Stochastic surrender with asymmetric information. An alternative approach for the fair valuation of life insurance contracts.

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Abstract

In this paper, we study the valuation of life insurance contracts subject to surrender risk. In particular, our aim is to study how to model in a realistic and tractable way the surrender time. In the academic literature, the surrender time has traditionally been modeled as an optimal stopping time with respect to the filtration generated by the prices of the financial assets. We argue we should avoid modeling the surrender time in this way and propose an alternative model based on random times that admits a so-called hazard characterization. Technicalities set aside, the main difference with respect to the traditional model, is the way we model the insurer's and policyholder's information. Our model allows us to model explicitly, or at least implicitly, an asymmetry of information whereas the traditional model assumes a symmetric information.

We also study the impact of the absence of arbitrage hypothesis and show it implies a constraint on the distribution of the surrender time.

Keywords : Fair value, Surrender time, Life insurance, martingale measure, Enlargement of filtration, hazard process.

1 Introduction.

In this paper, we study the valuation of life insurance contracts subject to surrender risk in an arbitrage-free framework. Two broad approaches are usually distinguished in the literature. In the first one, the surrender time is defined exogenously whereas in the second one, it is endogenously defined. The first approach consists, roughly, in using a priori fixed probabilities for the surrender time without trying to find a structural explanation for the

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surrender decision. These probabilities can be defined like death probabilities are, that is through a (time-varying) deterministic instantaneous mortality (here, surrender) rate. In probability theory, these instantaneous rates are more commonly known as intensities. A very general form of this kind of models can be found in the literature on disability insurance. This approach has been mainly followed by practitioners but not really by academics. There are mainly two related reasons. The first one is the deterministic intensities used in these models imply the independence between the surrender time and the evolution of economic factors, which is obviously not a realistic assumption. The second reason is more fundamental. Contrary to dying or contracting a disease, the surrender is a decision taken freely by the policyholder. The surrender is thus not purely random, even though it can be influenced by random events. In order to answer to these drawbacks, the academic literature on fair valuation of insurance contracts has modeled the surrender time as an optimal stopping time with respect to the filtration generated by the prices of the financial assets (see Bacinello [2], [3], [4], [5] and Grosen and Jorgensen [10]). In the following, we will called this kind of optimal stopping time model, the traditional model because of its widespread acceptance in the academic literature. Albizzati and Geman [1] described also a kind of ad-hoc model where both approaches are melted together.

Our contribution is threefold. Firstly, we study the impact of introducing the surrender decision on the hypothesis of absence of arbitrage on the financial market and vice versa. We prove, under market completeness, the equivalence between the absence of arbitrage and what is called in probability theory, the (H)-hypothesis. We give the constraint this (H)-hypothesis implies on the distribution of the surrender time.

Secondly, we argue we should avoid using stopping times with respect to the filtration generated by the financial prices, if we are to model a realistic surrender time, mainly because it implicitly assumes the insurance company and the policyholder have the same set of information and assumes the surrender decision is accordingly based on this set only. On the contrary, we argue a policyholder takes his surrender decision on a larger set of information, which is in part, not available to the insurer. We propose an alternative model based on random times that admit a so-called hazard process. These random times allow us to model explicitly, or at least implicitly, an asymmetry of information. This class of random times is extremely large since essentially any random time not adapted to the filtration generated by the prices of financial assets, belongs to this class. We show this class of random times includes exogenously as well as endogenously defined surrender times, so that we can reconcile the two main approaches under the asymmetric information assumption. In particular, our model includes the deterministic intensity models as well as the stochastic intensity models which have been recently introduced in the fair valuation literature to model stochastic mortality (see Dahl [7]). In this last paper, the stochastic intensity is assumed to be Markovian, independent of the financial market and driven by a brownian motion. Our model does not rely on these simplifying assumptions. Actually, intensities does not even have to exist. Thirdly, we give general fair valuation formulas in our framework.

For the sake of simplicity, we assume there is no mortality risk.

This paper is organized as follows. In section 2, we present the theoretical framework we will use throughout this paper. In particular, in subsection 2.1, we present the financial market and in subsection 2.2, we describe the life insurance contract we are going to study. In subsection 2.3, we describe how is modeled the information shared by the policyholder and the insurer and give a general (risk neutral based) valuation formula for an arbitrary model of surrender time. In subsection 2.4, we show the equivalence between the absence of arbitrage and the (H)-hypothesis and explain the implications on the surrender time. In subsection 2.5, we present how the surrender time is modeled in the literature on fair valuation, we give its strengths and weaknesses and introduce our alternative model based on a hazard process. In subsection 2.6, we show how the general risk neutral valuation formulas can be simplified in our setting. In subsection 2.7, we study a particular model based on a brownian filtration. In section 3, we apply our framework to the valuation of a single premium unit-linked contract in a general semimartingale market model with stochastic term structure and stochastic surrender.

This paper is largely inspired by the financial literature on default risk models and owes a lot to Jeanblanc's and Rutkowski's paper [11].

2 The Theoretical Framework.

2.1 The Financial market.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A perfect frictionless financial market is defined on this space. We assume there is one locally risk free asset denoted $S_t^0 = S^0(t, \omega)$ and s risky assets $S_t^i = S^i(t, \omega), i = 1 \cdots s$ following real cádlág stochastic processes. The price of the locally risk free asset is assumed to follow a strictly positive, continuous path of finite variation process so that we can always find a continuous, finite variation process D such that $S_t^0 = e^{D_t}$. The discounted values of these assets are noted

$$X_t^i = \frac{S_t^i}{S_t^0}, i = 0, \cdots s$$

Even though this assumption is not necessary, it is often assumed there exists a positive predictable stochastic process $r_t = r(t, \omega)$ called the instantaneous risk free rate such that the value of the locally risk free asset evolves according to the following differential equation

$$dS_t^0 = r_t S_t^0 dt$$

In this case, we have $D_t = \int_0^t r_u du$. Let $\mathbb{F} = (\Im_t)_{0 \le t \le T}$ be the filtration generated by the stochastic processes S^i for $i = 0, \ldots s$. We assume this filtration respects the usual hypotheses. Intuitively, it models the information one has about the financial market as it evolves over time. In our context, we will assume this filtration models the information shared by the insurers and the policyholders.

We assume furthermore there is no arbitrage opportunity in this financial market with regard to the information set $\mathbb{F}$. This assumption is equivalent to the existence of at least one probability measure Q, called a local martingale measure, equivalent to P for which the discounted prices X_t^i, $i = 0 \cdots s$ are $(\mathbb{F}, Q)$-local martingales. According to Schweizer [17], this absence of arbitrage implies the discounted prices X^i follow $\mathbb{F}$-special semimartingales under P.

Finally, we assume this financial market is complete with regard to $\mathbb{F}$.

2.2 The Life Insurance Contract.

In life insurance, the traditional premium principle rests on the equivalence between the present value of the payments stream of the insurer and the present value of the payments stream of the policyholder. In this section, we describe the way we model these payments. Ultimately, the actual payoffs the insurer gets or pays, depend on the surrender decision. The date at which the surrender occurs is denoted τ. It is a strictly positive random variable ($\mathcal{F}$-measurable) on Ω. In other words, $\tau = \tau(\omega)$ is a random time. Up to now, no other assumption is made. Notice we do not assume τ is a $\mathbb{F}$-stopping time. In the following, we will use the following notation : t_0 will denote the initial date of the insurance contract, T the term of the contract and t an arbitrary date between t_0 and T. Without loss of generality, we will assume $t_0 = 0$.

For the sake of simplicity, we do not take the mortality into account.

2.2.1 Payments stream of the insurer.

We mainly need three building blocks. The first one is the payoff the insurer has to pay at the term T of the contract if the policyholder has not surrendered. This payoff is assumed to be a $\Im_T$-measurable random variable and is noted $g(T, \omega)$. At the term of the contract, the insurer has to pay

$$g(T, \omega)1_{\{\tau > T\}}$$

For a traditional participating life insurance contract, this could be simply the guaranteed capital. For an annuity contract, this could be the present value (at time T) of the annuities. For a unit-linked contract, this could be, for example, the value of the units bought, with or without a guarantee.

The second building block is the payoffs the insurer has to pay as long as the policyholder has not surrendered. We model these payoffs through their cumulated value up to time t, $C(t, \omega)$, assumed to be a right continuous increasing $\mathbb{F}$-adapted process. If we define $H_t = H_t(\omega) = 1_{\{\tau(\omega) \leq t\}}$, the cumulated payoff up to surrender is given by:

$$C(T, \omega)1_{\{\tau > T\}} + C(\tau_-, \omega)1_{\{t_0 < \tau \leq T\}} = \int_{t_0}^{T} (1 - H_u) dC(u, \omega)$$

where we assume $C(t_0, \omega) = 0$ and $C(T, \omega) = C(T_-, \omega)$. For example, this could be the payments of a constant or a time-dependent annuity. In this case and in most real-life contract, $C(\cdot, \omega)$ will be a.s. a step-wise function where the heights of the steps equal the values of the payments.

The third building block is the amount the policyholder gets back if he surrenders before the term T. This amount is assumed to be a bounded $\mathbb{F}$-predictable stochastic process. It's noted $R(\tau,\omega)$. We can also write it by

$$1_{\{t_0<\tau\leq T\}}R(\tau,\omega)=\int_{t_0}^{T}R(u,\omega)dH_u$$

For a traditional participating contract, the policyholder could be allowed, for example, to get back the premiums he has already paid, with a time-dependent penalty or not, and with or without a guaranteed interest rate on these premiums. For a unit-linked contract, he could for example, gets back the value of the units he has bought with or without penalty, or could be allowed to get back the premiums he has paid to buy these units.

The total liabilities of the insurer are then given by the sum of those three payoffs.

