Optimal Dividends in the Brownian Motion Model with Credit and Debit Interest

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Introduction

- Problem: How should the dividends be paid to the shareholders, if the aim is to maximize the expectation of the discounted dividends until possible ruin of the company?

- De Finetti (1957): Discrete time model with gains of $+1$ and -1.

- Gerber and Shiu (2004): A Brownian motion with constant drift.

- This paper: The surplus earns interest at the constant force $\rho > 0$.

The Model

Let $X(t)$ denote the surplus of the company, if no dividends are paid. Then the surplus process follows the following dynamics:

$$dX(t) = (\mu + \rho X(t))dt + \sigma dW(t), \quad t \geq 0, \tag{1}$$

we assume that dividends are paid to the shareholders according to a *barrier strategy*, say with parameter $b > 0$. Whenever the (modified) surplus is about to go above the level b, the excess (or "overflow") will be paid as dividends. This scheme can be formalized as follows.

Let

$$M(t) = \max_{0 \le s \le t} X(s) \tag{2}$$

be the observed maximum of the surplus process, and let $D(t)$ denote the aggregate dividends by time t. Then, according to the barrier strategy with parameter b,

$$D(t) = \max(M(t) - b, 0). \tag{3}$$

The modified surplus at time t is $X(t) - D(t)$, and

$$T = \inf\{t : X(t) - D(t) = 0\} \tag{4}$$

is the *time of ruin*, which is certain.

Now

$$D = \int_0^T e^{-\delta t} dD(t), \tag{5}$$

is the discounted value of the dividends until ruin. We are interested in $V(x;b)$, $0 \le x \le b$, the expectation of D, considered a function of the initial surplus x.

$V(x;b)$ satisfies the following second-order differential equation,

$$\frac{\sigma^2}{2} V''(x;b) + (\mu + \rho x)V'(x;b) - \delta V(x;b) = 0, \quad 0 < x < b, \tag{6}$$

in combination with the boundary conditions

$$\begin{cases} V(0;b) = 0, \\ V'(b;b) = 1. \end{cases} \tag{7}$$

Let the function $g(x)$ be a solution of the differential equation

$$\frac{\sigma^2}{2} g''(x) + (\mu + \rho x) g'(x) - \delta g(x) = 0, \quad x > 0, \tag{8}$$

combined with the boundary condition

$$g(0) = 0. \tag{9}$$

The function $g(x)$ is unique apart from a constant factor. From (6) - (7) we obtain the *factorization formula*,

$$V(x; b) = \frac{g(x)}{g'(b)}, \quad 0 \leq x \leq b. \tag{10}$$

From (3) it follows that

$$V(x;b) = x - b + V(b;b), \quad x > b, \tag{11}$$

which has the following interpretation: if the initial surplus x exceeds the dividend barrier b, the difference is immediately paid out as dividends.

For given $x > 0$, $b > 0$,

$$\lim_{\sigma \to \infty} V(x;b) = x. \tag{12}$$

Kummer's confluent hypergeometric equation

We define $t = -\frac{1}{2}[\frac{1}{\sigma}\sqrt{\frac{2}{\rho}}(\mu + \rho x)]^2$ and the function $h(t)$ by the requirement that $g(x) = h(t)$, then

$$t\,h''(t) + \left(\frac{1}{2} - t\right)h'(t) + \frac{\delta}{2\rho}h(t) = 0, \quad t < -\frac{\mu^2}{\rho\sigma^2}. \tag{13}$$

Thus the function $h(t)$ satisfies the Kummer's equation.

