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# **Legal Valuation Portfolio in Non-Life Insurance**

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## Topic of the presentation

Construction of a legal valuation portfolio to value the runoff situation of insurance risks.

### Motivation

- New risk based solvency requirements
- Protect policyholder respectively the injured from the consequences of insolvency of an insurance company

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## Notation

$Y_{k,j}$  payments for accident year  $k$  in development period  $j$

where  $1 \leq k \leq K$  and  $1 \leq j \leq J \leq K$

$$X_{k,j} = \sum_{l=1}^j Y_{k,l}$$

cumulative payments for accident year  $k$

$$Y_t := \sum_{k+j=t+1} Y_{k,j}$$

payments of the accounting year  $t$

$\{\mathcal{F}_t\}_t$

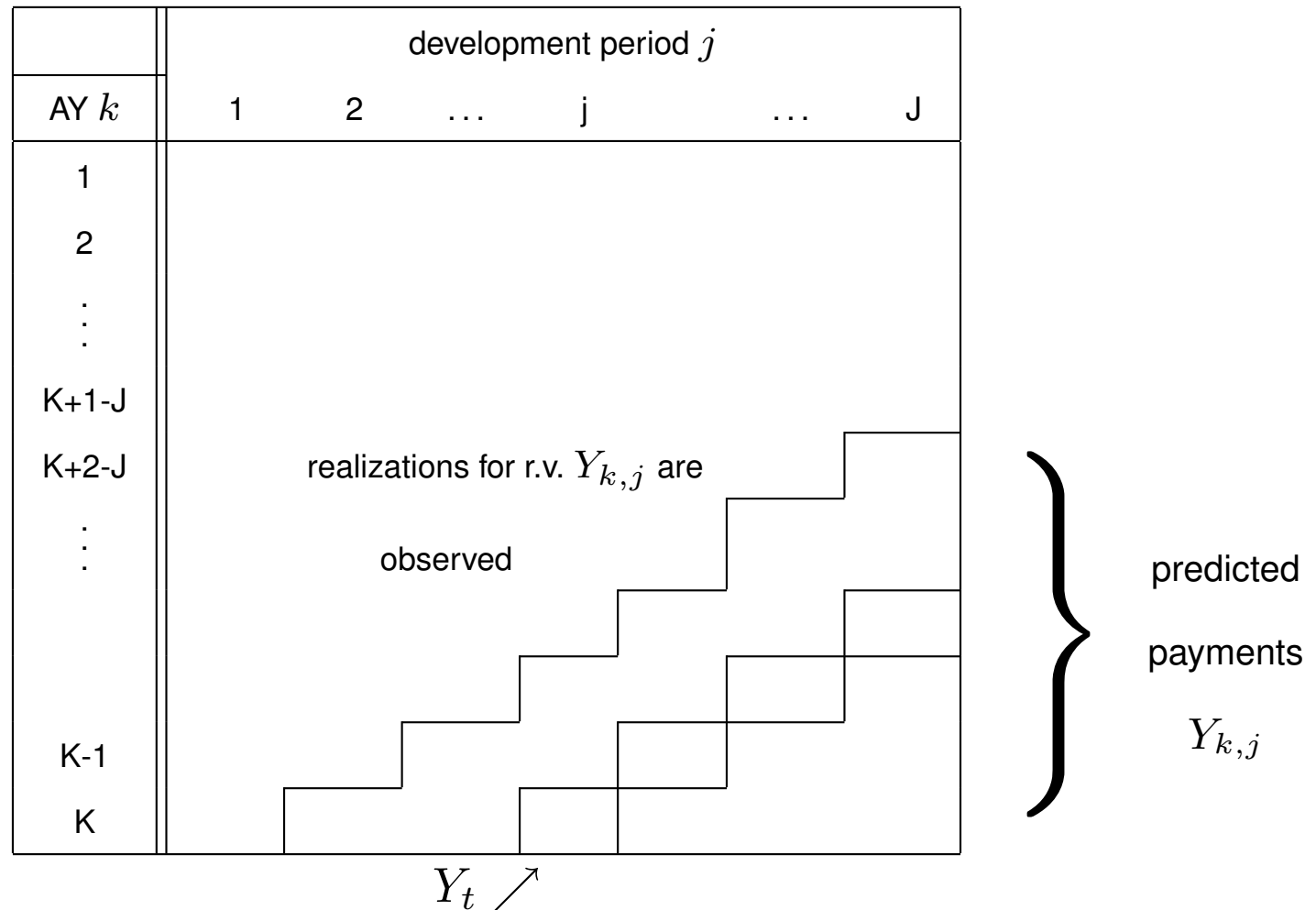
Filtration of information for accounting years  $t = 1, \dots, K + J - 1$