2.2.2 Payments stream of the policyholder.

The policyholder is committed to pay premiums periodically as long as he has not surrendered. This is similar to the second building block of the preceding subsection. We model these premiums through their cumulated value up to time t, $CP(t,\omega)$, assumed to be a $\mathbb{F}$-adapted right continuous increasing process. The cumulated payoff up to surrender is given by:

$$CP(T,\omega)1_{\{\tau>T\}}+CP(\tau_-,\omega)1_{\{t_0<\tau\leq T\}}=\int_{t_0}^{T}(1-H_u)dCP(u,\omega)$$

Where we assume $CP(t_{0-},\omega)=0$ and $CP(T,\omega)=CP(T_-,\omega)$. For the sake of simplicity, we'll usually consider these premiums are paid at N fixed discrete dates t_i with i=0...N-1. In this case, $CP(\cdot,\omega)$ is a step wise function. The premium paid at time t_i is noted $P(t_i,\omega)$ and is equal to $P(t_i,\omega)=CP(t_i)-CP(t_{i-})$. $P(t_i,\omega)$ is then a random variable assumed to be $\Im_{t_i}$-measurable. The payments stream of the insured becomes then

$$\sum_{i=0}^{N-1}P(t_i,\omega)1_{\{\tau>t_i\}}$$

This way of describing an insurance contract is fairly general. As we've seen, it can accommodate unit-linked products as well as traditional participating contracts.

2.3 Risk Neutral valuation.

In order to calculate the present values of these payoffs, we rely on the risk neutral valuation principle. In this setting, the value at date t of the liabilities of the insurer or of the policyholder, is given by the expected discounted payoffs under a risk neutral measure Q, conditionally on the information known at date t. Since our aim is to determine the fair value of an insurance contract from the point of view of an insurer, this expectation should be taken conditionally on the information detained by this insurer. In our context, this information occupies a central place. So, before proceeding with the valuation formulas, we first describe precisely how this information is modeled.

2.3.1 The Information.

We have already defined the process $H_t = 1_{\{\tau \leq t\}}$. Let us introduce a few other definitions.

Definition 1 *Let $\mathbb{H} = (\hbar_t)_{0 \leq t \leq T}$ be the filtration generated by the process H_t.*

The filtration $\mathbb{H}$ models the information an insurance company has over the fact one of its policyholder has already surrendered or not.

Definition 2 *Let $\mathbb{G} = (G_t)_{0 \leq t \leq T}$ be the filtration defined as $G_t = (\Im_t \vee \hbar_t)$ pour $0 \leq t \leq T$*

The filtration $\mathbb{G}$ models altogether the evolution of the information over the financial market and over the fact the policyholder has surrendered. $\mathbb{F}$ is thus a subfiltration of $\mathbb{G}$, i.e., $\mathbb{F} \subseteq \mathbb{G}$. We assume this is all the information an insurer has. In particular, it means the insurer has no private information about its policyholders except for the fact they have already surrendered or not. According to this assumption, $\mathbb{G}$ is thus the filtration under which we should take our expectations in our valuation formulas.

Notice that if we assume τ is an $\mathbb{F}$-stopping time, $\mathbb{H} \subset \mathbb{F}$. This means that it is sufficient to observe the financial market to know if one has surrendered or not. In this case, the filtration $\mathbb{H}$ does not offer any additional information and the filtrations $\mathbb{F}$ and $\mathbb{G}$ coincide. Finally, in any case, τ is a $\mathbb{G}$-stopping time and H_t is $\mathbb{G}$-adapted.

2.3.2 Present value of the insurer payment streams.

By applying the risk neutral valuation principle, we have the following proposition for the present value of the total liabilities.

Proposition 1 *The value at date t, L_t^C, of the total liabilities of the insurer is given by*

$$\begin{aligned} L_t^C &= E^Q[e^{-(D_T - D_t)} g(T,\omega) 1_{\{\tau > T\}} + \int_t^T e^{-(D_u - D_t)} (1 - H_u) dC(u,\omega) \\ &+ \int_t^T e^{-(D_u - D_t)} R(u,\omega) dH_u | G_t] \end{aligned} \tag{1}$$

Proof 1 *This is a straightforward application of the risk neutral valuation principle to the three building blocks described in the preceding section.*

2.3.3 Present value of the policyholder payment streams.

As far as the present value A_t of the liabilities of the policyholder is concerned, we have according to the risk neutral valuation principle, the following proposition :

Proposition 2 *The value at time t, A_t, of the liabilities of the policyholder is given by*

$$A_t = E^Q[\int_t^T e^{-(D_u - D_t)} (1 - H_u) dCP(u,\omega) | G_t]$$

If we assume the premiums are paid at N fixed discrete dates, we have :

$$A_t = E^Q[\sum_{i=\lceil t \rceil}^{N-1} e^{-(D_{t_i}-D_t)} P(t_i,\omega) 1_{\{\tau > t_i\}} | G_t] \tag{2}$$

where $\lceil t \rceil = inf\{i | t_i > t\}$

Proof 2 *Straightforward.*

In order to have a fair valued contract at the initial date t_0 of the contract, the present values of the liabilities of both parties must be equal : $L_{t_0} = A_{t_0}$. At a later date $t \geq t_0$, the difference $V_t = A_t - L_t$ gives the fair value of the contract for the insurer.

It's worth noticing we can imagine alternative ways of modeling the payment when one surrenders. In the model we described above, the payment is made at the exact time of surrender. But we can also consider the payment is made at a discrete time. For example, it is probably more realistic to consider the policyholder will get the amount R at the end of the week or the month during which he surrenders. Let's denote t_j the different time of payment with $j = 1 \cdots K$ and $t_K = T$. In this case, the payoff at time t_j is given by :

$$R(t_j, \omega) 1_{\{t_{j-1} < \tau \leq t_j\}}$$

The present value of the commitments of the insurer in the discrete case are then

$$\begin{aligned} L_t^D &= E^Q[e^{-(D_T - D_t)} g(T,\omega) 1_{\tau > T} + \int_t^T e^{-(D_u - D_t)} (1 - H_u) dC(u,\omega) \\ &+ \sum_{j=\lceil t \rceil}^{k} e^{-(D_{t_j} - D_t)} R(t_j,\omega) 1_{\{t_{j-1} < \tau \leq t_j\}} | G_t] \end{aligned}$$

This model can reveal to be a useful alternative to the one describe above. It should also be a good approximation of L_t^C as the intervals $]t_{j-1}, t_j]$ get shorter. We could also have discretized the process $C(\cdot,\omega)$ as we did for $CP(\cdot,\omega)$. Notice we distinguish the continuous time payment and the discrete time payment by the subscript C or D.

2.4 Enlargement of filtration and the (H) hypothesis.

In section 2.1, we assumed there is no arbitrage in the financial market with respect to the information set $\mathbb{F}$. This is a standard assumption in the financial literature. But in the last section, we actually made a stronger assumption: by using the risk neutral valuation principle, we assumed the existence of an equivalent measure under which the discounted prices X^i are $(\mathbb{G}, Q)$-local martingales. Accordingly, we implicitly assumed there is no arbitrage, not only, under the filtration $\mathbb{F}$ but also under the new filtration $\mathbb{G}$. Is this a reasonable economic assumption ? Does the observation of surrenders should affect the hypothesis of absence of arbitrage in the financial market ? We think the answer to this question is obvious: we cannot expect the observation of surrenders will give rise to arbitrage in the financial market if there isn't initially. Accordingly, the discounted prices X^i should also be $(\mathbb{G}, Q)$-local martingales. As we can see the fact that local martingales under a certain filtration are still local martingales under an enlarged one, is closely related to the fact that there is no arbitrage if we increase our information set from an initial filtration to an enlarged one.

At this point, we must be cautious. Indeed, in general, going from an initial filtration $\mathbb{F}$ to an enlarged one $\mathbb{G}$, $(\mathbb{F}, Q)$-local martingales are not necessarily $(\mathbb{G}, Q)$-local martingales. Accordingly, the absence of arbitrage under the filtration $\mathbb{F}$ does not necessarily imply the absence of arbitrage under the filtration $\mathbb{G}$. In the following, we derive necessary and sufficient condition for the absence of arbitrage assumption to hold under $\mathbb{G}$. As we'll see, it implies a constraint on the distribution of the random time τ.

Let's first recall a definition and a couple of results.

Definition 3 ***: (H)-Hypothesis***
Let $(\Omega, \mathcal{F}, Q)$ be a probability space. Let $\mathbb{G}$ be a filtration and $\mathbb{F}$ be an arbitrary sub-filtration of $\mathbb{G}$ i.e. for every t, $\Im_t \subseteq G_t$.
The (H)-hypothesis is enounced as : for every t, $\Im_\infty$ and G_t are conditionally independent with respect to $\Im_t$

The (H)-hypothesis is, in general, equivalent to the invariance of (local) martingales for $\mathbb{F}$ and $\mathbb{G}$. More precisely, according to Brémaud and Yor [6], we have

Lemma 1 *The following assertions are equivalent*

1. *The (H)-hypothesis is verified.*
2. *Every $(\mathbb{F}, Q)$-local martingale is a $(\mathbb{G}, Q)$-local martingale.*

In particular, in our setting, we have the following Lemma

Lemma 2 *The (H)-hypothesis is equivalent to*

$$Q(\tau \leq t|\Im_\infty) = Q(\tau \leq t|\Im_t) \ \forall t$$

See [11] for a proof.
We can now prove the relations between the (H)-hypothesis and the absence of arbitrage under $\mathbb{G}$.

Proposition 3 *If there is no arbitrage in our financial market under $\mathbb{F}$ and $Q(\tau \leq t|\Im_\infty) = Q(\tau \leq t|\Im_t)$ $\forall t$, then there is no arbitrage under $\mathbb{G}$.*

Proof 3 *Since there is no arbitrage under $\mathbb{F}$, there exists an equivalent (to P) measure Q such that the discounted prices X^i are $(\mathbb{F}, Q)$-local martingales. By lemma 1 and 2, if Q is such that $Q(\tau \leq t|\Im_\infty) = Q(\tau \leq t|\Im_t)$, these discounted prices X^i are also $(\mathbb{G}, Q)$ local martingales so that our financial market is arbitrage free under $\mathbb{G}$.*

The following proposition gives almost the converse.

