Optimal Investment Strategy for a Non-Life Insurance Company

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Probability space

$$\left(\Omega, \Im, \mathbf{P}\right) \qquad \mathbf{F} = \left(\Im_t\right)_{0 \le t \le T} \qquad \Im_T = \Im$$

The filtration satisfies the usual hypotheses of

- completeness
- right continuity

Insurance risk process

- Collective risk model

$$J(t) = \sum_{i=1}^{N(t)} Y_i$$

- $N(t)$ Poisson process with intensity λ
- $\{Y_i, i \in \mathbf{N}\}$ sequence of positive iid random variables

$$\int_0^\infty y^4 p(dy) < \infty \quad \mathbf{E}Y_i = \mu$$

- The process $J(t)$ is $\mathbf{F}$-adapted, RCLL

Insurance risk process

- Poisson stochastic integral

$$J(t) = \int_0^t \int_0^\infty yM(ds, dy)$$

$$M(t, A) = \#\left\{0 \le s \le t, \Delta J(t) \in A\right\}$$

$$\Delta J(t) = J(t) - J(t-)$$

- $M(t, A)$ random variable, Poisson distributed with $\lambda tp(A)$
- The proces $\widetilde{M}(t, A) = M(t, A) - \lambda tp(A)$ is a martingale

Financial market

- Risk-free asset

$$\frac{dB(t)}{B(t)} = rdt\ ,\ B(0) = 1$$

- Risky assets $i = 1, 2, .., n$

$$\frac{dS_i(t)}{S_i(t)} = a_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t),\ \ S_i(0) = s_i > 0$$

$\mathbf{W}(t) = \left(W_1(t), \ldots, W_n(t)\right)^{\mathrm{T}}$ standard Brownian motion,

$\mathbf{F}$ -adapted

Insurer's wealth process

θ_i fraction of available wealth invested in the risky asset i

θ_0 fraction of available wealth invested in the risk-free asset

$$dX(t) = \sum_{i=1}^{n} \theta_i(t) X(t-) \frac{dS_i(t)}{S_i(t)} + \left(1 - \sum_{i=1}^{n} \theta_i(t)\right) X(t-) \frac{dB(t)}{B(t)} - dJ(t)$$

$$dX(t) = X(t-)\theta(t)^{\mathrm{T}} \pi dt + X(t-)\theta(t)^{\mathrm{T}} \Sigma d\mathbf{W}(t) + \\ + X(t-) r dt - dJ(t), \qquad X(0) = x_0$$

Admissible strategies

$\{\theta_1(t), ..., \theta_n(t), 0 < t \le T\}$ predictable process with respect to filtration $\mathbf{F}$

$$\mathbf{P}\left(\int_0^T \theta_i(t)^2 X(t-)^2 \, dt < \infty\right) = 1, \qquad i = 1, ..., n$$

$$\mathbf{P}\left(\int_0^T \theta_0(t) X(t-) \, dt < \infty\right) = 1$$

The process $\{X(t), 0 \le t \le T\}$ is an $\mathbf{F}$-adapted semimartingale, RCLL

Insurer's wealth process

Levy-type stochastic integral

$$dX(t) = \left(X(t-)\theta(t)^{\mathrm{T}}\pi + X(t-)r - \int_{0<y<1} y\lambda p(dy) \right) dt +$$

$$+ X(t-)\theta(t)^{\mathrm{T}}\Sigma d\mathbf{W}(t) - \int_{y\geq 1} yM(dt, dy) +$$

$$- \int_{0<y<1} y\tilde{M}(dt, dy), \quad X(0) = x_0$$

Reserve

Insurer's risk profile

$$R(t) = \mathbf{E}\left[\int_t^T e^{-\hat{\delta}(s-t)} d\hat{J}(t) \,\middle|\, \mathfrak{I}_t\right], \quad 0 \le t < T$$

$$\hat{J}(t) = \sum_{i=1}^{\hat{N}(t)} \hat{Y}_i$$

$$R(t) = \frac{\hat{\mu}\hat{\lambda}}{\hat{\delta}}\left(1 - e^{-\hat{\delta}(T-t)}\right), \quad \hat{\mu} \ge \mu, \ \hat{\lambda} \ge \lambda, \ \hat{\delta} \le r$$