We conclude that

$$g(x) = h(t) = \alpha\,(-t)^{1-c}\,e^t\,M\,(1-a, 2-c; -t) + \beta\,e^t\,U(c-a, c; -t), \tag{14}$$

for certain coefficients α and β, with

$$t = -\frac{1}{\rho\sigma^2}(\mu + \rho x)^2 \tag{15}$$

and a and c given by

$$c = \frac{1}{2}, \qquad a = -\frac{\delta}{2\rho}. \tag{16}$$

We may set

$$\begin{aligned} \alpha &= U\left(c-a,\, c;\, \frac{\mu^2}{\rho\sigma^2}\right), \\ \beta &= -\left(\frac{\mu^2}{\rho\sigma^2}\right)^{1-c} M\left(1-a,\, 2-c;\, \frac{\mu^2}{\rho\sigma^2}\right). \end{aligned}$$

In the limiting situation $\sigma = 0$, we have

$$X(t) = xe^{\rho t} + \mu \bar{s}_{\overline{t}|},$$

where $\bar{s}_{\overline{t}|}$ is calculated at the force of interest ρ. Now the determination of $V(x;b)$ is an exercise of compound interest. Let t_0 be the time when $X(t_0) = b$. Then we find that

$$V(x;b) = e^{-\delta t_0} V(b;b) = \left(\frac{\mu + \rho x}{\mu + \rho b}\right)^{\delta/\rho} \frac{\mu + \rho b}{\delta} \tag{17}$$

after simplification.

When $\rho = 0$, by (2.11) - (2.15) of Gerber and Shiu (2004),

$$V(x;b) = \frac{e^{rx} - e^{sx}}{re^{rb} - se^{sb}}, \tag{18}$$

where $r = (-\mu + \sqrt{\mu^2 + 2\delta\sigma^2})/\sigma^2$ and $s = (-\mu - \sqrt{\mu^2 + 2\delta\sigma^2})/\sigma^2$.

Example 1. Assume that $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 0.5$. We calculate $V(x;10)$ for selected values of x and ρ.

The results are exhibited in Table 1.

Table 1. The influence of ρ and x on $V(x; 10)$ with $\delta = 4\%$

	$V(x; 10)$ with $\sigma = 0.5$ ($\sigma = 5$)			
x	$\rho = 0$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.2	13.63 (0.36)	15.25 (0.38)	16.90 (0.39)	18.57 (0.41)
0.4	16.47 (0.72)	18.42 (0.75)	20.40 (0.77)	22.41 (0.80)
0.6	17.15 (1.07)	19.17 (1.11)	21.23 (1.15)	23.31 (1.20)
0.8	17.39 (1.42)	19.44 (1.47)	21.53 (1.53)	23.63 (1.58)
1.0	17.55 (1.76)	19.62 (1.82)	21.72 (1.89)	23.85 (1.96)
2.0	18.27 (3.38)	20.41 (3.51)	22.59 (3.64)	24.78 (3.78)
4.0	19.79 (6.30)	22.06 (6.53)	24.35 (6.77)	26.67 (7.01)
6.0	21.43 (8.87)	23.80 (9.17)	26.19 (9.47)	28.59 (9.79)
8.0	23.20 (11.16)	25.64 (11.50)	28.09 (11.85)	30.54 (12.21)
10	25.12 (13.24)	27.59 (13.60)	30.05 (13.96)	32.52 (14.34)

Example 2. Assume that $\mu = 1$, $b = 10$, $\delta = 4\%$, $\rho = 2\%$. We calculate $V(x; 10)$ for selected values of x and σ. The results are displayed in Table 2.