Proposition 4 *Assume our financial market is arbitrage free and complete under $\mathbb{F}$. If there is no arbitrage in the financial market under $\mathbb{G}$ and let's denote by Q an arbitrary local martingale measure under $\mathbb{G}$ (equivalent to P) then $Q(\tau \leq t|\Im_\infty) = Q(\tau \leq t|\Im_t)$ $\forall t$.*

Proof 4 *Since the market is assumed arbitrage free under $\mathbb{F}$ and $\mathbb{G}$, there exists martingale measures R and Q such that the discounted prices X are $(\mathbb{F}, R)$-local martingales and $(\mathbb{F}, Q)$-local martingales.*

We first show Q and R coincide on $\mathbb{F}$. Let's denote the restrictions on G_t and $\Im_t$ of the Radon Nikodym derivatives of Q and R by the following stochastic processes

$$\varepsilon_t^{Q,G} = \frac{dQ}{dP}_{|G_t}, \varepsilon_t^{Q,F} = \frac{dQ}{dP}_{|\Im_t} = E^P[\varepsilon_t^{Q,G}|\Im_t], \varepsilon_t^{R,F} = \frac{dR}{dP}_{|\Im_t}$$

In order to prove, Q and R coincide on $\mathbb{F}$, we have to prove $\varepsilon_t^{Q,F}$ and $\varepsilon_t^{R,F}$ are indistinguishable or equivalently $\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}}$ and 1 are indistinguishable.
Notice $\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}}$ is a $(R, \mathbb{F})$ martingale. Indeed we have :

$$E^R[\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}}|\Im_s] = E^P[\frac{\varepsilon_t^{R,F}\varepsilon_t^{Q,F}}{\varepsilon_s^{R,F}\varepsilon_t^{R,F}}|\Im_s] = \frac{\varepsilon_s^{Q,F}}{\varepsilon_s^{R,F}}$$

Let U be an arbitrary $\mathbb{F}$-stopping time.

On one hand, we have :

$$E^R[(\frac{\varepsilon_U^{Q,F}}{\varepsilon_U^{R,F}})^2] \geq E^R[\frac{\varepsilon_U^{Q,F}}{\varepsilon_U^{R,F}}]^2 = (\frac{\varepsilon_0^{Q,F}}{\varepsilon_0^{R,F}})^2 = 1$$

Thanks to Jensen's inequality and the fact $\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}}$ is a $(R, \mathbb{F})$-martingale.

On the other hand, we have :

$$\begin{aligned} E^R[(\frac{\varepsilon_U^{Q,F}}{\varepsilon_U^{R,F}})^2] &= E^R[\frac{\varepsilon_U^{Q,F}}{\varepsilon_U^{R,F}}\frac{E^P[\varepsilon_U^{Q,G}|\Im_U]}{\varepsilon_U^{R,F}}] \\ &= E^P[\frac{\varepsilon_U^{Q,F}}{\varepsilon_U^{R,F}}E^P[\varepsilon_U^{Q,G}|\Im_U]] \\ &= E^P[\frac{\varepsilon_U^{Q,F}}{\varepsilon_U^{R,F}}\varepsilon_U^{Q,G}] \end{aligned}$$

By definition of market completeness, the set of processes X under R, has the predictable representation property : any $(\mathbb{F}, R)$-local martingales can be represented as a stochastic integral with respect to the $(\mathbb{F}, R)$-local martingales X for a locally bounded $\mathbb{F}$-predictable integrand. Since $\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}}$ is a $(R, \mathbb{F})$ martingale, it has such a representation i.e. we have

$$\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}} = \frac{\varepsilon_0^{Q,F}}{\varepsilon_0^{R,F}} + \int_0^t \varphi_u dX_u \quad (3)$$

for a locally bounded $\mathbb{F}$*-predictable process* φ*. According to [15] theorem 33 chapter IV, if H is a locally bounded* $\mathbb{F}$*-predictable process and X a semimartingale for* $\mathbb{F}$ *and* $\mathbb{G}$ *then* φ *is a locally bounded* $\mathbb{G}$ *predictable process and the stochastic integrals* $H_{\mathbb{F}} \cdot X$ *and* $H_{\mathbb{G}} \cdot X$ *exist and are equal. Accordingly,* $\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}}$ *is still given, under* $\mathbb{G}$*, by the same stochastic integral (3). Moreover since the market under* $\mathbb{G}$ *is arbitrage free,* $\varepsilon_t^{Q,G}$ *is* $(P, \mathbb{G})$*-orthogonal to X and so it is to any stochastic integrals with respect to X. Accordingly,* $\varepsilon_t^{Q,G}$ *and* $\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}}$ *are* $(P, \mathbb{G})$*-orthogonal. By definition of orthogonality,* $\frac{\varepsilon_t^{Q,F}}{\varepsilon_t^{R,F}}\varepsilon_t^{Q,G}$ *is a* $(P, \mathbb{G})$*-local martingale. Furthermore, Since it is even a strictly positive process, it is also a* $(P, \mathbb{G})$*-supermartingale. So we have*

$$E^P[\frac{\varepsilon_U^{Q,F}}{\varepsilon_U^{R,F}}\varepsilon_U^{Q,G}] \leq \frac{\varepsilon_0^{Q,F}}{\varepsilon_0^{R,F}}\varepsilon_0^{Q,G} = 1$$

Since both inequality holds for any $\mathbb{F}$*-stopping time,* $\varepsilon_t^{Q,F}$ *and* $\varepsilon_t^{R,F}$ *are indeed indistinguishable.*

We can now prove the (H)-hypothesis is verified. Since Q and R coincide on $\mathbb{F}$*, the set of processes X inherit the predictable representation property under Q on* $\mathbb{F}$ *: any* $(Q, \mathbb{F})$*-local martingales M can be represented as a stochastic integral with respect to the* $(Q, \mathbb{F})$*-local martingales X for a locally bounded* $\mathbb{F}$*-predictable process. As we just explained, M is also equal to this stochastic integral under* $\mathbb{G}$ *. Since X are* $(\mathbb{G}, Q)$*-local martingales, M is also a* $(\mathbb{G}, Q)$*-local martingales. The (H)-hypothesis is then verified thanks to lemma 1.*

Thanks to lemma 3 and 4, if we accept the market completeness under $\mathbb{F}$, the (H)-hypothesis is equivalent to the absence of arbitrage. Notice though, lemma 3 does not require market completeness, the (H)-hypothesis is thus a sufficient condition for the no arbitrage condition even under market incompleteness. The necessity of the (H)-hypothesis under market incompleteness is still to prove.

Actually, in all generality, this (H)-hypothesis is not necessarily invariant if we go from the measure Q to an equivalent one, like the physical measure P for example, see [11] for a counter example due to Kusuoka. If we nevertheless assume this condition is also true under the physical measure P, we have $P(\tau \leq t|\Im_\infty) = P(\tau \leq t|\Im_t)$ $\forall t$, then in our setting, this condition is intuitively very clear : it tells us the information over the future evolution of the financial market (after t) does not give us more information on the surrender before t than does the information on the financial market up to time t. As a consequence, It also tells us that the fact that one surrenders will not affect the evolution of the financial market. This implication is very natural. Accordingly, even though, strictly speaking, the (H)-hypothesis is only crucial under Q, it seems to be harmless to equally impose this (H)-hypothesis under P.

Notice when τ is independent of $\mathbb{F}$, this (H)-hypothesis is directly respected since in this case, $Q(\tau \leq t|\Im_\infty) = Q(\tau \leq t) = Q(\tau \leq t|\Im_t)$ $\forall t$. We have this assumption when the surrender time is exogenously defined as described in the introduction. This assumption appears also in the stochastic mortality literature. The arguments of this section shows as expected, the mortality cannot induce any arbitrage opportunity, under this independence assumption.

Finally, when τ is an $\mathbb{F}$-stopping time, $\mathbb{F}$ and $\mathbb{G}$ coincide so that there is no enlargement of filtration. The risk neutral valuation framework can then be use safely. Indeed, in this case, the condition $Q(\tau \leq t|\Im_\infty) = Q(\tau \leq t|\Im_t)$ is trivially respected.

2.5 The surrender models.

Up to now, we have not yet made any assumption on the date τ at which the surrender occurs. The formulas given above are all very general and do not depend on the way we model τ. In the first subsection, we describe how τ is traditionally modeled in the literature on fair valuation of insurance contracts (see [2], [3], [10]). We give the cons and pros. In the second subsection, we present an alternative model based on a $\mathbb{F}$-hazard characterization of τ and explain how it could answer to the drawbacks of the traditional model.

2.5.1 τ as an optimal $\mathbb{F}$-stopping time.

The literature on fair valuation of insurance contracts has relied mainly on the following definition of τ :

$$\tau = inf\{t|R(t,\omega) \geq -V(t,\omega)\}$$

where $V(t,\cdot) = A_t - L_t$ is the fair value of the insurance contract at time t if one hasn't surrendered before t and doesn't surrender at time t. This definition comes from the optimal stopping time literature, in general, and from the financial literature on the optimal exercise time of American style options, in particular. This definition tells us it's optimal for the policyholder to surrender when the fair value of his contract is inferior or equal to the amount he can get by immediately surrendering this contract.

At first, this definition seems very appealing from a theoretical point of view. By defining the surrender behavior in this way, we endogenously determine the time of surrender. Indeed, this surrender time actually pops up from the very characteristics of the insurance contract and in the meantime, depends intrinsically on the evolution of the financial market. Unfortunately, a second look reveals also some drawbacks.

Theoretical implications. The most obvious drawback is simply, it doesn't realistically model a policyholder's surrender behavior.

Firstly, since τ is defined as an optimal stopping time, the policyholder is assumed to be able to process all the information, to make all the required complex calculations and is able to act in a perfectly rational manner. Obviously, this is not the way people behave, not even probably the best actuaries.

Secondly, policyholders are not only assumed to act rationally but are also assumed to take their decisions on exactly the same set of information $\mathbb{F}$. As a consequence, it means, for a given generation, all the policyholders would behave as one : we would observe a period without any surrender and suddenly everybody would surrender at the exact same time. Even though, the surrender decisions among different policyholders are indeed correlated in the real world, a perfect correlation is not realistic. This also means no room is left for idiosyncratic information that could trigger the decision of surrender. Notice, this critique has nothing to do with the fact τ is defined as "optimal" but does come from the fact that τ is an $\mathbb{F}$-stopping time. Notice also, the information $\mathbb{F}$ is assumed to be known by the insurer so that this definition of τ does not allowed for any asymmetry of information between the policyholders and the insurer.