Wealth path dependent disutility optimization

Find an investment strategy which minimizes the quadratic loss function

$$\mathbf{E}\left[\int_0^T \left\{ \left(R(s) - X(s) \right)^2 + \alpha \left(R(s) - X(s) \right) \right\} ds + \right.$$

$$\left. + \beta \left(X(T)^2 - \alpha X(T) \right) \right]$$

Dynamic Programming Principle

$$V(t,x) = \inf_{\theta(\cdot)\in\mathbf{R}^n} \mathbf{E}[\int_t^T \left\{\left(R(s) - X(s)\right)^2 + \alpha\left(R(s) - X(s)\right)\right\}ds +$$

$$+ \beta\left(X(T)^2 - \alpha X(T)\right) \Big| X(t) = x], \qquad 0 \le t < T$$

$$V(t-,x) = \inf_{\theta(t)\in\mathbf{R}^n} \{ \left(R(t-) - x\right)^2 dt + \alpha\left(R(t-) - x\right)dt +$$

$$+ \mathbf{E}\left[V(t, X(t)) \Big| X(t-) = x\right]\}$$

Ito's formula

$$dV(t, X(t)) = \frac{\partial V}{\partial t}(t, X(t-))dt + \frac{\partial V}{\partial x}(t, X(t-))\left\{ X(t-)\theta(t)^{\mathrm{T}}\pi + X(t-)r \right\}dt +$$

$$+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t, X(t-))X(t-)^2\theta(t)^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\theta(t)dt + \frac{\partial V}{\partial x}(t, X(t-))X(t-)\theta(t)^{\mathrm{T}}\Sigma d\mathbf{W}(t) +$$

$$+\int_0^\infty \left\{ V(t, X(t-) - y) - V(t, X(t-)) \right\} M(dt, dy)$$

If there exists a function $V(t,x)\in C^{1,2}([0,T],\mathbf{R})$ satisfying Hamilton-Jacobi-Bellman equation

$$0=\left(R(t)-x\right)^2+\alpha\left(R(t)-x\right)+V_t+V_x xr+\int_0^\infty\left\{V(t,x-y)-V(t,x)\right\}\lambda p(dy)+$$
$$+\inf_{\theta\in\mathbf{R}^n}\left[V_x x\theta^{\mathrm{T}}\pi+\frac{1}{2}V_{xx}x^2\theta^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\theta\right]$$

with the boundary condition $V(T,x)=\beta\left(x^2-\alpha x\right)$, such that the processes

$$\int_0^t\int_0^\infty\left\{V(s,X(s-)-y)-V(s,X(s-))\right\}\tilde{M}(ds,dy)$$

$$\int_0^t\frac{\partial V}{\partial x}(s,X(s-))X(s-)\theta_i(s)dW_j(s),\qquad i,j=1,\ldots,n$$

are martingales, and there exists an admissible control $\theta^*(\cdot)$ for which the infimum is reached, then

$$V(t,x)=\inf_{\theta(\cdot)\in\mathbf{R}^n}\mathbf{E}\left[\int_t^T\left\{\left(R(s)-X(s)\right)^2+\alpha\left(R(s)-X(s)\right)\right\}ds+\beta\left(X(T)^2-\alpha X(T)\right)\middle|\,X(t)=x\right]$$

and $\theta^*(\cdot)$ is the optimal control for the problem.