$\mathcal{D}_K$

realizations  $x_{k,j}$  of the cumulative payments  $X_{k,j}$  at time  $t = K$ ,

i.e.  $\{x_{k,j} \mid 1 \leq k \leq K \text{ and } 1 \leq j \leq \min\{J, K - k + 1\}\}$

## Upper trapezoid and lower triangle



## Legal valuation portfolio (VaPo)

$E_t^{(K)} := E [Y_t | \mathcal{F}_K]$     expected payments for accounting year  $t > K$

$\rho_t^K$     risk measure, depending on the information up to time  $K$

$Z^{(t)}$     Zero Coupon Bond maturing at time  $t$

$i$     risk-adjusted cost of capital rate

instrument/basis

Number of units

$Z^{(K+1)}$

$E_{K+1}^{(K)} + \rho_{K+1}^{(K)}$

$Z^{(K+2)}$

$E_{K+2}^{(K)} + i \cdot \rho_{K+2}^{(K)}$

$\vdots$

$\vdots$

$Z^{(K+\tau)}$

$E_{K+\tau}^{(K)} + i \cdot \rho_{K+\tau}^{(K)}$

$\vdots$

$\vdots$

$Z^{(K+J-1)}$

$E_{K+J-1}^{(K)} + i \cdot \rho_{K+J-1}^{(K)}$

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## Remarks

1)  $E_t^{(K)}$  should contain LAE to guarantee a smooth runoff of the liabilities

—→ see Appendix A of the submitted paper; here ULAE are not taken into consideration

2) For the numeric example we use a risk measure of the form  $\beta \cdot \sigma$ .

Motivation: if  $Y_t \sim N(\mu_t, \sigma_t^2)$  then the VaR and the TailVaR are both of the form  $\mu_t + \beta \cdot \sigma_t$ .

—→ Stochastic Model

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## Stochastic Model

**Assumption 1** For  $1 \leq j \leq J$  and  $1 \leq k \leq K$  let

$$X_{k,j} = f_{j-1} \cdot X_{k,j-1} + \sigma_{j-1} \cdot \sqrt{X_{k,j-1}} \cdot \varepsilon_{k,j},$$

with  $\sigma_{j-1} > 0$  and  $\varepsilon_{k,j}$  independent random variables with  $E[\varepsilon_{k,j}] = 0$  and  $\text{Var}[\varepsilon_{k,j}] = 1$ .

MACK assumptions for  $1 \leq k \leq K, 1 \leq j \leq J$ :

- 1)  $E(X_{k,j} | X_{k,1}, \dots, X_{k,j-1}) = f_{j-1} \cdot X_{k,j-1}$
- 2)  $(X_{k,1}, \dots, X_{k,J})$  and  $(X_{l,1}, \dots, X_{l,J})$  are independent for  $k \neq l$
- 3)  $\text{Var}(X_{k,j} | X_{k,1}, \dots, X_{k,j-1}) = \sigma_{j-1}^2 \cdot X_{k,j-1}$ .