Thirdly, since to our knowledge, all the continuous-time literature assumes $\mathbb{F}$ to be the brownian filtration, τ is then even a $\mathbb{F}$-predictable stopping time. This means intuitively that, at each time t, the insurance company is able to predict if its policyholders are going to surrender or not during the next small interval of time dt. So actually, there is never an unexpected surrender. Obviously, in the real world, an insurance company cannot perfectly predict if someone is going to surrender or not, because the decision is taken, in part, on information that are not accessible to the insurance company (even though the evolution of the financial market plays a role in the decision and can give the insurer a valuable information on the probability of surrender). One could argue we could overcome this "predictability" problem simply by using a model of financial prices with an unpredictable component, such as a Lévy process with jumps for example. In this case, we would have indeed unpredictable surrenders while keeping τ a $\mathbb{F}$-stopping time. The trouble with this approach is the "unpredictability" of the surrender would come from the characteristics of the financial market and not from the surrender decision itself.

Fourthly, when τ is a $\mathbb{F}$-stopping time, if the financial market is complete, so is the insurance market. It gives the false impression that the surrender risk can be, at least theoretically, perfectly hedge away and therefore, that the surrender is not really risky for an insurance company. As we said, it misses the point that surrender is by and large an unpredictable event and that for this reason, surrender is risky for an insurance company. From a risk management point of view, it is clear we cannot rely and take decisions on this kind of model. For pricing purposes, one could still maybe argue all these drawbacks are really no big deal since such a definition of τ still gives us an upper bound of the fair value of a life insurance contract. The trouble is using different models for pricing and risk management, that are by nature, not consistent with each other, is probably not very appropriate to manage an insurance company.

Finally, the value $V(t,\omega)$ that we compare to the surrender value $R(t,\omega)$, should be the value of the contract from the policyholder perspective. Unfortunately, most of the time, implicitly, the value V used in the literature is the value of the contract from the insurer perspective (see also Bacinello [5] for a discussion of this point). The value of the contract from the policyholder perspective depends actually on his own information and its own risk aversion. It is thus a subjective value. By definition, a subjective value is different from a policyholder to another and is unknown for the insurer. Accordingly, this argument implies once again the existence of an asymmetry of information which is really the point of this paper.

In conclusion, contrary to the traditional model of τ, we argue the surrender decision should be random with respect to the shared information $\mathbb{F}$, even though it shouldn't be independent of it. This information should actually influenced the randomness of the surrender decision. This leads to the conclusion the $\mathbb{F}$-measurability of the surrender decision is really the crucial point. In other words, in order to model realistically this decision, we argue we should avoid the hypothesis that τ is an $\mathbb{F}$-stopping time.

Practical implications. Up to now, we have focused our attention on the theoretical implications of defining τ as an optimal $\mathbb{F}$-stopping time. But we can also mention some drawbacks related to the numerical techniques usually used to price these contracts. Actually, the pricing of an American option is a very difficult problem. Even in the simplest situation, the Black-Scholes financial market and a plain-vanilla standard American put option, no closed form solution is known. In order to find the price of an American option, we usually rely on one of the following numerical techniques : the binomial trees (and its extensions like the trinomial trees) or the finite difference methods, thanks to the fact we work backward in time in both cases. Monte Carlo is actually not very well suited since it works forward. Unfortunately, the binomial trees as well as the finite difference methods have their own problems.

Firstly, they cannot easily deal with multiple sources of uncertainty. The trouble is the most interesting and promising financial models includes nowadays multi-factors models of the term structure, stochastic volatility models or jump-diffusion models for stock prices, etc.... For example, we can indeed easily construct a binomial tree when we consider a single stock which follows a geometric brownian motion, and a constant interest rate, but it becomes rapidly much more involved if we want to study a slightly more realistic financial market model. And so it is with the finite-difference method as well. The trouble is, in our opinion, to obtain valuable results, we need in life insurance, to rely on more sophisticated financial market models. The reason is a life insurance contract is basically a bundle of long term options and it's well known that for long term options, assuming constant interest rate has no more, in contrast with short term options, a second order effect. In the Black and Scholes equation, the volatility we should consider isn't the volatility of the stock but the volatility of the forward price which includes the volatility of a zero-coupon bond with time to maturity equals to the term of the options. For long term options, this last volatility is no more negligible and can make a big difference on the option value. Moreover, as far as the traditional participating life insurance contracts are concerned, the evolution of the yield curve should be, intuitively, the main economic driver of the surrender decision. It is because the interest rates move, that it becomes valuable for the policyholder to surrender or not. Assuming constant interest rate, we entirely miss this crucial effect and the results obtained are at the very least, doubtful. To conclude, for traditional participating contracts as well as unit-linked contracts, we argue it is crucial to use sophisticated and realistic multi-factor models of the term structure. Moreover, as far as the unit-linked contracts are concerned, we think it's time to go beyond the basic Black and Scholes model and try to introduce models with fat-tails distributions like the stochastic volatility models or Levy processes models. All these models are unfortunately not easily dealt with by binomial trees or finite difference methods.

Secondly, binomial trees and finite difference method cannot easily deal with path-dependent payoffs. But insurance contracts have notoriously very complex payoffs which comes from the very exotic path-dependent options embedded in these contracts. Monte-Carlo method would be welcome to help us to deal with these complex options but as we said, it is here not suited due to the American characteristic of the surrender option.

To conclude, the binomial tree is a simple method that is very well suited for simple market model but is very difficult to use in more involved financial models and payoffs, which are actually the interesting ones.

In the next subsection, we're going to present a framework that aims simultaneously at modeling the surrender behavior in a more realistic manner and at offering in some special cases, potentially more tractable expressions for the value of insurance contracts.

2.5.2 τ as a random time characterized by a $\mathbb{F}$-hazard process.

In the preceding subsection, we saw that using optimal stopping times to model the surrender decisions can be very cumbersome if not intractable in most interesting cases. Furthermore, we saw it does not model realistically neither the behavior of the policyholder nor the unpredictable component of the surrender decision for the insurance company (at least when $\mathbb{F}$ is the brownian filtration). We also saw it was crucial to avoid to define τ as an $\mathbb{F}$-stopping time, but it was also crucial to allow the probability of surrender to depend on the information $\mathbb{F}$. In this subsection, we present an alternative model of τ which is here described as a random time that admits a characterization by a $\mathbb{F}$-hazard process.

Let's first introduce our definition of τ and the definition of the associated hazard process.

Definition 4 *Let τ be a non negative random variable defined on $(\Omega, \mathcal{F}, P)$. Assume $P(\tau = 0) = 0$ and $P(\tau > t) > 0, \forall t \in \mathbb{R}_+$. We note $F_t = P(\tau \leq t | \Im_t)$, for all $t \in \mathbb{R}_+$.*

We define the $(\mathbb{F}, P)$-hazard process of τ, denoted by Γ by

$$F_t = 1 - e^{-\Gamma_t}$$

for all $t \in \mathbb{R}_+$.

In order to have Γ_t well-defined for all t, we have to impose the condition $F_t < 1$ because $\Gamma_t = -ln(1 - F_t)$. This excludes τ of being a $\mathbb{F}$-stopping time. In other words, a $\mathbb{F}$-stopping time cannot admit a $\mathbb{F}$-hazard process characterization. Notice that F_t is a $\mathbb{F}$-adapted stochastic process and admits a right continuous modification. Accordingly, Γ_t is also a $\mathbb{F}$-adapted càdlàg stochastic processes with $\Gamma_t < \infty$ $\forall t$ and $\Gamma_0 = 0$.

As already explained, when τ is an $\mathbb{F}$-stopping time, we can tell from the observation of the financial market if a policyholder has already surrendered or not. By modeling τ as defined above, we excludes this situation. Here, the surrender decision depends necessarily on elements outside the information $\mathbb{F}$ shared by the insurer and the policyholder. These elements are idiosyncratic information or events and can be different from a policyholder to another (unless the surrender decision would still be perfectly correlated). Accordingly, in defining τ as above, we implicitly model an asymmetry of information between the policyholder and the insurer. For the same reason, the surrender decision is indeed unpredictable for the insurer, even when $\mathbb{F}$ is a brownian filtration. This unpredictability here does not come from the characteristics of the financial market but from the asymmetry of information itself. Notice that even though the surrender decision is unpredictable, the probability of surrender is stochastic and depends on the evolution of the financial market (through Γ_t). By defining τ in this way, we indeed answer to the first set of critiques formulated against the traditional modeling of the surrender decision, while retaining one of its key aspects.

Examples. Let us now give some examples of such random times.

1. Probably the most simple random time having a hazard characterization is the time of the first jump of an homogeneous poisson process N_t, independent of $\mathbb{F}$, with constant intensity $\lambda > 0$. Indeed we have :

$$\begin{aligned} F_t &= 1 - P(\tau > t | \Im_t) \\ &= 1 - P(\tau > t) \\ &= 1 - P(N_t = 0) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

   The hazard process is here an increasing positive deterministic function, equals to $\Gamma_t = \lambda t$.

2. The time of the first jump of inhomogeneous poisson process independent of $\mathbb{F}$, with hazard function $\Lambda(t)$ (a deterministic cádlág function), is also such a random time.

$$\begin{aligned} F_t &= 1 - P(N_t = 0) \\ &= 1 - e^{-\Lambda(t)} \end{aligned}$$

   Again, the hazard process Γ is here an increasing positive deterministic function $\Gamma_t = \Lambda(t)$. More formally, this function $\Lambda(t)$ is actually a measure defined on the measurable space $(\mathbb{R}, \mathbb{B})$ where $\mathbb{B}$ is the Borel σ-algebra on $\mathbb{R}$. If this measure is absolutely continuous with regard to the Lebesgue measure, we can find the intensity $\lambda(t)$ of the poisson process, which is a positive function, given by

$$\Lambda(t) = \int_0^t \lambda(u) du$$

3. The definition of τ given above also includes the time of the first jump of a Cox process. This process is a counting process that generalizes the Poisson process to the case where the measure Λ is a random measure on $(\mathbb{R}, \mathbb{B})$. Such a counting process N_t with a random measure is also known as a doubly stochastic poisson process. We can define a Cox process in the following manner. N_t follows a Cox process if and only if conditionally on $\Im_t$, N_t follows an inhomogeneous Poisson process for all t.

   The hazard function of the inhomogeneous process can be given, conditionally on $\Im_t$, by the realization of an increasing positive cádlág $\mathbb{F}$-adapted stochastic process Λ_t. Since Λ_t is assumed $\mathbb{F}$-adapted, Λ_t is indeed, conditionally on $\Im_t$, a deterministic function. Furthermore, since each path is a.s. increasing, cádlág and positive, Λ_t induced, conditionally on $\Im_t$, a deterministic measure on $(\mathbb{R}, \mathbb{B})$ and unconditionally, a random measure on the same space. See Grandell [9] for a more formal description.