Table 2. The influence of σ and x on $V(x;10)$ with $\delta = 4\%$

	$V(x;10)$ with $\rho = 2\%$ ($\rho = 6\%$)			
x	$\sigma = 0$	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
0.2	21.00 (29.47)	7.28 (10.22)	0.98 (1.34)	0.39 (0.45)
0.4	21.17 (29.71)	12.17 (17.07)	1.91 (2.61)	0.77 (0.90)
0.6	21.34 (29.94)	15.47 (21.66)	2.81 (3.83)	1.15 (1.34)
0.8	21.51 (30.17)	17.71 (24.75)	3.67 (5.00)	1.52 (1.77)
1.0	21.68 (30.40)	19.25 (26.84)	4.49 (6.12)	1.89 (2.19)
2.0	22.53 (31.53)	22.42 (31.02)	8.10 (11.00)	3.64 (4.21)
4.0	24.30 (33.75)	24.50 (33.58)	13.44 (18.01)	6.76 (7.78)
6.0	26.13 (35.89)	26.34 (35.73)	17.12 (22.55)	9.47 (10.80)
8.0	28.03 (37.97)	28.24 (37.81)	19.84 (25.63)	11.85 (13.36)
10	30.00 (40.00)	30.21 (39.84)	22.02 (27.90)	13.96 (15.53)

The optimal barrier

For given $x > 0$, we want to find b which maximizes $V(x;b)$. From (10) and (11) we obtain

$$\frac{\partial}{\partial b}V(x;b) = \begin{cases} -V(b;b)\,\dfrac{g''(b)}{g'(b)} & \text{if} \quad 0 < b < x, \\ \\ -V(x;b)\,\dfrac{g''(b)}{g'(b)} & \text{if} \quad b \geq x. \end{cases} \qquad (19)$$

From (8) and (9) it follows that

$$\frac{g''(0)}{g'(0)} = -\frac{2\mu}{\sigma^2} < 0.$$

Hence $\frac{\partial}{\partial b}V(x;b)$ is positive for small values of b. Because $V(x;b) \to 0$ for $b \to \infty$, we gather that $V(x;b)$ attains its maximum for a finite and positive value of b.

According to (19), the first order condition is that

$$g''(b) = 0. \tag{20}$$

It turns out that this equation has a unique solution $b = b^*$.

We note that

$$V''(x; b) = \frac{g''(x)}{g'(b)} \quad \text{for} \quad 0 < x < b$$

and hence

$$V''(b^*; b^*) = 0. \tag{21}$$

Thus, if we set $x = b = b^*$ in (6) and use the second condition in (7), we see that

$$\mu + \rho\, b^* - \delta V(b^*; b^*) = 0,$$

from which it follows that

$$V(b^*;\, b^*) = \frac{\mu + \rho\, b^*}{\delta}. \tag{22}$$

Thus, $V(b^*; b^*)$ is identical to the present value of a *perpetuity*, where the payment rate is the sum of the drift and the interest on the initial capital. For $\rho = 0$, formula (22) can be found as formula (7.1) in Gerber and Shiu (2004). In this case, there is a closed form expression for the optimal barrier:

$$b^* = \frac{2}{r - s} \log\left(-\frac{s}{r}\right), \tag{23}$$

where r and s are the same as in (18).

The difference $V(x; b^*) - x$ is the maximal return on the investment x. Formula (12) shows that in the limit $\sigma \to \infty$ (*extreme volatility business*), the maximal return becomes zero. In particular, we have $V(b^*; b^*) = b^*$ in the limit. From this and (22) we obtain a linear equation for the limiting value of b^*. We find that

$$\lim_{\sigma \to \infty} b^* = \frac{\mu}{\delta - \rho}. \tag{24}$$

This result generalizes formula (7.3) in Gerber and Shiu (2004).

Example 3. Assume that $\mu = 1, \delta = 4\%$. We calculate the optimal barrier b^* for selected values of ρ and σ. The results are shown in Table 3.

Table 3. The influence of ρ and σ on b^* with $\delta = 4\%$

	Optimal Barrier b^*				
σ	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.05	0.02476	0.02492	0.02511	0.02562	0.02648
0.10	0.08514	0.08580	0.08656	0.08855	0.09198
0.20	0.28484	0.28739	0.29033	0.29814	0.31161
0.50	1.31399	1.32847	1.34534	1.39034	1.46887
5	19.00860	20.49930	22.17000	26.18760	31.74960
50	24.91700	28.44770	33.13750	49.34760	95.14190
500	24.99920	28.57020	33.33130	49.99330	99.94670
∞	25	28.57140	33.33333	50	100

Example 4. As in Example 1, we assume that $\mu = 1$, $\delta = 4\%$, $\sigma = 0.5\,(\sigma = 5)$. Now we calculate $V(x; b^*)$ for the same combinations of x and ρ as in Table 1. The results are shown in Table 4.

