{Y}$ managed by a property-casualty insurance company (i.c.).



Introduction

Tools used in our model.

- The (*actual*) ***business*** of the i.c.:

Business={Portfolio, Operative structure}.

- The ***operative structure****:*

it is the set of the constraints and rules imposed on the portfolio's management by the insurance market and the regulatory authority and of the criteria adopted by the i.c.

- The ***state of the business****:*

it is described by using the loss exceedance probability (LEP) curve and some vectors defined as state variables.

- The ***state variables***' measures:

they are used to establish the ordering criteria.

Outline of the paper

- Portfolio of risks Y run by the i.c. at time $t = 0^-$.
- The description of the business $B(Y)$.
- The state of the business $B(Y)$.
- The graphic representation of the state of the business $B(Y)$.
- The classification of Y.
- The ordering of Y.
- The new unity of risk X introduced at time $t = 0$.
- The new portfolio $\tilde{Y}$.
- The state of the business $B(\tilde{Y})$.
- The graphic representation of $B(Y)$ and $B(\tilde{Y})$.
- The classification of X.
- The ordering of X.

Portfolio of risks Y run by the i.c. at time $t = 0^-$

Notations and assumptions.

- $Y = \{Y_1, \ldots, Y_l\}, \quad l >> 1, \quad Y_i$ independent components.
- The risk's analysis is over a time horizon of one year.
- The composition of Y does not change over $[0,1]$.
- The total claim amount in one year is represented by S_Y.
- F_{S_Y} is known and defined on $[0, M(S_Y)], \quad M(S_Y) \in (0, \infty)$.
- The i.c. operates out of a continuous time economic environment. In particular the model does not include taxes, commissions, investment incomes, dividend payout to shareholders, inflation.

The description of the business $B(Y)$

The ***business*** relative to portfolio Y :

$$B(Y) = \left(Y, \Theta(Y)\right).$$

- $\Theta(Y)$ is the ***operative structure*** relative to portfolio Y.
- Criteria for defining $\Theta(Y)$.

1. Insurance market's constraint.
2. Criteria adopted by the i.c.
3. Conditions imposed on the i.c. by the regulatory authority (r.a.).

The description of the business $B(Y)$

Independently of the principles used to assess the single pure premiums $P(Y_i)$, the premium income over one year can be expressed by

$$P(Y) = E[S_Y] + c(Y), \quad c(Y) = \eta(Y)E[S_Y].$$

1. Insurance market's constraint.

$$\eta(Y) \equiv \eta \in (0, M_\eta],$$

where $M_\eta > 0$ is the maximum of η in the competitive insurance maket (i.m.) during $[0,1]$.

The description of the business $B(Y)$

2. Criteria adopted by the i.c.

- The i.c. fixes $\varepsilon_0 \in (0,1)$ as the maximum acceptable ruin probability per year.
- The i.c. selects the premium calculation principles.
- Having expressed the free reserve proportional to the pure premium,

$$u(Y) = \alpha P(Y), \ \ \alpha \equiv \alpha(Y) > 0,$$

the i.c. fixes the maximum for α, $M_\alpha > 0$, lower than a value M_M linked to the i.m.

The description of the business $B(Y)$

3. Conditions imposed on the i.c by the r.a. (I).

- The i.c. must operate within the maximum acceptable ruin probability per year $\varepsilon^* \in (0, \varepsilon_0]$ to which corresponds the maximum acceptable loss $MAL^*(Y)$.
- The i.c. must have a minimum free reserve

$$u^* = \alpha^*(1+\eta^*)E[S_Y].$$

The ***minimum acceptable capital structure*** is

$$CS^*(Y) = u^* + (1+\eta^*)E[S_Y] = h^* E[S_Y],$$

where $h^* = (1+\alpha^*)(1+\eta^*)$ is the ***minimum acceptable actuarial capitalization factor***.

The description of the business $B(Y)$

3. Conditions imposed on the i.c by the r.a. (II).

- The only authorized state of the business is the ***acceptable state:***

$$\frac{CS(Y)}{CS^*(Y)} \geq 1, (\alpha, \eta) \in [m_\alpha, M_\alpha] \times [m_\eta, M_\eta].$$

As particular case, it includes the ***stable state:***

$$\frac{CS(Y)}{CS^*(Y)} \geq 1 + \varphi_0, (\alpha, \eta) \in [m_\alpha, M_\alpha] \times [m_\eta, M_\eta],$$

where $\varphi_0 > 0$ is independent on the portfolio and $CS(Y) = u(Y) + P(Y)$ is the capital structure of the i.c.