   According to this definition, conditionally on $\Im_t$, the distribution of the time of the first jump of a Cox process is given by

$$\begin{aligned} F_t &= 1 - P(N_t = 0 | \Im_t) \\ &= 1 - e^{-\Lambda_t} \end{aligned}$$

Once again, this random time admits an increasing, positive $\mathbb{F}$-hazard process Γ_t given by $\Gamma_t = \Lambda_t$. Unconditionally, we also have

$$\begin{aligned} P(\tau \leq t) &= E^P[1 - e^{-\Lambda_t}] \\ &= 1 - E^P[e^{-\Lambda_t}] \end{aligned}$$

If Λ_t is absolutely continuous with respect to the Lebesgue measure, it exists a unique positive $\mathbb{F}$-adapted predictable process λ_t called the $(\mathbb{F}, P)$-intensity process of N, such that

$$\Lambda_t = \int_0^t \lambda_u du$$

A few elements are worth underlining. Firstly, in the two first examples, the hazard process is a deterministic function and τ cannot depend on the financial market. In the case of the Cox process, the hazard process is a stochastic process which allows the probability of the surrender time to depend on the evolution of the financial market.

Secondly, the examples of random times given here are, all three, the time of the first arrival of a counting process. But we'd like to highlight that these are just special cases of our definition and do not represent all the random times that admit a hazard process. Actually, as long as we construct a random time with $P(\tau \leq t|\Im_t) \neq 1$, we can find a process Γ_t.

Thirdly, in these three examples, the hazard functions or the hazard process are increasing since each three induces a (deterministic or random) measure on the Borel sets. Our definition of τ does not depend on this assumption. As we said, these examples are just special cases of our definition and in general we can find random times admitting positive non necessarily increasing hazard process. So, Γ_t increasing is not a necessary assumption. However, the (H) hypothesis, we've seen in section 2.4, implies the process $\hat{F}_t$ defined as $\hat{F}_t = Q(\tau \leq t|\Im_t)$, is an increasing process. The $(\mathbb{F}, Q)$-hazard process $\hat{\Gamma}_t$ defined as $\hat{\Gamma}_t = -ln(1 - \hat{F}_t)$ similarly to the definition 4, is then also an increasing process and if $\hat{\lambda}_t$ defined as $\hat{\Gamma}_t = \int_0^t \hat{\lambda}_u du$, exists, then it is a positive process. So, **$\hat{\Gamma}_t$ increasing is a necessary condition for constructing τ but Γ_t increasing, is not** since as we said, the (H) hypothesis is not necessarily true under P and has not to be[1]. This justify the fact we can focus, under Q, on Cox processes to model the surrender time.

Construction of τ. The hypothesis made on τ in our definition, are very weak and allows us to construct τ in a great variety of ways. Actually, the two broad approaches described in the introduction (the exogenous and endogenous one) are compatible with our definition of τ.

The first one consists in describing explicitly the surrender behavior and the asymmetry of information between the policyholder and the insurance company. In other words, it consists in describing the surrender from the policyholder point of view. For example, we can generalize the traditional modeling of surrender time. We saw the decision of surrender was taken on a larger set of information that includes idiosyncratic information. Let's denote

[1]In section 2.4, we argued it is probably a reasonable and harmless assumption to assume the (H) hypothesis is respected under P too. In this case, Γ_t is obviously also an increasing process.

this policyholder's information set by a filtration $\mathbb{J}$ with $\mathbb{F} \subset \mathbb{J}$. We could consider τ to be an $\mathbb{J}$-optimal stopping time instead of a $\mathbb{F}$-optimal stopping time. For example, we could model a stochastic mortality which would depend on information related to the health of the policyholder and only known by himself and not the insurer. In this case, we could still defined τ as $\tau = inf\{t|R(t,\omega) \geq -V(t,\omega)\}$ but where $-V(t,\omega)$ is the value of the insurance contract for the policyholder but, this time, given his own information $\mathbb{J}$. In this case, τ would be endogenously defined but still not $\mathbb{F}$-measurable. Accordingly, we could then derive a probability of surrender $\hat{F}_t \neq 1 \; \forall \; t$ such that $\hat{\Gamma}_t = -ln(1-\hat{F}_t)$ has a sense. Even if this model would probably be too complex to deal with, it's interesting to realize we can introduce a generalization of the traditional model in our setting. Another simpler example of this approach is the following. Let assume one surrenders when the value of a given financial asset S_t reaches a certain level, E. This level at which one surrenders, is specific to each policyholder and is unknown by the insurer. The insurance contract portfolio is then heterogenous with respect to this level. Finally, let assume this level follows a unit exponential distribution under Q and that E is independent of $\mathbb{F}$. In this case, we have $\hat{F}_t = 1 - e^{-sup_{0<s\leq t} S_s}$ so that the $(\mathbb{F}, Q)$-hazard process is given by $\hat{\Gamma}_t = sup_{0<s\leq t} S_s$ which is indeed $\mathbb{F}$-adapted, cádlág, positive and increasing. Our point in giving these examples is to show our definition allows a structural model of τ or in other words, allows an endogenous definition of the surrender time.

The second approach considers the insurer point of view and is closer to an exogenous specification of τ. It simply consists in specifying directly the process followed by $\hat{\Gamma}$ without necessarily trying to describe explicitly the actual surrender behavior nor the asymmetry of information. In this approach, we implicitly assume it's better to try to find a robust probabilistic description of the surrender time than trying to explicitly model an elusive human behavior. One way to construct the random time τ given an hazard process, is the following. Let E be a random variable ($\mathcal{F}$-measurable) defined on Ω, but which is independent of the filtration $(\Im_t)_{0\leq t\leq T}$. E is assumed to have an unit exponential distribution under Q. Let $\hat{\Lambda}$ be an $\mathbb{F}$-adapted increasing positive process with $\hat{\Lambda}_0 = 0$ and $\hat{\Lambda}_t < \infty$. If τ is defined as

$$\tau = inf\{t|\hat{\Lambda}_t \geq E\}$$

then τ is a random time with a $\mathbb{F}$-hazard process under Q given by $\hat{\Gamma} = \hat{\Lambda}$. Indeed, we have :

$$\begin{aligned} Q(\tau \leq t|\Im_t) &= Q(\hat{\Lambda}_t \geq E|\Im_t) \\ &= 1 - e^{-\hat{\Lambda}_t} \end{aligned}$$

Alternatively, we can also start with a positive $\mathbb{F}$-adapted predictable stochastic process $\hat{\lambda}$ and define τ as

$$\tau = \inf\{t| \int_{u=0}^{t} \hat{\lambda}_u du \geq E\}$$

In this case, τ is a random time with $(\mathbb{F}, Q)$-hazard process given by $\hat{\Gamma}_t = \int_{u=0}^{t} \hat{\lambda}_u du$ and with a $(\mathbb{F}, Q)$-intensity process equals to $\hat{\lambda}$. Indeed, in both cases, the random times constructed can be seen as the time of the first jump of Cox processes. Both of these constructions also offer an easy way to simulate the random time τ. We simply have to simulate the hazard process and an independent exponential random variable. The two methods described above are standard constructions which have the advantage of being very convenient but we can go further at the cost of constructing random times less tractable. For example, even if $\hat{\Lambda}_t$ is not an increasing process, we can still define τ in the same way. Let $\hat{\Lambda}$ be an $\mathbb{F}$-adapted positive process with $\hat{\Lambda}_0 = 0$. We can define $\tau = inf\{t|\hat{\Lambda}_t \geq E\}$. In this case, we have

$$\begin{aligned} Q(\tau \leq t|\Im_t) &= Q(\sup_{0<s\leq t} \hat{\Lambda}_s \geq E|\Im_t) \\ &= 1 - e^{-sup_{0<s\leq t}\hat{\Lambda}_s} \end{aligned}$$

The random time τ admits a $(\mathbb{F}, Q)$-hazard process characterization with the hazard process $\hat{\Gamma}_t = sup_{0<s\leq t}\hat{\Lambda}_s$. This random time can still be seen as the time of the first jump of a Cox process since $sup_{0<s\leq t}\hat{\Lambda}_s$ is an increasing cádlág positive $\mathbb{F}$-adapted process. Obviously, this form is less tractable than the other two, but can be sometime justified. Notice the second example of the first approach can be seen as an application of this last construction. Accordingly, the 2 approaches we gave, are not necessarily antagonist.

In practice, one would probably prefer to specify directly $\hat{\lambda}$ instead of $\hat{\Gamma}$ since the first one has an intuitive meaning the second lacks. It is well known that the intensity parameter λ or the intensity function $\lambda(t)$ of a Poisson process corresponds to what is known in life insurance as the instantaneous mortality rate. When it exists, we have the well-known following expression :

$$\lambda(t) = -\frac{d\ln(1 - P(\tau \leq t))}{dt} = \lim_{\Delta t \to 0} \frac{P(t < \tau \leq t + \Delta t|\tau > t)}{\Delta t}$$

So informally, $\lambda(t)dt$ is equal to the probability one surrenders during the next interval dt if one hasn't yet surrendered before time t. This conditional probability is here deterministic for all t. When it exists, we have a similar result for the $(\mathbb{F}, Q)-$intensity process

$$\hat{\lambda}_t = \lim_{\Delta t \to 0} \frac{Q(t < \tau \leq t + \Delta t|\tau > t \cup \Im_t)}{\Delta t}$$

So informally, $\hat{\lambda}_t dt$ is again equal to the probability one surrenders, under Q, during the next interval dt if one has not surrendered before time t and knowing the information $\Im_t$. Obviously, this probability is stochastic and for a given time t, its value depends on the financial market. To model this instantaneous probability, we can make an explicit reference to the financial asset prices, by specifying $\hat{\lambda}$ as a function of these prices. In most applications, in order to get tractable formulas, we'll restrict ourself to markovian intensity processes. Let $\hat{\lambda}(\cdot, \cdot)$ be a function from $\mathbb{R} \times \mathbb{R}^{s+1} \to \mathbb{R}_+$. In this case, we'll define the intensity process of τ as:

$$\hat{\lambda}_t(\omega) = \hat{\lambda}(t, S_t(\omega))$$

Notice this stochastic process is indeed $\mathbb{F}$-adapted.