Table 4. The influence of ρ and x on $V(x;b^*)$ with $\delta = 4\%$

	$V(x;b^*)$ with $\sigma = 0.5$ ($\sigma = 5$)			
x	$\rho = 0$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.2	19.16 (0.42)	19.42 (0.48)	19.68 (0.56)	19.96 (0.67)
0.4	23.16 (0.84)	23.45 (0.95)	23.76 (1.11)	24.08 (1.32)
0.6	24.11 (1.25)	24.41 (1.42)	24.73 (1.65)	25.05 (1.97)
0.8	24.46 (1.66)	24.76 (1.88)	25.07 (2.18)	25.40 (2.56)
1.0	24.68 (2.06)	24.99 (2.33)	25.30 (2.70)	25.63 (3.22)
2	25.69 (3.96)	25.99 (4.48)	26.30 (5.21)	26.63 (6.20)
4	27.69 (7.39)	27.99 (8.34)	28.30 (9.67)	28.63 (11.51)
6	29.69 (10.39)	29.99 (11.71)	30.30 (13.55)	30.63 (16.09)
8	31.69 (13.07)	31.99 (14.69)	32.30 (16.94)	32.63 (20.06)
10	33.69 (15.51)	33.99 (17.37)	34.30 (19.96)	34.63 (23.55)

The distribution of the time of ruin under a barrier strategy

The *Laplace transform* of the probability density function of the time of ruin T is

$$L(x; b) = E[e^{-\delta T}] \tag{25}$$

$L(x; b)$ can be characterized as the solution of second order differential equation

$$\frac{\sigma^2}{2} L''(x; b) + (\mu + \rho x) L'(x; b) - \delta L(x; b) = 0, \quad 0 < x < b, \tag{26}$$

in conjunction with the boundary conditions

$$\begin{cases} L(0; b) = 1, \\ L'(b; b) = 0. \end{cases} \tag{27}$$

Example 5. Assume $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 3$. We calculate the expected time of ruin $E[T]$ for different values of ρ and x. The results are displayed in Table 5.

Table 5. The influence of ρ and x on $E[T]$ with $\delta = 4\%$ and $\sigma = 3$

	$E[T]$					
x	$\rho = 0$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 4\%$	$\rho = 6\%$	$\rho = 8\%$
0.2	1.605	1.701	1.805	2.039	2.314	2.637
0.4	3.132	3.320	3.523	3.981	4.517	5.148
0.6	4.584	4.859	5.157	5.827	6.614	7.538
0.8	5.963	6.322	6.710	7.583	8.608	9.811
1.0	7.274	7.713	8.166	9.252	10.502	11.970
2.0	12.900	13.676	14.514	16.398	18.604	21.193
4.0	20.454	21.656	22.952	25.857	29.243	33.199
6.0	24.579	25.973	27.473	30.823	34.711	39.234
8.0	26.507	27.962	29.525	30.010	37.045	41.728
10	27.025	28.488	30.058	33.559	37.611	42.311

Life after ruin

We assume that whenever the surplus is negative, interest is debited at a force $\tau > \delta$. Now we assume that the company has to go out of business, when the surplus is at the critical level

$$\lambda = -\frac{\mu}{\tau}. \tag{28}$$

Example 6. Assume that $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 0.5$ ($\sigma = 5$) and $\tau = 6\%$. We calculate $V(x; 10)$ for the same combinations of x and ρ as in Table 6.