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## Estimators for $f_{j-1}$ and $\sigma_{j-1}^2$

$$\hat{f}_{j-1} := \frac{\sum_{k=1}^{K-j+1} X_{k,j}}{S_{j-1}} \quad \text{with} \quad S_{j-1} := \sum_{k=1}^{K-j+1} X_{k,j-1} \quad (\text{Chain-ladder estimator})$$

$$\hat{\sigma}_{j-1}^2 := \frac{1}{K-j} \cdot \sum_{k=1}^{K-j+1} X_{k,j-1} \cdot \left( \frac{X_{k,j}}{X_{k,j-1}} - \hat{f}_{j-1} \right)^2 \quad \text{for } 1 \leq j \leq J-1$$

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## Estimators for cumulative and non-cumulative payments

For

$$E [X_{k,j} | \mathcal{D}_K] = x_{k,k^*} \cdot f_{k^*} \cdot \dots \cdot f_{j-1}$$

where  $k^* := K - k + 1$ , we use the estimator

$$\widehat{X}_{k,j} := x_{k,k^*} \cdot \widehat{f}_{k^*} \cdot \dots \cdot \widehat{f}_{j-1}.$$

Using  $Y_{k,j} = X_{k,j} - X_{k,j-1}$  and  $\widehat{Y}_{k,j} := \widehat{X}_{k,j} - \widehat{X}_{k,j-1}$  we obtain

- $E [Y_{k,j} | \mathcal{D}_K] = x_{k,k^*} \cdot f_{k^*} \cdot \dots \cdot f_{j-2} \cdot (f_{j-1} - 1)$
- $\widehat{Y}_{k,j} = x_{k,k^*} \cdot \widehat{f}_{k^*} \cdot \dots \cdot \widehat{f}_{j-2} \cdot (\widehat{f}_{j-1} - 1).$

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## Prediction error for single accident years

$$E \left[ \left( \hat{X}_{k,j} - X_{k,j} \right)^2 \middle| \mathcal{D}_K \right] = E \left[ \underbrace{\left( X_{k,j} - E [X_{k,j} | \mathcal{D}_K] \right)^2}_{\text{process variance}} \middle| \mathcal{D}_K \right] + \underbrace{\left( \hat{X}_{k,j} - E [X_{k,j} | \mathcal{D}_K] \right)^2}_{\text{estimation error}}$$

where the observations in the upper trapezoid

$$\mathcal{D}_K = \{x_{k,j} \mid 1 \leq k \leq K \text{ and } 1 \leq j \leq \min\{J, K - k + 1\}\}$$

are given.

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**Estimator of the process variance for  $X_{k,j}$**

$$\widehat{E} \left[ \left( X_{k,j} - E [X_{k,j} | \mathcal{D}_K] \right)^2 \middle| \mathcal{D}_K \right] := \underbrace{(\widehat{X}_{k,j})^2 \cdot \sum_{l=k^*}^{j-1} \frac{\widehat{\sigma}_l^2}{\widehat{X}_{k,l} \cdot \widehat{f}_l^2}}_{=:\Gamma_{k,j}^2}.$$

## Estimation error for $X_{k,j}$

$$\left(\widehat{X}_{k,j} - E[X_{k,j} | \mathcal{D}_K]\right)^2 = x_{k,k^*}^2 \cdot \left(\widehat{f}_{k^*} \cdots \widehat{f}_{j-1} - f_{k^*} \cdots f_{j-1}\right)^2$$

In order to determine the estimation error, we average over the possible values  $\widehat{f}_{k^*}, \widehat{f}_{k^*+1}, \dots, \widehat{f}_{j-1}$ , given the upper trapezoid  $\mathcal{D}_K$ .