To conclude, our definition of τ as a random time characterized by a $(\mathbb{F}, Q)$-hazard process allows us to specify the surrender time in a great variety of way and seem to be a valuable alternative to the traditional modeling. Notice we constructed the distribution of τ directly under Q through the specification of $\hat{\Gamma}$. To find an explicit functional between $\hat{\Gamma}$ and Γ is not easy task in full generality. In section 2.7, we give a simple functional under certain simplifying assumptions.

2.6 Risk Neutral Valuation Formula With an $\mathbb{F}$-hazard process

In this section, we show how the risk neutral valuation formulas of sections 2.3.2 and 2.3.3 can be modified when we assumed τ is characterized by an $\mathbb{F}$-hazard process or an $\mathbb{F}$-intensity process. In the first subsection, we introduce a number of useful results that allows us to get rid of the indicator function by taking conditional expectations with respect to $\Im_t$ instead of G_t. In the second and third subsections, we apply these formulas to our valuation problem.

2.6.1 Conditional expectations with respect to $\Im_t$.

In this subsection, we assume the random time τ admits an $\mathbb{F}$-hazard process Γ under the probability measure considered.

Lemma 3 *Let X be a $\Im_s$-measurable random variable with $s \geq t$. We have*

$$E[1_{\{\tau>s\}}X|G_t] = 1_{\tau>t}E[e^{-(\Gamma_s-\Gamma_t)}X|\Im_t]$$

If moreover, Γ is absolutely continuous, it exists an $\mathbb{F}$-predictable measurable process λ, referred to as the $\mathbb{F}$-intensity process of τ, such that $\Gamma_t = \int_0^t \lambda_u du$. The last formula becomes

$$E[1_{\{\tau>s\}}X|G_t] = 1_{\tau>t}E[e^{-\int_t^s \lambda_u du}X|\Im_t]$$

Proof 5 *See [11] for a proof.*

In section 3 when we study unit-linked contracts, we'll actually only use this first lemma.

Lemma 4 *Let Z be a bounded $\mathbb{F}$-predictable process. Then for any $t < s \leq \infty$, we have*

$$E[\int_t^s Z_u dH_u|G_t] = 1_{\tau>t}e^{\Gamma_t}E[\int_t^s Z_u dF_u|\Im_t]$$

If F_t is a continuous increasing process, then

$$E[\int_t^s Z_u dH_u|G_t] = 1_{\tau>t}e^{\Gamma_t}E[\int_t^s e^{-\Gamma_u}Z_u d\Gamma_u|\Im_t]$$

If moreover, τ admits an $\mathbb{F}$-intensity process such that $\Gamma_t = \int_0^t \lambda_u du$

$$E[\int_t^s Z_u dH_u|G_t] = 1_{\tau>t}E[\int_t^s \lambda_u e^{-\int_t^u \lambda_v dv}Z_u du|\Im_t]$$

Proof 6 *See [11] for a proof.*

Lemma 5 *Let C be a right continuous $\mathbb{F}$-adapted process such that C_s is an integrable random variable. If F_t follows a process of finite variation then for every $t \leq s$*

$$E[\int_t^s (1-H_u)dC_u|G_t] = 1_{\tau>t}e^{\Gamma_t}E[\int_t^s e^{-\Gamma_u}dC_u|\Im_t]$$

If moreover, τ admits an $\mathbb{F}$-intensity process such that $\Gamma_t = \int_0^t \lambda_u du$ then

$$E[\int_t^s (1-H_u)dC_u|G_t] = 1_{\tau>t}E[\int_t^s e^{-\int_t^u \lambda_v dv}dC_u|\Im_t]$$

Proof 7 *A proof can be found in [11] for a $\mathbb{F}$-predictable process C but this assumption is actually not used in their proof.*

If the hypothesis (H) is respected under the probability measure considered, it implies F_t is an increasing process, and F_t is then indeed of finite variation.

2.6.2 Present value of the insurer payments stream.

In order to simplify formula (1), we can directly apply the 3 Lemmas of the last subsection. We have the following proposition.

Proposition 5 *Let's assume $\hat{F}_t$ is an increasing process[2], the present value of the insurer payments stream is given by*

$$\begin{aligned} L_t^C &= 1_{\tau>t}E^Q[e^{-(D_T-D_t)}e^{-(\hat{\Gamma}_T-\hat{\Gamma}_t)}g(T,\omega)|\Im_t] \\ &+ 1_{\tau>t}E^Q[\int_t^T e^{-(D_u-D_t)}e^{\hat{\Gamma}_t}R(u,\omega)d\hat{F}_u|\Im_t] \\ &+ 1_{\tau>t}E^Q[\int_t^T e^{-(D_u-D_t)}e^{-(\hat{\Gamma}_u-\hat{\Gamma}_t)}dC_u|\Im_t] \end{aligned} \quad (4)$$

If $\hat{F}_t$ is an increasing continuous process and τ admits an $(\mathbb{F},Q)$-intensity process $\hat{\lambda}$. If furthermore, an instantaneous risk free rate r_t exists, then the present value of the insurance commitments is given by

$$\begin{aligned} L_t^C &= 1_{\tau>t}E^Q[e^{-\int_t^T(r_u+\hat{\lambda}_u)du}g(T,\omega)|\Im_t] \\ &+ 1_{\tau>t}E^Q[\int_t^T \hat{\lambda}_u e^{-\int_{v=t}^u(r_v+\hat{\lambda}_v)dv}R(u,\omega)du|\Im_t] \\ &+ 1_{\tau>t}E^Q[\int_t^T e^{-\int_{v=t}^u(r_v+\hat{\lambda}_v)dv}dC_u|\Im_t] \end{aligned} \quad (5)$$

2.6.3 Present value of the policyholder payments stream.

Again, we can apply Lemma 1 to formula (2). We then have the following proposition.

Proposition 6 *Let's assume the premiums are paid at fixed discrete dates t_i with $i = 0\ldots n-1$, the present value of the policyholder's commitments is given by*

$$A_t = 1_{\tau>t}E^Q[\sum_{i=0}^{n-1} e^{-(D_{t_i}-D_t)}e^{-(\hat{\Gamma}_{t_i}-\hat{\Gamma}_t)}P(t_i,\omega)|\Im_t] \quad (6)$$

If τ admits an $(\mathbb{F},Q)$-intensity process and if an instantaneous risk free rate r_t exists, we have

$$A_t = 1_{\tau>t}E^Q[\sum_{i=0}^{n-1} e^{-\int_t^{t_i}(r_u+\hat{\lambda}_u)du}P(t_i,\omega)|\Im_t] \quad (7)$$

[2]this assumption is necessary if we want the (H) hypothesis to hold

It's worth stressing the importance of these expressions. Firstly, it generalizes well known results when τ is stochastic but independent of $\mathbb{F}$ like in the stochastic mortality models. Secondly, by taking expectations with respect to $\Im_t$ instead of G_t, we are allowed to replace the indicator function by a term similar to the discounting term used in stochastic term structure models. We see $\hat{\Gamma}_t$ plays the same role as D_t or when they exist, the intensity process $\hat{\lambda}_t$ is similar to the short term rate r_t. This means the value of a life insurance contract subject to surrender risk can be considered, except for the second term in equations (4) and (5), to be equal to the value of the same contract without surrender risk but under a modified stochastic term structure $D_t + \hat{\Gamma}_t$. This also means all the results developed in the extensive literature on option pricing models with stochastic interest rates can be applied, mutatis mutandis, to the valuation of life insurance contracts with stochastic surrender. In this literature, closed form solutions or quasi-closed form solutions have been developed in fairly complex situations including multi-factors models of the yield curve, stochastic volatility models, Levy processes,... When no quasi-closed form solution exists, we can also sometimes, depending how the τ is specified, rely on Monte Carlo simulations. From the point of view of the tractability, this is a huge improvement since as we said, to our knowledge, no simple numerical technique can be easily implemented in such models when τ is an optimal stopping time.

Thirdly, these expressions have also implications on the numerical point of view. If we decide to rely on Monte-Carlo simulations to find the value of these conditional expectations, it is more efficient to use the conditional expectations with respect to $\Im_t$ instead of those taken with respect to G_t. The reason is in the latter case, we have indicator functions in our expectations whereas, in the former case, we have not. Indicator functions are slower to simulate because the simulations which values are outside the set of the indicator functions does not improve the precision. In the former case, all the simulations count.

2.7 $\mathbb{F}$ as the Brownian filtration.

Up to now, we have made weak assumptions on the form of the different processes. In this section we're going to make more specific assumptions, namely we assume

1. the filtration $\mathbb{F}$ is the Brownian filtration
2. τ admits a continuous increasing $(\mathbb{F}, P)$-hazard process Γ_t.
3. the (H) hypothesis is respected under P so that the brownian motion is also a brownian motion with respect to $\mathbb{G}$ under P.

In this setting, we give a change of equivalent measure formula that allows us to go from the real measure P to an equivalent one. It also provides a simple functional between Γ and $\hat{\Gamma}$.

2.7.1 Change of measure and choice of the local martingale measure

As we have seen, the risk neutral valuation principle rests on the choice of a so-called local martingale measure Q equivalent to P, that is, Q is chosen such that discounted prices on the financial market are local martingales. The following theorem gives us the set of measures equivalent to P, that obviously includes the equivalent local martingale measure we should use in our valuation.

Lemma 6 *Any probability measure Q equivalent to P has a Radon-Nikodym derivative with respect to P, η_t, with the following representation :*

$$\eta_t = 1 + \int_{u=0}^{t} \xi_u dW_u + \int_{u=0}^{t} \zeta_u dM_u$$

where W_t is a $\mathbb{G}$ brownian motion under P, $M_t = H_t - \Gamma_{t\wedge\tau}$ is a $\mathbb{G}$-martingale under P and ξ_t and ζ_t are $\mathbb{G}$-predictable processes.

Since η_t is strictly positive, it can also always be written as :

$$\eta_t = 1 - \int_{u=0}^{t} \eta_{u-}\beta_u dW_u + \int_{u=0}^{t} \eta_{u-}\kappa_u dM_u$$

where β_t and $\kappa_t > -1$ are $\mathbb{G}$-predictable processes. It is well-known such stochastic differential equation has a unique solution given by the following product of Doleans exponentials :

$$\begin{aligned} \eta_t &= \mathcal{E}_t(\int_{u=0}^{\cdot} \kappa_u dM_u)\mathcal{E}_t(-\int_{u=0}^{\cdot} \beta_u dW_u) \\ &= (1+\kappa_\tau 1_{\{\tau\le t\}})exp(-\int_{u=0}^{t\wedge\tau} \kappa_u d\Gamma_u)exp(-\int_0^t \beta_u dW_u - \frac{1}{2}\int_0^t |\beta_u|^2 du) \end{aligned}$$

A Girsanov like theorem can be formulated in this setting.