Optimal investment strategy

$$\theta^*(t) = \left(g(t) - X^*(t-)\right)\frac{1}{X^*(t-)}\left(\Sigma\Sigma^{\mathrm{T}}\right)^{-1}\pi$$

$$g(t) = -\frac{b(t)}{2a(t)}$$

$$1 + a'(t) + \phi a(t) = 0, \qquad a(T) = \beta$$

$$-2R(t) - \alpha - 2\mu\lambda\, a(t) + b'(t) + \varphi b(t) = 0, \; b(T) = -\alpha\beta$$

$$\phi = 2r - \pi^{\mathrm{T}}\left(\Sigma\Sigma^{\mathrm{T}}\right)^{-1}\pi$$

$$\varphi = r - \pi^{\mathrm{T}}\left(\Sigma\Sigma^{\mathrm{T}}\right)^{-1}\pi$$

Insurer's wealth under the strategy

$$X^*(t) = g(t) - \frac{1}{Z(t)}\{ g(0) - x_0 + \int_0^t Z(s)\left(g'(s) - g(s)r\right)ds +$$

$$+ \int_0^t \int_0^\infty Z(s) y M(ds, dy) \}$$

$$Z(t) = \exp\left\{ -\left(r - \pi^T \left(\Sigma\Sigma^T\right)^{-1} \pi\right)t + \frac{1}{2}\left\|\Sigma^{-1}\pi\right\|^2 t + \left(\Sigma^{-1}\pi\right)^T \mathbf{W}(t) \right\}$$

$$m'(t) = \left(g(t) - m(t)\right)\pi^T \left(\Sigma\Sigma^T\right)^{-1} \pi + m(t)r - \lambda\mu, \quad m(0) = x_0$$

Optimal investment strategy

- Short-selling the asset

$$\theta^*(t) < 0 \Leftrightarrow X^*(t-) < 0 \vee X^*(t-) > g(t)$$

- Borrownig from a bank account

$$\theta^*(t) > 1 \Leftrightarrow 0 < X^*(t-) < \frac{g(t)}{1 + \frac{\sigma^2}{a - r}}$$

Optimal investment strategy

- the higher the reserve, the higher the fraction of the wealth invested in the risky asset (given the same positive level of available wealth) and the higher the expected value of the insurer's wealth
- the higher the value of alpha, the higher the fraction of the wealth invested in the risky asset (given the same positive level of available wealth) and the higher the expected value of the insurer's wealth

Simulation study

- One year policy, discretization one week
- Insurance risk process: $\lambda = 100$, Gamma distribution, expected value 100 and variance 5000
- Financial market $r = 4\%, a = 10\%, \sigma = 20\%, \hat{\delta} = 3{,}5\%$
- 1000 simulations