We propose the following estimator for the estimation error:

$$\left(\widehat{X}_{k,j} - \widehat{E[X_{k,j} | \mathcal{D}_K]}\right)^2 := \widehat{X}_{k,j}^2 \cdot \underbrace{\left(\prod_{l=k^*}^{j-1} \left(1 + \frac{\widehat{\sigma}_l^2}{\widehat{f}_l^2 \cdot s_l}\right) - 1\right)}_{=:\Delta_{k,j}^2}.$$

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## MACK's estimator of the estimation error

$$\left( \widehat{X}_{k,j} - \widehat{E} [X_{k,j} | \mathcal{D}_K] \right)^2 := \widehat{X}_{k,j}^2 \cdot \sum_{l=k^*}^{j-1} \frac{\widehat{\sigma}_l^2}{\widehat{f}_l^2 \cdot s_l}.$$

**Remark:** MACK's estimator can be seen as a linear approximation to the above estimator:

$$\begin{aligned} \Delta_{k,j}^2 &= \prod_{l=k^*}^{j-1} \left( 1 + \frac{\widehat{\sigma}_l^2}{\widehat{f}_l^2 \cdot s_l} \right) - 1 = 1 + \sum_{l=k^*}^{j-1} \frac{\widehat{\sigma}_l^2}{\widehat{f}_l^2 \cdot s_l} + \dots - 1 \\ &\approx \sum_{l=k^*}^{j-1} \frac{\widehat{\sigma}_l^2}{\widehat{f}_l^2 \cdot s_l}. \end{aligned}$$

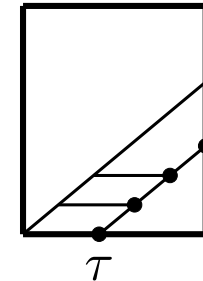
## Prediction error for accounting years (non-cumulative payments)

For our VaPo we consider pooled, non-cumulative payments  $Y_t = \sum_{k+j=t+1} Y_{k,j}$ .

Analogously as before we obtain

a) best-estimates  $\hat{Y}_t := \sum_{k+j=t+1} \hat{Y}_{k,j}$  for these non-cumulative payments

b) we estimate the prediction error (special attention has to be paid to the dependence when pooling accident years)



$$\begin{aligned}
 & E \left[ \left( \hat{Y}_{K+\tau} - Y_{K+\tau} \right)^2 \middle| \mathcal{D}_K \right] \\
 &= \underbrace{E \left[ \left( Y_{K+\tau} - E [Y_{K+\tau} | \mathcal{D}_K] \right)^2 \middle| \mathcal{D}_K \right]}_{\text{process variance}} + \underbrace{\left( \hat{Y}_{K+\tau} - E [Y_{K+\tau} | \mathcal{D}_K] \right)^2}_{\text{estimation error}}.
 \end{aligned}$$

## Estimator for the Prediction error for accounting years ( $Y_{K+\tau}$ )

We obtain the following estimator

$$\widehat{E} \left[ \left( \widehat{Y}_{K+\tau} - Y_{K+\tau} \right)^2 \middle| \mathcal{D}_K \right] := \sum_{\substack{k+j=K+1+\tau \\ \tau+1 \leq j \leq J}} \left( \Gamma_{k,j}^2 + \Gamma_{k,j-1}^2 \cdot (1 - 2 \cdot \widehat{f}_{j-1}) \right) \leftarrow \begin{array}{l} \text{process} \\ \text{variance} \end{array}$$

$$\left. \begin{array}{l} \text{estimation} \\ \text{error} \end{array} \right\} \begin{aligned} & + \sum_{k^*=1}^{J-\tau} \Delta_{k,k^*+\tau-1}^2 \cdot \widehat{Y}_{k,k^*+\tau}^2 + \left( \Delta_{k,k^*+\tau}^2 - \Delta_{k,k^*+\tau-1}^2 \right) \cdot \widehat{X}_{k,k^*+\tau} \\ & + 2 \cdot \sum_{k^*=1}^{J-\tau} \sum_{l^*>k^*}^{\min\{k^*+\tau-1, J-\tau\}} \left( \Delta_{l,k^*+\tau-1}^2 \cdot \widehat{Y}_{k,k^*+\tau} \cdot \widehat{Y}_{l,l^*+\tau} \right. \\ & \left. + \left( \Delta_{l,k^*+\tau}^2 - \Delta_{l,k^*+\tau-1}^2 \right) \cdot \widehat{X}_{k,k^*+\tau} \cdot \widehat{Y}_{l,l^*+\tau} \right). \end{aligned}$$

