

EM Algorithm for Bivariate Phase Distributions

Karl-Theodor Eisele

Phase distributions

- **are nice to handle.**
- **allow for explicit formulas, e.g. densities, higher moments , ruin probabilities.**
- **are closed under convex combinations, convolutions.**
- **Compound variables are phase variables, if the counting and the size variables are phase variables.**
- **allow dependent multivariate distributions.**
- **Matrix exponentials are nowadays easy to calculate.**
- **Simulations of Phase variables can be done directly by using the underlying homogeneous Markov process.**

Phase distributions

- are dense in the class of distributions on $[0, \infty)$ in the weak topology

however,

- have exponentially decreasing tail distributions; they can not be used for large- or extreme-value problems.

See e.g.:

- Encyclopedia of Actuarial Science, 2004, “Phase type distributions” and “Phase method”
- Rolski et al.: Stochastic Processes for Insurance and Finance, 2000.
- Hipp, Ch.: Lectures in Strasbourg, 2000-2005

Definitions (continuous case)

Let $X(t)$ be a homogeneous Markov chain with finite state space $D' = D \cup \{0\}$,

$$D = \{ 1, \dots, d \},$$

having $0 \in D$ as an absorbing and attracting state . The 'life time'

$$T = \inf \{ t \geq 0, X(t) \notin D \}$$

is a phase variable.

We call the complement C of an absorbing subset of D' inaccessible (e.g. D).

If (C_1, \dots, C_m) are inaccessible subsets, then (T_1, \dots, T_m) with

$$T_v = \inf \{ t \geq 0, X(t) \notin C_v \}$$

is a multivariate phase variable.

Common distributions

Let $\tilde{L} = (l_{jk})_{0 \leq j, k \leq d}$ be the stationary generator of $X(t)$ where $l_{jk} \geq 0$ is the intensity of the transition from state j to k , $j \neq k$,

$$l_{jj} = - \sum_{j \neq k} l_{jk} ,$$

$L = (l_{jk})_{1 \leq j, k \leq d}$ the submatrix of \tilde{L} and $\tilde{\pi} = (\pi_0, \pi_1, \dots, \pi_d) = (\pi_0, \pi)$ be the starting distribution. Then

Theorem 1. For $0 \leq \mu < m$ and $0 \leq t_1 \leq \dots \leq t_{m-\mu}$. we get

$$\begin{aligned} & \mathbb{P}(T_1 = \dots = T_\mu = 0 \text{ and } T_{\mu+\nu} > t_\nu \text{ for } \nu = 1, \dots, m - \mu) \\ &= \pi \Delta^{A(\mu)} e^{Lt_1} \Delta^{C_{\mu+1}} e^{L(t_2-t_1)} \Delta^{C_{\mu+2}} \dots e^{L(t_m-t_{m-1})} \cdot \Delta^{C_m} \eta \end{aligned}$$

where

$$\begin{aligned} \eta &= (1, \dots, 1) \in \mathbb{R}^d \\ \Delta_{pq}^C &= \begin{cases} 1 & \text{if } p = q \in C \\ 0 & \text{else} \end{cases} \quad \text{and} \\ A(\mu) &= D \setminus \left(\bigcup_{\nu=1}^{\mu} C_\nu \right) . \end{aligned}$$

Corollary 2. For $m = 2$, $\mathcal{C} = \{C_1, C_2\}$ and $\nu = 1, 2$, we have

(i) for $0 < t_1 < t_2$

$$\mathbb{P}(T_\nu > t_1, T_{3-\nu} > t_2) = \pi \cdot e^{L \cdot t_1} \cdot \Delta^{C_\nu} \cdot e^{L \cdot (t_2 - t_1)} \cdot \Delta^{C_{3-\nu}} \cdot \eta,$$

(ii) for $0 < t$

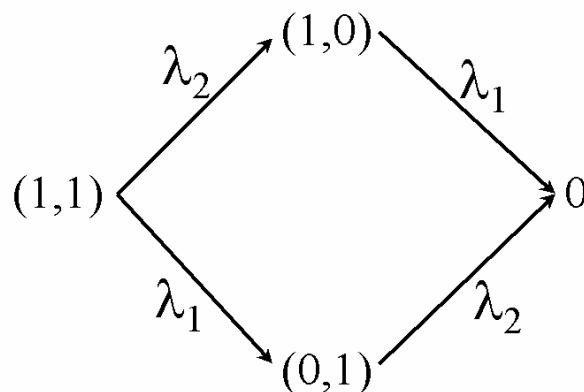
$$\mathbb{P}(T_1 = T_2 \in dt) = -\pi \cdot e^{L \cdot t} \cdot \Delta^{C_1 \cap C_2} \cdot L \cdot \eta \cdot dt.$$