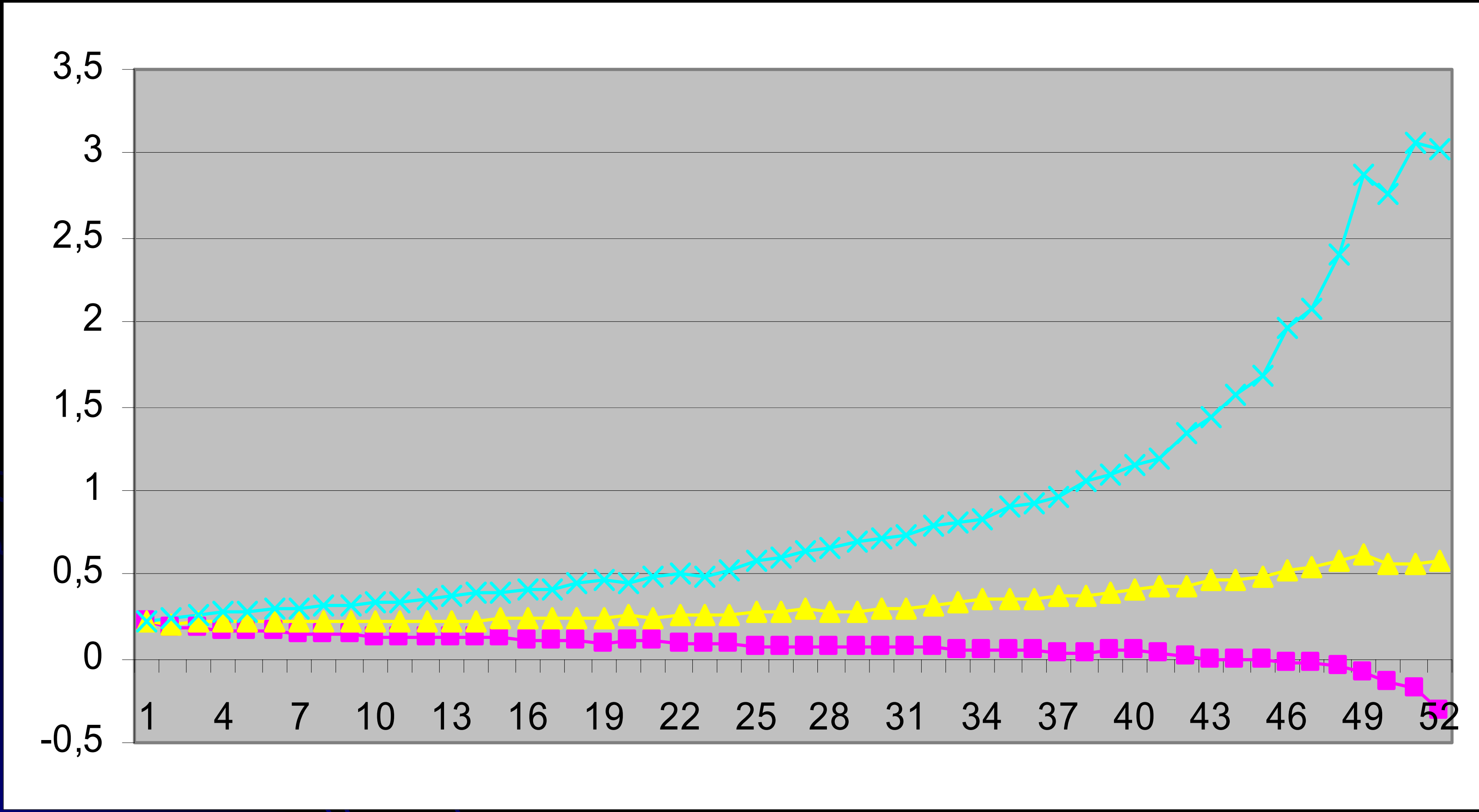
Simulation study

- the higher the risk allowance in the reserve, the lower the fraction of the wealth invested in the risky asset, the higher the expected terminal wealth and the lower the ruin probability
- the higher the value of alpha, the higher the fraction of the wealth invested in the risky asset, the higher the expected terminal wealth and the higher the ruin probability
- the value of beta has only marginal effect on the results

Simulation study

- the ruin probability is lower and the expected terminal wealth is higher compared with the situation when the insurer has only a bank account at its disposal
- The distribution of the terminal wealth under the optimal strategy has lighter left tail (5% percentile) and heavier right tail (95% percentile) compared with the distribution of the terminal wealth in the case of the risk free investment strategy