The description of the business $B(Y)$

- The acceptable and the stable states of the business are equivalent to a non negative and to a strictly positive **capacity**, respectively, having defined the capacity of the i.c. relative to the portfolio Y by

$$C_Y(\alpha,\eta) = CS(Y) - CS^*(Y),$$

where

$$(\alpha,\eta) \in [m_\alpha, M_\alpha] \times [m_\eta, M_\eta].$$

The description of the business $B(Y)$

- The ***operative structure*** $\Theta(Y)$ expresses all the previous constraints, criteria and rules and it can be represented by the following set

$$\Theta(Y) = \left\{\varepsilon^*, M_\alpha, M_\eta, \varphi_0; \alpha(Y), \eta(Y), h(Y), \varepsilon \in (0, \varepsilon^*]\right\}.$$

The graphic representation of the state of the business $B(Y)$

- To each d.f. F_{S_Y} corresponds a loss exceedance probability (LEP) curve.
- We trace the LEP relative to Y on the $(H, 0, \varepsilon)$-plane, where

$$H = S_Y / E[S_Y].$$

- We represent the state of the business $B(Y) = (Y, \Theta(Y))$ on the LEP curve through the points

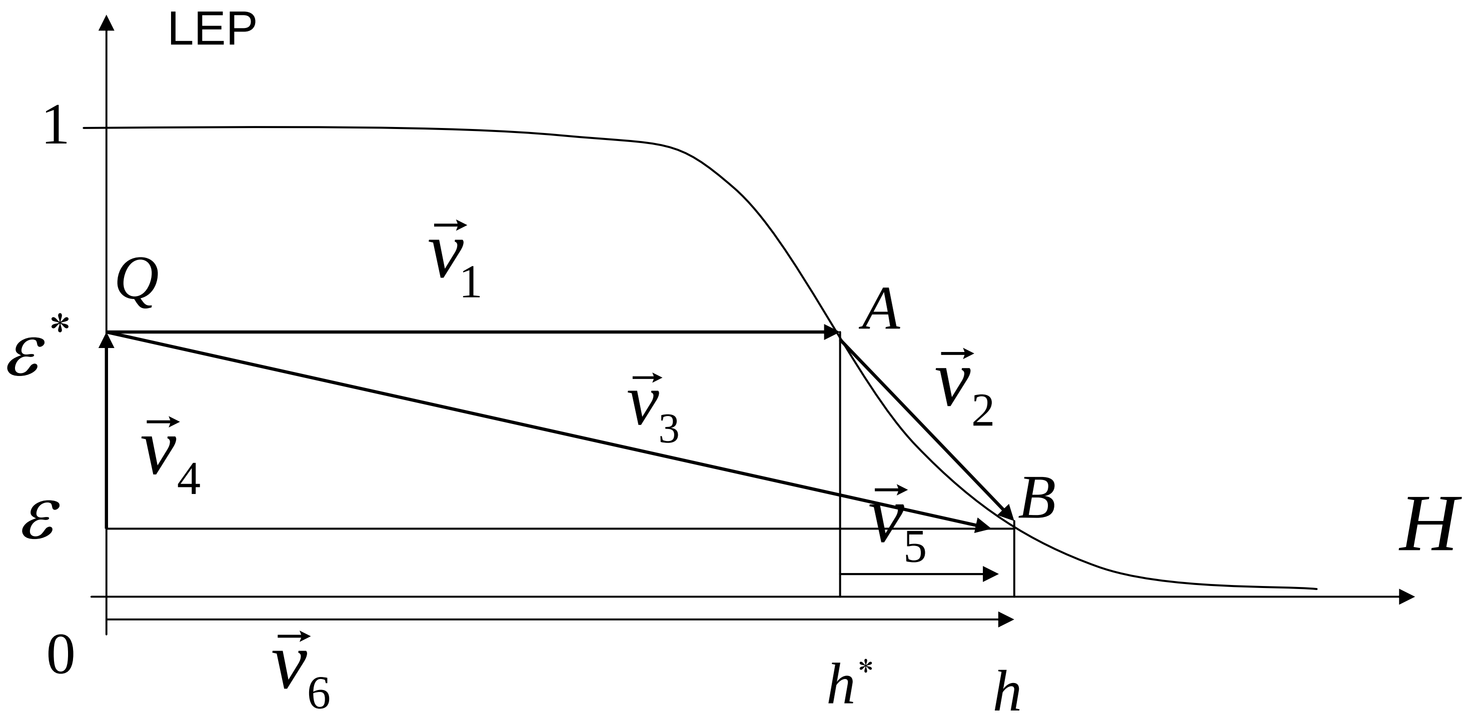
$$A = (h^*, \varepsilon^*), \qquad B = (h, \varepsilon), \qquad Q = (0, \varepsilon^*).$$