**Example: Run-off-trapezoid (non cumulative payments)**

AY $k$	$j = 1$	2	3	4	5	6	7	8	9	10
1	12666	5450	868	199	257	22	18	10	4	1
2	11890	6766	824	257	46	-25	0	0	2	0
3	11618	7784	1113	257	53	9	6	74	2	0
4	13442	11807	1558	327	120	33	-3	1	0	30
5	11643	6125	744	764	99	7	5	0	-10	8
6	11419	7298	792	327	-5	22	8	-1	12	
7	11168	7489	1089	357	52	27	24	34		
8	11020	5942	551	161	35	24	9			
9	10285	5300	519	133	203	12				
10	13933	11110	1297	430	113					
11	11812	9018	1606	141						
12	10013	5832	486							
13	11684	6384								
14	10646									

### Example: Estimated non cumulative payments

	development period $j$									
AY $k$	2	3	4	5	6	7	8	9	10	
6										7
7								2		7
8							14	1		<b>6</b>
9						7	13	<b>1</b>		6
10					19	11	<b>22</b>	2		10
11				106	16	<b>9</b>	18	2		8
12			242	78	<b>12</b>	7	13	1		6
13		896	281	<b>91</b>	14	8	16	2		7
14	6719	861	<b>270</b>	87	13	8	15	1		7

$\tau$	1	2	3	4	5	6	7	8	9
$E_{K+\tau}^{(K)}$	8013	1270	<b>412</b>	134	46	33	23	9	7

**Prediction error of  $\hat{Y}_{K+\tau}$  and the estimator of  $\sigma_{K+\tau}^{(K)}$**

Process variance for accounting year $K + \tau$								
15	16	17	18	19	20	21	22	23
2296569	107372	40002	8340	1184	989	858	155	103

Estimation error for accounting year $K + \tau$								
15	16	17	18	19	20	21	22	23
161265	8592	3466	825	157	139	113	26	18

Prediction error of $\hat{Y}_{K+\tau}$								
15	16	17	18	19	20	21	22	23
2457834	115964	43468	9165	1341	1129	971	181	121

$\hat{\sigma}_{K+\tau}^{(K)} := \sqrt{\hat{E} \left[ \left( \hat{Y}_{K+\tau} - Y_{K+\tau} \right)^2 \middle  \mathcal{D}_K \right]}$								
15	16	17	18	19	20	21	22	23
1568	341	208	96	37	34	31	13	11

**Example of the valuation portfolio:** with  $\rho_{K+\tau}^{(K)} = 2.326 \cdot \widehat{\sigma}_{K+\tau}^{(K)}$   
 $i = 6\%$ .

$Z(K+\tau)$	$E_{K+\tau}^{(K)}$	$\widehat{\sigma}_{K+\tau}^{(K)}$	$\rho_{K+\tau}^{(K)}$	$\rho_{K+\tau}^K$ or $i \cdot \rho_{K+\tau}^K$	number of units
$Z(K+1)$	8013	1568	3647	3647	11660
$Z(K+2)$	1270	341	792	48	1318
$Z(K+3)$	412	208	485	29	441
$Z(K+4)$	134	96	223	13	148
$Z(K+5)$	46	37	85	5	51
$Z(K+6)$	33	34	78	5	37
$Z(K+7)$	23	31	72	4	27
$Z(K+8)$	9	13	31	2	10
$Z(K+9)$	7	11	26	2	8
sum	9946			3755	13701

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## Different Term structures

	1	2	3	4	5	6	7	8	9
BPV	0.88%	1.14%	1.36%	1.57%	1.75%	1.91%	2.05%	2.18%	2.29%
3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%
0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

## Prices of the legal VaPo

	BPV	3.5%	0.0%
price of the VaPo	13528	13131	13701
difference in %	1.26%	4.16%	0.00%