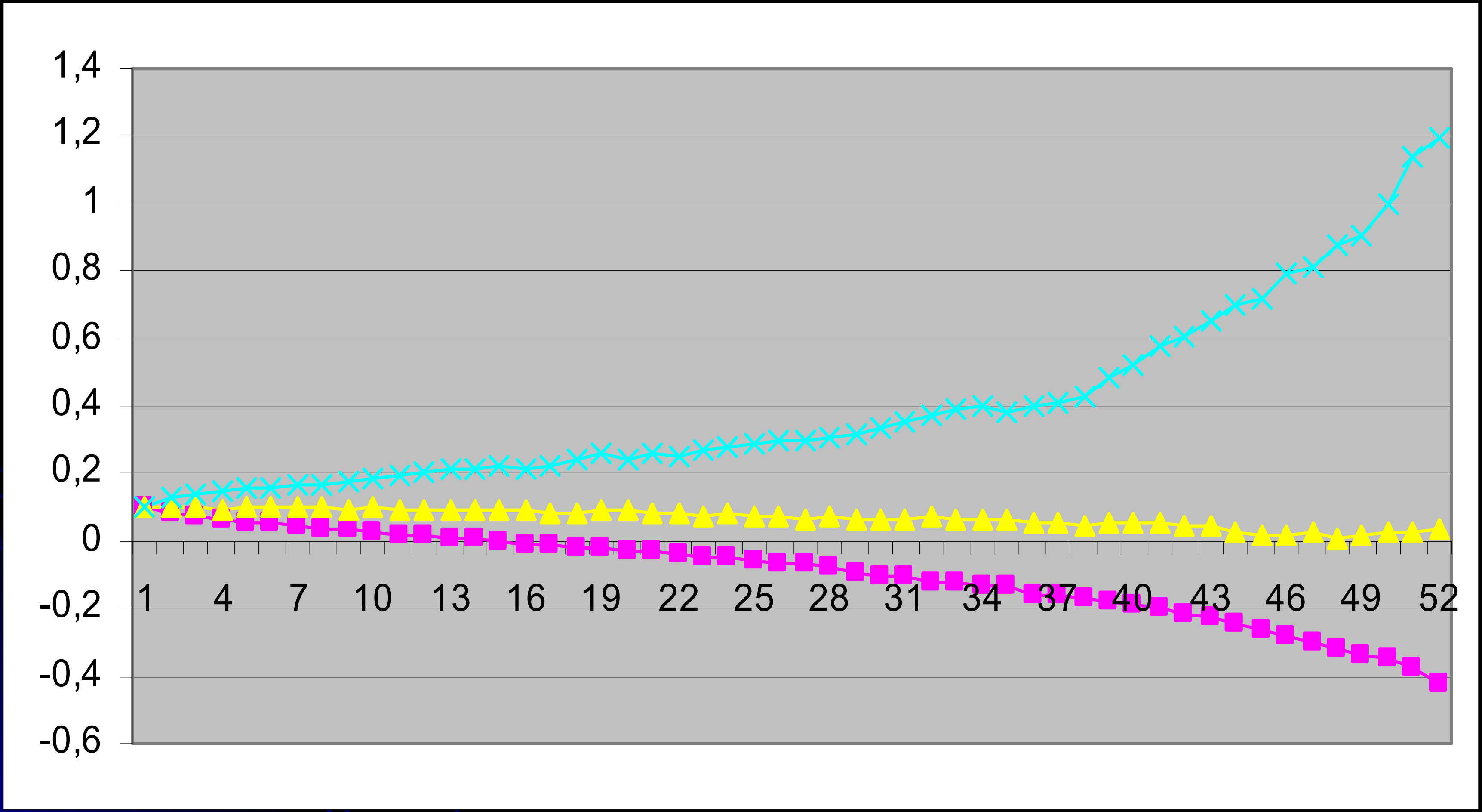
Reserve=15%, alfa=6000, beta=1 (percentiles 15th,50th,85th)



Reserve=15%, alfa=6000, beta=1

Percentiles	Risk-free+risky investment	Risk-free investment
5%	-619,66	-644,46
15%	420,48	236,66
30%	1183,56	952,05
50%	1735,93	1593,28
70%	2361,60	2269,56
85%	2906,07	2904,24
95%	3723,49	3209,12
Mean	1688,13	1579,77
Deviation	1262,64	1297,11
Ruin probability	0,096	0,116