Lemma 7 *Any probability measure Q equivalent to P has a Radon-Nikodym derivative given by*

$$\frac{dQ}{dP}_{|G_t} = \eta_t = \varepsilon_t^1 \times \varepsilon_t^2$$

with

$$\varepsilon_t^1 = (1+\kappa_\tau 1_{\{\tau\le t\}})exp(-\int_{u=0}^{t\wedge\tau} \kappa_u d\Gamma_u)$$

$$\varepsilon_t^2 = exp(-\int_{u=0}^{t} \beta_u dW_u - \frac{1}{2}\int_{u=0}^{t} |\beta_u|^2 du)$$

where κ and β are $\mathbb{G}$-predictable process.

Furthermore, the process

$$\hat{W}_t = W_t + \int_{u=0}^{t} \beta_u du$$

follows a brownian motion with respect to $\mathbb{G}$ under Q and the process $\hat{M}_t$ given by

$$\hat{M}_t = H_t - \int_{u=0}^{t\wedge\tau} (1+\kappa_u) d\Gamma_u$$

is a $\mathbb{G}$-martingale under Q.

The proofs of lemmas 6 and 7 can be found in [11].

Unfortunately, this set of equivalent measure is too large. As we said, it is crucial our local martingale measure respects the (H)-hypothesis under Q and we've seen this property is not necessarily invariant from an equivalent measure to another, so the equivalent measures respecting (H) form only a subset of all the equivalent measures. A sufficient condition for the equivalent measure Q to respect the (H)-hypothesis, is β and κ are not $\mathbb{G}$-predictable but $\mathbb{F}$-predictable processes. We then have the following proposition.

Proposition 7 *Let Q be an equivalent measure to P. If the processes β and κ given in lemma 7, are $\mathbb{F}$-predictable then the (H)-hypothesis is verified under Q.*

Proof 8 *We have to prove*

$$Q(\tau \leq t|\Im_\infty) = Q(\tau \leq t|\Im_t) \tag{8}$$

Let's first consider the left-hand term of this equality. We have

$$\begin{aligned} Q(\tau \leq t|\Im_\infty) &= \frac{E^P[\eta_\infty 1_{\tau\leq t}|\Im_\infty]}{E^P[\eta_\infty|\Im_\infty]} = \frac{E^P[\varepsilon^1_\infty \varepsilon^2_\infty 1_{\tau\leq t}|\Im_\infty]}{E^P[\varepsilon^1_\infty\varepsilon^2_\infty|\Im_\infty]} \\ &= \frac{E^P[(1+\kappa_\tau 1_{\{\tau\leq\infty\}})exp(-\int_0^{\infty\wedge\tau}\kappa_u d\Gamma_u)1_{\tau\leq t}|\Im_\infty]}{E^P[\varepsilon^1_\infty|\Im_\infty]} \\ &= \frac{E^P[(1+\kappa_\tau 1_{\{\tau\leq t\}})exp(-\int_0^{t\wedge\tau}\kappa_u d\Gamma_u)1_{\tau\leq t}|\Im_\infty]}{E^P[\varepsilon^1_\infty|\Im_\infty]} \\ &= \frac{E^P[(1+\kappa_\tau 1_{\{\tau\leq t\}})exp(-\int_0^{t\wedge\tau}\kappa_u d\Gamma_u)1_{\tau\leq t}|\Im_t]}{E^P[\varepsilon^1_\infty|\Im_\infty]} \end{aligned}$$

The third equality comes from the assumption ε^2_∞ is $\Im_\infty$-measurable. The last one comes from the fact the (H)-hypothesis is verified for P and that this hypothesis implies for any random variable A, G_t-measurable, we have $E^P[A|\Im_\infty] = E^P[A|\Im_t]$.
If we now consider the denominator of this last equation, we have :

$$\begin{aligned} E^P[\varepsilon^1_\infty|\Im_\infty] &= \int_{v=0}^{\infty}(1+\kappa_v)exp(-\int_0^v \kappa_u d\Gamma_u)dP(\tau\leq v|\Im_\infty) \\ &= \int_{v=0}^{\infty}(1+\kappa_v)exp(-\int_0^v \kappa_u d\Gamma_u)dP(\tau\leq v|\Im_t) \\ &= E^P[(1+\kappa_\tau 1_{\{\tau\leq\infty\}})exp(-\int_0^{\infty\wedge\tau}\kappa_u d\Gamma_u)|\Im_t] \\ &= E^P[\varepsilon^1_\infty|\Im_t] \\ &= E^P[E^P[\varepsilon^1_\infty|G_t]|\Im_t] \\ &= E^P[\varepsilon^1_t|\Im_t] \end{aligned}$$

The first equality comes from the assumption κ is $\mathbb{F}$-adapted and thus $\Im_\infty$-measurable. The second equality comes from the fact the (H)-hypothesis is verified under P. The last one is due *to the fact ε^1 is a $\mathbb{G}$-martingale. So finally, we get for the left-hand side of (8)*

$$Q(\tau \leq t|\Im_\infty) = \frac{E^P[\varepsilon_t^1 1_{\tau\leq t}|\Im_t]}{E^P[(\varepsilon_t^1|\Im_t]}$$

As far as the right-hand side of (8) is concerned, we have :

$$\begin{aligned} Q(\tau \leq t|\Im_t) &= \frac{E^P[\eta_t 1_{\tau\leq t}|\Im_t]}{E^P[\eta_t|\Im_t]} = \frac{E^P[\varepsilon_t^1\varepsilon_t^2 1_{\tau\leq t}|\Im_t]}{E^P[\varepsilon_t^1\varepsilon_t^2|\Im_t]} \\ &= \frac{E^P[\varepsilon_t^1 1_{\tau\leq t}|\Im_t]}{E^P[\varepsilon_t^1|\Im_t]} \text{ since } \varepsilon_t^2 \text{ is } \Im_t\text{-measurable.} \end{aligned}$$

The proposition is then proved.

We already said that in all generality, it is not an easy task to find an explicit functional between Γ and $\hat{\Gamma}$. In the case of the brownian filtration and when β and κ are $\mathbb{F}$-predictable, we can find a simple relation between Γ and $\hat{\Gamma}$. We have the following proposition.

Proposition 8 *Let Q be an equivalent measure to P. If The processes β and κ of lemma 7 are $\mathbb{F}$-predictable then the process $\hat{\Gamma}_t$ defined by*

$$\hat{\Gamma}_t = \int_{u=0}^t (1+\kappa_u)d\Gamma_u$$

is the $\mathbb{F}$-hazard process of τ under Q. If τ admits an $\mathbb{F}$-intensity process λ under P, then the process $\hat{\lambda}_t$ defined by

$$\hat{\lambda}_t = (1+\kappa_t)\lambda_t$$

is the $\mathbb{F}$-intensity of τ under Q.

Proof 9 *We have*

$$e^{-\hat{\Gamma}_t} = Q(\tau > t|\Im_t) = \frac{E^P[\varepsilon_t^1 1_{\tau>t}|\Im_t]}{E^P[\varepsilon_t^1|\Im_t]}$$

since ε_t^2 is $\mathbb{F}$-adapted. Let's first consider the numerator. We have

$$\begin{aligned} E^P[\varepsilon_t^1 1_{\tau>t}|\Im_t] &= E^P[(1+\kappa_\tau 1_{\tau\leq t})exp(-\int_0^{t\wedge\tau}\kappa_u d\Gamma_u)1_{\tau>t}|\Im_t] \\ &= E^P[exp(-\int_0^t \kappa_u d\Gamma_u)1_{\tau>t}|\Im_t] \\ &= exp(-\int_0^t \kappa_u d\Gamma_u)E^P[1_{\tau>t}|\Im_t] = exp(-\int_0^t \kappa_u d\Gamma_u)e^{-\Gamma_t} \\ &= e^{-\int_0^t(1+\kappa_u)d\Gamma_u} \end{aligned}$$

The third equality comes from the assumption κ is $\mathbb{F}$-adapted. Consider now the denominator, we have

$$
\begin{aligned}
E^P[\varepsilon_t^1|\Im_t] &= E^P[(1+\kappa_\tau 1_{\tau\leq t})exp(-\int_0^{t\wedge\tau}\kappa_u d\Gamma_u)|\Im_t] \\
&= E^P[1_{\tau>t}exp(-\int_0^{t\wedge\tau}\kappa_u d\Gamma_u) + (1+\kappa_\tau)1_{\tau\leq t}exp(-\int_0^{t\wedge\tau}\kappa_u d\Gamma_u)|\Im_t] \\
&= E^P[1_{\tau>t}exp(-\int_0^{t}\kappa_u d\Gamma_u) + (1+\kappa_\tau)1_{\tau\leq t}exp(-\int_0^{\tau}\kappa_u d\Gamma_u)|\Im_t] \\
&= exp(-\int_0^{t}\kappa_u d\Gamma_u)E^P[1_{\tau>t}|\Im_t] + E^P[(1+\kappa_\tau)1_{\tau\leq t}exp(-\int_0^{\tau}\kappa_u d\Gamma_u)|\Im_t] \\
&= exp(-\int_0^{t}(1+\kappa_u)d\Gamma_u) + \int_0^t(1+\kappa_v)exp(-\int_0^{v}\kappa_u d\Gamma_u)dF_v \\
&= exp(-\int_0^{t}(1+\kappa_u)d\Gamma_u) + \int_0^t exp(-\int_0^{v}(1+\kappa_u)d\Gamma_u)(1+\kappa_v)d\Gamma_v \\
&= exp(-\int_0^{t}(1+\kappa_u)d\Gamma_u) - \int_0^t d[exp(-\int_0^{v}(1+\kappa_u)d\Gamma_u)] \\
&= exp(-\int_0^{t}(1+\kappa_u)d\Gamma_u) - [exp(-\int_0^{t}(1+\kappa_u)d\Gamma_u) - 1] \\
&= 1
\end{aligned}
$$

The sixth equality comes from the fact Γ is assumed continuous.