The graphic representation of the state of the business $B(Y)$

A, B, Q define on the $(H, 0, \varepsilon)$ -plane the ***state variables***, that is, the following six vectors:

$$
\begin{cases}
\vec{v}_1 = (h^*, 0), \\
\vec{v}_2 = (h - h^*, \varepsilon - \varepsilon^*), \\
\vec{v}_3 = (h, \varepsilon - \varepsilon^*), \\
\vec{v}_4 = (0, \varepsilon^* - \varepsilon), \\
\vec{v}_5 = (h - h^*, 0), \\
\vec{v}_6 = (h, 0).
\end{cases}
$$

The graphic representation of the state of the business $B(Y)$



The graphic representation of the state of the business $B(Y)$

- One of the three possible systems of basic state equations is

$$\begin{cases} \vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0, \\ v_2 + \vec{v}_4 - \vec{v}_5 = 0, \\ \vec{v}_3 + \vec{v}_4 - \vec{v}_6 = 0. \end{cases}$$

The classification of Y

- The classification is based on the state of the business and is graphically expressed only through the vector $\vec{v}_4$.

  Definition.

  The management of portfolio Y

1. is ***authorized*** if either $\operatorname{sgn}\vec{v}_4 > 0$ (the stable state) or $|\vec{v}_4| = 0$ (the acceptable state),
2. It is ***not authorized*** if $\operatorname{sgn}\vec{v}_4 < 0$.

The ordering of Y

- The ordering criteria $oc_i = oc_i(\Theta)$ for portfolio Y are defined by the measures $\rho(\vec{v}_i)$ of the vectors $\vec{v}_i,\ i = 1,\ldots,6.$
- The measures of the vectors are:

$$\rho(\vec{v}_i) = |\vec{v}_i|,\ i = 1,6,$$

$$\rho(\vec{v}_i) = \operatorname{sgn}\vec{v}_4 \cdot |\vec{v}_i|,\ i = 2,3,4,$$

if either $\operatorname{sgn}\vec{v}_4 > 0$ or $\operatorname{sgn}\vec{v}_4 < 0.$

$$\rho(\vec{v}_5) = \operatorname{sgn}\vec{v}_5 \cdot |\vec{v}_5|,$$

where $\operatorname{sgn}\vec{v}_5 > [<]\, 0$ when $h > [<]\, h^*.$

If $|\vec{v}_4| = 0,$ then $\vec{v}_1 = \vec{v}_3 = \vec{v}_6,\quad \vec{v}_2 = \vec{v}_5 = 0.$

The ordering of Y

- Let Y_1, Y_2 be two portfolios whose states of the business are defined by the vectors $\vec{v}_i(Y_1), \vec{v}_i(Y_2), i = 1, \ldots, 6.$

  Definition. Y_1 precedes Y_2 in the $oc_i(\Theta(\cdot))$ - order, i.e. $Y_1 \leq_{oc_i(\Theta(\cdot))} Y_2$,

  iff

$$\rho(\vec{v}_i(Y_1)) \leq \rho(\vec{v}_i(Y_2)), i = 1, \ldots, 6.$$

  Y_1 precedes Y_2 in the $\Theta(\cdot)$ -order, i.e. $Y_1 \leq_{\Theta(\cdot)} Y_2$,

  iff

$$\rho(\vec{v}_i(Y_1)) \leq \rho(\vec{v}_i(Y_2)), \quad \forall\, i = 1, \ldots, 6.$$

The new unity of risk X introduced at time $t = 0$

Notations and assumptions.