Example

Let Z_1 and Z_2 be two independent exponential variables with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, set

$$T_1 = Z_1 \text{ and } T_2 = \max(Z_1, Z_2) .$$

(T_1, T_2) are two phase variables with $D' = \{0, (1, 1), (1, 0), (0, 1)\}$ and transition intensities



$C_1 = \{(1, 1), (1, 0)\}$ and $C_2 = D = \{(1, 1), (1, 0), (0, 1)\}$ are inaccessible subsets of D' .

$$L = \begin{pmatrix} -\lambda_1 - \lambda_2 & \lambda_2 & \lambda_1 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_2 \end{pmatrix}$$

and

$$\mathbb{P}(T_1 > t_1, T_2 > t_2) = \begin{cases} e^{-\lambda_1 t_2} + e^{-\lambda_2 t_2} (e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2}) & \text{if } 0 < t_1 < t_2 \\ e^{-\lambda_1 t_1} & \text{if } 0 < t_2 < t_1 \end{cases}$$

with marginal distributions

$$F_1(t_1) = \mathbb{P}(T_1 \leq t_1) = 1 - e^{-\lambda_1 t_1} \text{ and}$$

$$F_2(t_2) = \mathbb{P}(T_2 \leq t_2) = (1 - e^{-\lambda_1 t_2}) \cdot (1 - e^{-\lambda_2 t_2}) .$$

In the special case $\lambda_1 = \lambda_2$, the copula has the explicit form

$$C(u, v) = (u \cdot v^{1/2}) \wedge v .$$

EM algorithm for bivariate phase variables (continuous case)

The EM algorithm is based on the entropy inequality

$$g_{\chi_1}(T) \geq g_{\chi_0}(T)$$

for the marginal density

$$g_{\chi}(T) = \int f_{\chi}(\xi) \cdot \mu(d\xi | T(\xi) = T)$$

of a random variable ξ whose density $f_{\chi}(\xi)$ with respect to some σ -finite measure $\mu(d\xi)$ depends on some parameter set χ , and where

$$\chi_1 = \arg \max_{\chi} \mathbb{E}_{\chi_0} (\log f_{\chi}(\xi) | T(\xi) = T) .$$

Let $(D' = \{0, \dots, d\}, \mathcal{C} = \{C_1, C_2\}, \tilde{L}, \tilde{\pi},)$.

be the characteristics of the process $X(t)$ where C_ν are inaccessible sets of D , kept fixed in the sequel, and the parameter set $\chi = (L, \pi)$. A trajectory is given by $\xi = (j_1, t_1, j_2, t_2, \dots, j_m, t_m)$, where j_1 is the initial state, t_1 the time $X(t)$ spends in j_1 , then $X(t)$ jumps to j_2 and stays there a time t_2 , and so on.

The density of such a trajectory is

$$f_\chi(\xi) = \pi_{j_1} \cdot (-l_{j_1 j_1}) \cdot \exp(t_1 l_{j_1 j_1}) \cdot p(j_1, j_2) \cdot (-l_{j_2 j_2}) \cdot \exp(t_2 l_{j_2 j_2}) \cdot \dots \cdot (-l_{j_m j_m}) \cdot \exp(t_m l_{j_m j_m}) \cdot p(j_m, j_{m+1}) .$$

where $j_{m+1} = 0$ and for $j \neq k$, $p(j, k) = l_{jk} / (-l_{jj})$ is the conditional probability of a jump from j to k , given that a jump takes place.

The log-density function is

$$\begin{aligned} \ln(f_{\chi}(\xi)) &= \ln(\pi_{j_1}) + \sum_{\nu=1}^m t_{\nu} \cdot l_{j_{\nu}j_{\nu}} + \sum_{\nu=1}^m \ln(l_{j_{\nu}j_{\nu+1}}) \\ &= \sum_{j=1}^d \ln \pi_j \cdot N_j + \sum_{j=1}^d l_{jj} \cdot Z_j + \sum_{j=1}^d \sum_{k=0 (!), k \neq j}^d \ln l_{jk} \cdot N_{jk} \end{aligned}$$

with $N_j = \mathbf{1}_{\{j\}}(j_1)$

$N_{jk} = \#\{(j_{\nu}, j_{\nu+1}) = (j, k), \text{ avec } \nu \leq m\}$, number of jumps from j to k

$Z_j = \sum_{\nu=1}^m t_{\nu} \cdot \mathbf{1}_{\{j\}}(j_{\nu})$, the local time of j .