Table 6. The influence of ρ and x on $V(x;10)$ with $\delta = 4\%$ and $\tau = 6\%$

	$V(x;10)$ with $\sigma = 0.5$ ($\sigma = 5$)			
x	$\rho = 0$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
-10	9.12 (8.09)	10.19 (8.35)	11.29 (8.61)	12.39 (8.88)
-8	10.89 (10.44)	12.17 (10.77)	13.47 (11.11)	14.79 (11.46)
-6	12.52 (12.73)	13.99 (13.13)	15.49 (13.54)	17.01 (13.97)
-4	14.04 (14.95)	15.70 (15.42)	17.38 (15.91)	19.09 (16.41)
-2	15.49 (17.09)	17.32 (17.63)	19.17 (18.19)	21.05 (18.76)
0	16.87 (19.16)	18.86 (19.77)	20.88 (20.39)	22.93 (21.03)
2.0	18.27 (21.15)	20.41 (21.82)	22.59 (22.51)	24.78 (23.22)
4.0	19.79 (23.10)	22.06 (23.83)	24.35 (24.57)	26.67 (25.34)
6.0	21.43 (25.04)	23.80 (25.81)	26.19 (26.60)	28.59 (27.41)
8.0	23.20 (26.98)	25.64 (27.78)	28.09 (28.60)	30.54 (29.44)
10	25.12 (28.96)	27.59 (29.77)	30.05 (30.60)	32.52 (31.45)

Example 7. Assume that $\mu = 1$, $b = 10, \delta = 4\%$, $\sigma = 0.5\,(\sigma = 5)$ and $\rho = 2\%$. We calculate $V(x;10)$ for selected values of x and τ.

Table 7. The influence of ρ and x on $V(x;10)$ with $\delta = 4\%$ and $\rho = 2\%$

	$V(x;10)$ with $\sigma = 0.5$ $(\sigma = 5)$			
x	$\tau = 6\%$	$\tau = 7\%$	$\tau = 8\%$	$\tau = 10\%$
-10	11.29 (8.61)	10.37 (6.06)	8.89 (3.77)	0 (0)
-8	13.47 (11.11)	13.00 (8.82)	12.41 (6.74)	10.19 (3.27)
-6	15.49 (13.54)	15.27 (11.49)	15.01 (9.63)	14.35 (6.47)
-4	17.38 (15.91)	17.29 (14.08)	17.20 (12.40)	16.98 (9.56)
-2	19.17 (18.19)	19.15 (16.55)	19.13 (15.04)	19.08 (12.48)
0	20.88 (20.39)	20.88 (18.90)	20.88 (17.53)	20.88 (15.20)
2.0	22.59 (22.51)	22.59 (21.13)	22.59 (19.86)	22.59 (17.70)
4.0	24.35 (24.57)	24.35 (23.27)	24.35 (22.07)	24.35 (20.04)
6.0	26.19 (26.60)	26.19 (25.34)	26.19 (24.19)	26.19 (22.23)
8.0	28.09 (28.60)	28.09 (27.37)	28.09 (26.25)	28.09 (24.33)
10	30.05 (30.60)	30.05 (29.38)	30.05 (28.26)	30.05 (26.36)

Example 8. As in Example 3, we assume that $\mu = 1$ and $\delta = 4\%$. But now business (and dividends) can go on after ruin according to a debit rate of interest of $\tau = 6\%$.

Table 8. The influence of ρ and σ on b^* with $\delta = 4\%$ and $\tau = 6\%$

	Optimal Barrier b^*				
σ	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.05	0.00051	0.00057	0.00064	0.00087	0.00137
0.10	0.00203	0.00226	0.00256	0.00347	0.00549
0.20	0.00812	0.00905	0.01023	0.01388	0.02199
0.50	0.05113	0.05698	0.06439	0.08731	0.13817
5	5.11239	5.70392	6.45109	8.72959	13.49200
50	8.28724	9.46708	11.03840	16.51990	32.75470
500	8.33287	9.52324	11.11030	16.66520	33.32740
∞	8.33333	9.52381	11.11111	16.66667	33.33333