- $X = \{X_1, \ldots, X_m\},\ m \geq 1,\ X_h$ dependent components.
- S_X is the total claim amount of X per year.
- F_{S_X} is known and defined on

$$[0, M(S_X)],\ M(S_X) > 0.$$

The new portfolio $\tilde{Y}$

- The i.c. runs the portfolio $\tilde{Y} = \{Y; X\}$ in $[0,1]$.
- The state of the business is

$$B(\tilde{Y}) = (\tilde{Y}, \tilde{\Theta}), \quad \tilde{\Theta} = \Theta(\tilde{Y}).$$

- The operative structure relative to portfolio $\tilde{Y}$ is represented by

$$\tilde{\Theta} = \{\varepsilon^*, M_\alpha, \tilde{M}_\eta, \varphi_0; \alpha(\tilde{Y}), \eta(\tilde{Y}), h(\tilde{Y}), \tilde{\varepsilon} \in (0, \varepsilon^*]\}.$$

The state of the business $B(\tilde{Y})$

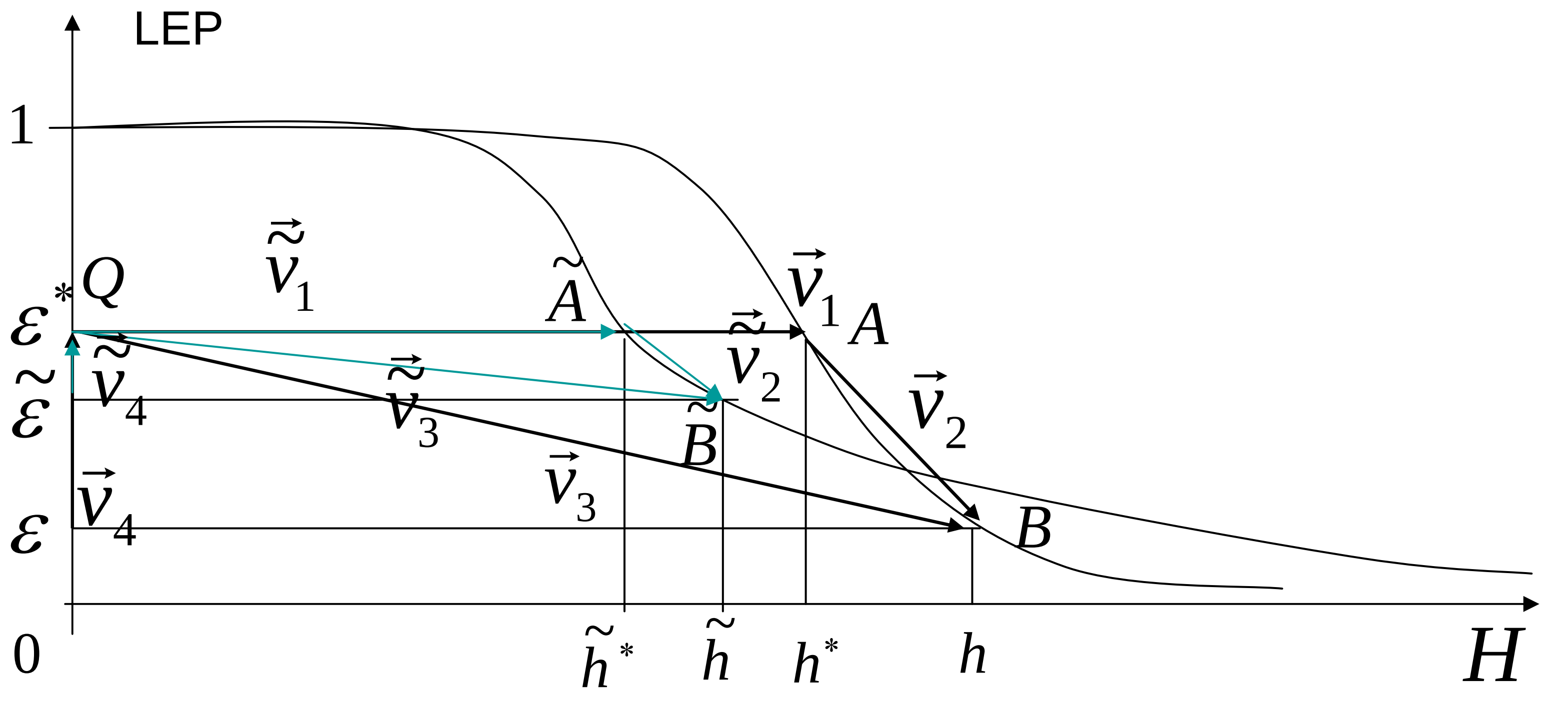
- The state of the business $B(\tilde{Y})$ is represented on the $(H,0,\varepsilon)$- plane through the points