For observations $T_1 = (T_{11}, T_{12}), \dots, T_n = (T_{n1}, T_{n2})$, the maximization problem is

$$\max_{\chi=(L,\pi)} \mathbb{E}_{\chi_0} \left(\sum_{j=1}^d \ln \pi_j \cdot N_j + \sum_{j=1}^d l_{jj} \cdot Z_j + \sum_{j=1}^d \sum_{k=0 (!), k \neq j}^d \ln l_{jk} \cdot N_{jk} \middle| T_1, \dots, T_n \right).$$

Starting with some initial parameters $\chi^{(0)} = (L^{(0)}, \pi^{(0)})$, the maximization problem boils down to the recursion procedure

$$\pi_j^{(\nu+1)} = \pi_j^{(\nu)} \cdot \frac{1 - \pi_0}{n} \sum_{i=1}^n \mathbb{E}_{(\pi^{(\nu)}, L^{(\nu)})} (N_j | (T_i)) \text{ for } 1 \leq j \leq d \text{ and}$$

$$l_{jk}^{(\nu+1)} = \frac{\sum_{i=1}^n \mathbb{E}_{(\pi^{(\nu)}, L^{(\nu)})} (N_{jk} | (T_{i1}, T_{i2}))}{\sum_{i=1}^n \mathbb{E}_{(\pi^{(\nu)}, L^{(\nu)})} (Z_j | (T_i))},$$

$$l_{j0}^{(\nu+1)} = \frac{\sum_{i=1}^n \mathbb{E}_{(\pi^{(\nu)}, L^{(\nu)})} (N_{j0} | (T_i))}{\sum_{i=1}^n \mathbb{E}_{(\pi^{(\nu)}, L^{(\nu)})} (Z_j | (T_i))},$$

$$l_{jj}^{(\nu+1)} = - \sum_{k=0, k \neq j}^d l_{jk}^{(\nu)}.$$

In the case $0 < T_\nu < T_{3-\nu}$ we get

$$\mathbb{E}_\chi (N_j | (T_1, T_2)) = \pi_j \begin{cases} \frac{\eta_j \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu) \cdot L \cdot \eta}}{\pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu) \cdot L \cdot \eta}} & \text{if } j \in C_1 \cap C_2 \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}_\chi (N_{jk} | (T_1, T_2)) = l_{jk} \begin{cases} \frac{\int_0^{T_\nu} \pi \cdot e^{L \cdot u} \cdot \Delta(j, k) \cdot e^{L \cdot (T_\nu - u)} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu) \cdot L \cdot \eta} \cdot du}{\pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu) \cdot L \cdot \eta}} & \text{if } j, k \in C_1 \cap C_2 \\ \frac{\pi \cdot e^{L \cdot T_\nu} \cdot \Delta(j, k) \cdot e^{L \cdot (T_{3-\nu} - T_\nu) \cdot L \cdot \eta}}{\pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu) \cdot L \cdot \eta}} & \text{if } j \in C_\nu, k \in C_{3-\nu} \setminus C_\nu \\ \frac{\int_{T_\nu}^{T_{3-\nu}} \pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (u - T_\nu)} \cdot \Delta(j, k) \cdot e^{L \cdot (T_\nu - 3 - u) \cdot L \cdot \eta} \cdot du}{\pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu) \cdot L \cdot \eta}} & \text{if } j, k \in C_{3-\nu} \setminus C_\nu \\ 0 & \text{else} \end{cases}$$

and

$$\mathbb{E}_X(N_{j0} | (T_1, T_2)) = I_{j0} \begin{cases} \frac{\pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu)} \cdot L \cdot \eta_j}{\pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu)} \cdot L \cdot \eta} & \text{if } j \in C_{3-\nu} \setminus C_\nu \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}_X(Z_j | (T_1, T_2)) = \begin{cases} \frac{\int_0^{T_\nu} \pi \cdot e^{L \cdot u} \cdot \Delta(j, j) \cdot e^{L \cdot (T_\nu - u)} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu)} \cdot L \cdot \eta \cdot du}{\pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu)} \cdot L \cdot \eta} & \text{if } j \in C_1 \cap C_2 \\ \frac{\int_{T_\nu}^{T_{3-\nu}} \pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (u - T_\nu)} \cdot \Delta(j, j) \cdot e^{L \cdot (T_\nu - 3 - u)} \cdot L \cdot \eta \cdot du}{\pi \cdot e^{L \cdot T_\nu} \cdot (L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L) \cdot e^{L \cdot (T_{3-\nu} - T_\nu)} \cdot L \cdot \eta} & \text{if } j \in C_{3-\nu} \setminus C_\nu \\ 0 & \text{else} \end{cases}$$