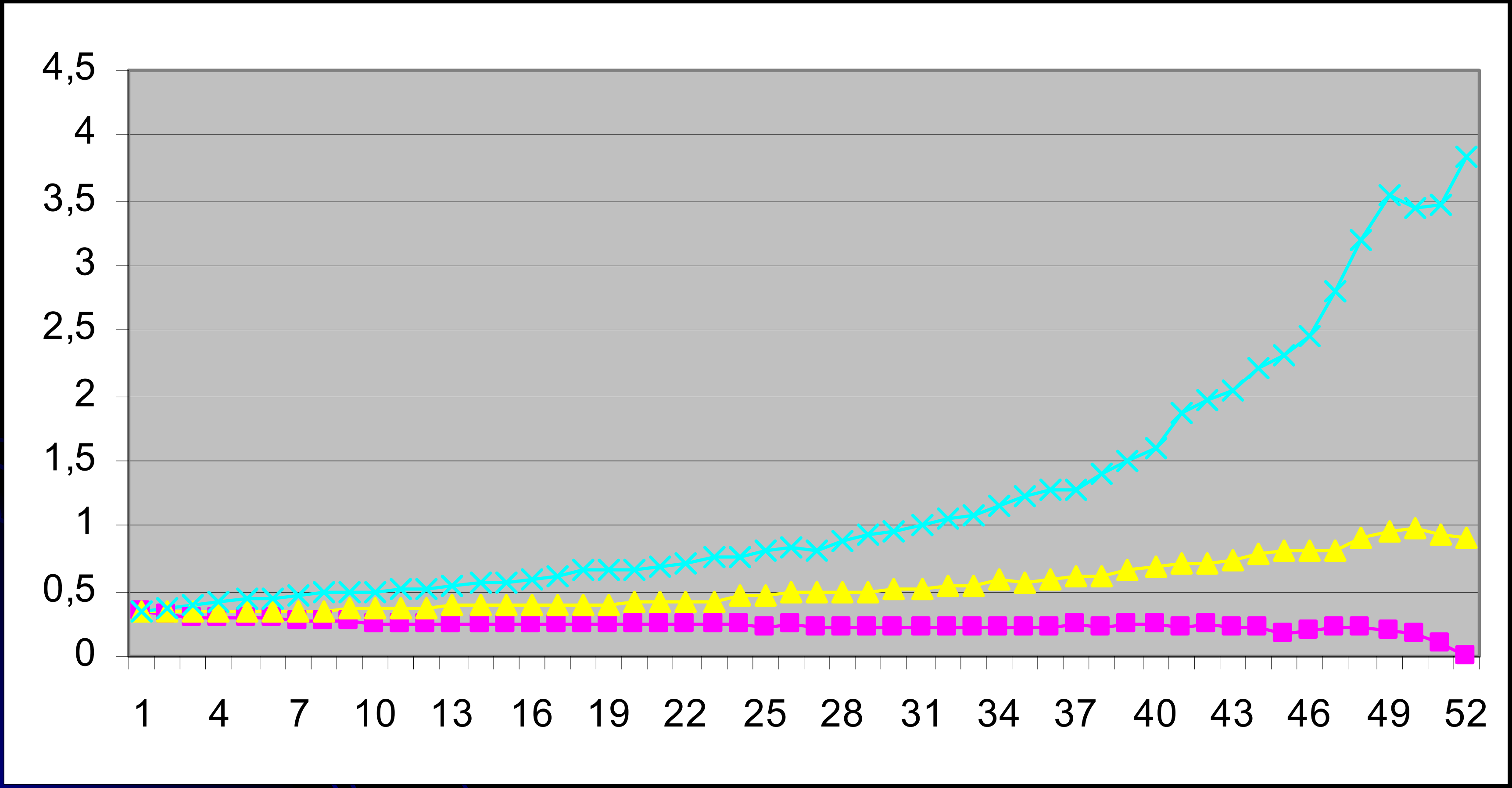
Reserve=25%, alfa=6000, beta=1 (percentiles 15th,50th,85th)



Reserve=25%, alfa=6000, beta=1

Percentiles	Risk-free+risky investment	Risk-free investment
5%	632,18	427,53
15%	1364,94	1178,12
30%	1978,17	1817,23
50%	2694,00	2515,11
70%	3225,25	3181,54
85%	3850,07	3816,71
95%	4469,84	4166,16
Mean	2600,06	2498,84
Deviation	1162	1239,28
Ruin probability	0,014	0,016

Reserve=15%, alfa=8000, beta=1 (percentiles 15th,50th,85th)



Reserve=15%, alfa=8000, beta=1

Percentiles	Risk-free+risky investment	Risk-free investment
5%	-600,35	-644,46
15%	569,16	236,66
30%	1294,73	952,05
50%	2039,13	1593,28
70%	2646,94	2269,56
85%	3323,38	2904,24
95%	3931,54	3209,12
Mean	1886,31	1579,77
Deviation	1462,39	1297,11
Ruin probability	0,114	0,116

Thank you for your attention

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