$$\tilde{A} = \left(\tilde{h}^*, \varepsilon^*\right), \tilde{B} = \left(\tilde{h}, \tilde{\varepsilon}\right), Q = \left(0, \varepsilon^*\right),$$

by which we define the new state variables

$$\begin{cases} \tilde{\vec{v}}_1 = \vec{v}_1 + \overrightarrow{A\tilde{A}}, & \tilde{\vec{v}}_2 = \vec{v}_2 + \overrightarrow{BF} + \overrightarrow{BG} - \overrightarrow{A\tilde{A}}, \\ \tilde{\vec{v}}_3 = \vec{v}_3 + \overrightarrow{BF} + \overrightarrow{BG}, & \tilde{\vec{v}}_4 = \vec{v}_4 - \overrightarrow{BG}, \\ \tilde{\vec{v}}_5 = \vec{v}_5 + \overrightarrow{BF} - \overrightarrow{A\tilde{A}}, & \tilde{\vec{v}}_6 = \vec{v}_6 + \overrightarrow{BF}. \end{cases}$$

The graphic representation of $B(Y)$ and $B(\tilde{Y})$



The classification of X

Assumption. The state of the business $B(Y)$ is authorized and X is completely retained by the i.c.

- The classes and sub-classes of X are

normal risk	capacity generator risk
	capacity risk
dangerous risk	great risk
	catastrophic risk
	mega-catastrophic risk

The classification of X

Definition.

- X is a normal risk iff $B(\tilde{Y})$ is authorized, i.e., $\tilde{h} \geq \tilde{h}^*$.
- X is a dangerous risk iff $\tilde{h} < \tilde{h}^*$.

Definition. Let X be a normal risk.

- X is a capacity generator risk iff

$$\tilde{h} > \tilde{h}^* + \tau(h - h^*),$$

- X is capacity risk iff

$$\tilde{h}^* \leq \tilde{h} < \tilde{h}^* + \tau(h - h^*).$$

where

$$\tau = E[S_Y]/E[S_{\tilde{Y}}].$$

The classification of X

Definition. Let X be a dangerous risk.

- X is a great risk iff

$$\tilde{h} < \tilde{h}^* \le h(M_\alpha, \tilde{M}_\eta).$$

- X is a catastrophic risk iff

$$h(M_\alpha, \tilde{M}_\eta) < \tilde{h}^* \le h(M_M, \tilde{M}_\eta).$$

- X is a mega-catastrophic risk iff

$$h(M_M, \tilde{M}_\eta) < \tilde{h}^*.$$

The ordering of X

- Let $B(Y)$ be a state of the business.
- Let X_1, X_2 be two unities of risk and let $\tilde{Y}_1 = \{Y; X_1\}$, $\tilde{Y}_2 = \{Y; X_2\}$ be the corresponding new portfolios.

Definition. X_1 precedes X_2 in the $oc_i(B(Y))$ - order, i.e. $X_1 \leq_{oc_i(B(Y))} X_2$,

iff

$$\tilde{Y}_1 \leq_{oc_i(\Theta(\cdot))} \tilde{Y}_2, \quad i = 1, \ldots, 6.$$

X_1 precedes X_2 in the $B(Y)$ -order, i.e. $X_1 \leq_{B(Y)} X_2$,

iff

$$\rho(\vec{v}_i(\tilde{Y}_1)) \leq \rho(\vec{v}_i(\tilde{Y}_2)), \quad \forall\, i = 1, \ldots, 6.$$

The ordering of X

Proposition. If $\tilde{Y}_1 \leq_{oc_i(\Theta(\cdot))} \tilde{Y}_2, \quad i = 1,4,5,6,$
then

$$\rho(\vec{v}_i(\tilde{Y}_1)) \leq \rho(\vec{v}_i(\tilde{Y}_2)), \quad \forall\, i = 1,\ldots,6,$$

that is,

$$X_1 \leq_{B(Y)} X_2.$$