As we said, η is the Radon-Nikodym derivative of Q with respect to P. The problem is now to find the processes β_t and κ_t we should use in our valuation formulas.

Let's start with β_t first. If we had no surrender risk, η would reduce only to ε_2 and we would get back to the standard case extensively studied in financial economics. In this case, we should choose β_t such that the discounted financial prices are local martingales under the associated equivalent measure. β_t would then correspond to the price of market risk. There is no difference in our setting. This is the essence of the first part of the demonstration of proposition 4 where we proved the restriction of Q on $\mathbb{F}$ coincide with R (at least in a complete market). Accordingly, we should simply choose β_t as if there was no surrender risk. Let's finally notice since the financial market is complete, the local martingale measure of the financial market is unique and so is β_t.

The choice of κ_t is more problematic. It determines how the hazard or the intensity process should be adjusted for the pure risk of surrender. The value of κ_t we should use, is the price of the pure surrender risk. Unfortunately, we don't have any market on which assets affected by surrender risk are traded. So, even if the financial market is complete, the "life insurance market" is not with respect to this surrender risk. The equivalent local martingale measure related to this risk is accordingly not unique and so is the choice of κ_t. We can neither extract the market price of this risk from the values of assets affected by surrender since as we just said there is no such market. However, this pure surrender risk correspond to the idiosyncratic risk component of the surrender decision. If we assume the idiosyncratic component is independent from a policyholder to another, we can argue this pure surrender risk can be diversified away. Accordingly, in a competitive market, no remuneration for this risk should be required and its price could be considered equal to 0. Another argument pleading for a pure surrender risk equal to 0, is to recognize the associated local martingale measure corresponds to the minimal martingale measure of Schweizer.

If we set $\kappa_t = 0 \; \forall t$, we see that τ admits the hazard process $\hat{\Gamma}_t = \Gamma_t$ or the intensity process $\hat{\lambda}_t = \lambda_t$ under Q. This last equality simply means the processes Γ or λ are not modified if we set the price of surrender risk equal to 0, but it does not mean the probabilities of surrender are equal under the local martingale measure and the physical measure. Indeed, these processes depend on the brownian motion and are accordingly adjusted, under the local martingale measure, by the price of market risk. In conclusion, by setting $\kappa = 0$, the idiosyncratic component of the surrender decision is not adjusted for the risk, but the common component of the surrender risk, which is linked to the financial market and is not diversifiable, is indeed adjusted for the risk through the market price of risk.

3 Application to unit-linked contracts

In this section, we apply the framework developed above to the valuation of unit-linked contracts. We study single premium contracts and periodic premium contracts. We do not make any specific assumptions on the distribution followed by our stochastic processes other than the discounted prices of the financial assets are, under Q, not only $\mathbb{F}$-martingales but also $\mathbb{G}$-martingales. We derived general formulas for the valuation of unit-linked contracts. You'll notice we only need lemma 3 of section 2.6.1 to derive these valuation formulas so that τ does not even need to admit an $(\mathbb{F}, Q)$-intensity process nor does $\hat{\Gamma}$ need to be continuous.

3.1 Single premium contract.

We assume a single premium P is paid at time t. With this premium, the policyholder acquires n units of a stock index with value $S(t)$ at time t, as well as a guarantee g at the term T of the contract. This guarantee could be for example a guaranteed rate on the initial value of the units, a guaranteed rate on the premium paid, etc...

When the policyholder decides to surrender, we assume he gets back a proportion $(1-\alpha)$ of the n units. So, α is here the penalty when one surrenders. As far as the timing of payment of the surrender value is concerned, we're going to study two different situations that will eventually, lead to the same result. In the first case, we assume the surrender occurs in continuous time and the payment is made at this precise time τ. In the second case, we still assume surrender occurs in continuous time but the payment are made at k discrete dates.

3.1.1 continuous time payment

The surrender value is here equal to $(1-\alpha)nS(\tau)$ and is paid at time τ. Using our notation, we have :

$$R^c(\tau) = (1-\alpha)nS(\tau)$$

$$g(T,\omega) = max(nS(T), g)$$

The value at date t of the liabilities is given by :

$$L_t^C = E^Q[e^{-(D_T-D_t)}g(T,\omega)1_{\tau>T} + e^{-(D_\tau-D_t)}R^c(\tau)1_{\{t<\tau\leq T\}}|G_t]$$

3.1.2 discrete time payment

Here, in case of surrender, the payment of the surrender value is made at k discrete times t_i with $i = 1, \ldots k$ In this situation, the surrender value at the date of surrender τ is still equal to $(1-\alpha)nS(\tau)$, but this amount is assumed to be invested in risk free asset up to the next time of payment. In other words, if surrender occurs between $]t_{i-1}, t_i]$, the policyholder will get back at time t_i an amount equals to $(1-\alpha)nS(\tau)$ compounded at the risk free rate up to t_i. Using the notation used before, we then have:

$$\begin{aligned} R^d(t_i,\omega) &= (1-\alpha)nS(\tau)e^{(D_{t_i}-D_\tau)}1_{\{t_{i-1}<\tau\leq t_i\}} \\ &= R^c(\tau,\omega)e^{(D_{t_i}-D_\tau)}1_{\{t_{i-1}<\tau\leq t_i\}} \end{aligned}$$

$$g(T,\omega) = max(nS(T), g)$$

Using the results described above, we can write :

$$\begin{aligned} L_t^D &= E^Q[e^{-(D_T-D_t)}g(T,\omega)1_{\tau>T} + \sum_{j=1}^{k} e^{-(D_{t_j}-D_t)}R^d(t_j)|G_t] \\ &= E^Q[e^{-(D_T-D_t)}g(T,\omega)1_{\tau>T} + \sum_{j=1}^{k} e^{-(D_{t_j}-D_t)}R^c(\tau)e^{(D_{t_j}-D_\tau)}1_{t_{j-1}<\tau\leq t_j}|G_t] \\ &= E^Q[e^{-(D_T-D_t)}g(T,\omega)1_{\tau>T} + \sum_{j=1}^{k} e^{-(D_\tau-D_t)}R^c(\tau)1_{t_{j-1}<\tau\leq t_j}|G_t] \\ &= E^Q[e^{-(D_T-D_t)}g(T,\omega)1_{\tau>T} + e^{-(D_\tau-D_t)}R^c(\tau)\sum_{j=1}^{k} 1_{t_{j-1}<\tau\leq t_j}|G_t] \\ &= E^Q[e^{-(D_T-D_t)}g(T,\omega)1_{\tau>T} + e^{-(D_\tau-D_t)}R^c(\tau)1_{\{t<\tau\leq T\}}|G_t] \\ &= L_t^C \end{aligned}$$

Since $L^C = L^D$, we simply write L from now on. We can simplify these equations as followed

$$\begin{aligned} L_t &= E^Q[e^{-(D_T-D_t)}1_{\{\tau>T\}}max(nS(T), g)|G_t] \\ &+ E^Q[e^{-(D_\tau-D_t)}(1-\alpha)nS(\tau)(1_{\{\tau>t\}} - 1_{\{\tau>T\}})|G_t] \\ &= \underbrace{E^Q[e^{-(D_T-D_t)}1_{\{\tau>T\}}max(nS(T), g)|G_t]}_{I} \\ &+ \underbrace{E^Q[(1-\alpha)nS(\tau)e^{-(D_\tau-D_t)}1_{\{\tau>t\}}|G_t]}_{II} \\ &- \underbrace{E^Q[(1-\alpha)nS(\tau)e^{-(D_\tau-D_t)}1_{\{\tau>T\}}|G_t]}_{III} \end{aligned}$$

The terms I, II and III can be simplified as follow :

$$\begin{aligned} I &= E^Q[e^{-(D_T-D_t)}1_{\{\tau>T\}}max(nS(T),g)|G_t] \\ &= 1_{\{\tau>t\}}E^Q[e^{-(D_T-D_t)}e^{-(\hat{\Gamma}_T-\hat{\Gamma}_t)}max(nS(T),g)|\Im_t] \end{aligned}$$

$$\begin{aligned} II &= E^Q[(1-\alpha)nS(\tau)e^{-(D_\tau-D_t)}1_{\{\tau>t\}}|G_t] \\ &= 1_{\{\tau>t\}}(1-\alpha)nE^Q[e^{-(D_\tau-D_t)}S(\tau)|G_t] \\ &= 1_{\{\tau>t\}}(1-\alpha)nS(t) \end{aligned}$$

In the last equality, we use the hypothesis that all $\mathbb{F}$-martingales are also G-martingales. The hypothesis (H) under Q is thus necessary to derive these formulas.

$$\begin{aligned} III &= (1-\alpha)nE^Q[e^{-(D_\tau-D_t)}S(\tau)1_{\{\tau>T\}}|G_t] \\ &= (1-\alpha)nE^Q[e^{-(D_T-D_t)}E^Q[S(\tau)e^{-(D_\tau-D_T)}1_{\{\tau>T\}}|G_T]|G_t] \\ &= (1-\alpha)nE^Q[e^{-(D_T-D_t)}1_{\{\tau>T\}}E^Q[S(\tau)e^{-(D_\tau-D_T)}|G_T]|G_t] \\ &= (1-\alpha)nE^Q[e^{-(D_T-D_t)}1_{\{\tau>T\}}S(T)|G_t] \\ &= 1_{\{\tau>t\}}(1-\alpha)nE^Q[e^{-(D_T-D_t)}e^{-(\hat{\Gamma}_T-\hat{\Gamma}_t)}S(T)|\Im_t] \end{aligned}$$

In the one but last equality we use the hypothesis that the discounted price of the financial asset is, under Q, a $\mathbb{G}$-martingale. Eventually, we have

$$\begin{aligned} L_t &= 1_{\{\tau>t\}}e^{(D_t+\hat{\Gamma}_t)}E^Q[e^{-(D_T+\hat{\Gamma}_T)}max(nS(T),g)|\Im_t] \\ &+ 1_{\{\tau>t\}}(1-\alpha)n\{S(t)-e^{(D_t+\hat{\Gamma}_t)}E^Q[e^{-(D_T+\hat{\Gamma}_T)}S(T)|\Im_t]\} \end{aligned} \quad (9)$$

When an intensity process and an instantaneous risk free rate exist, this last expression becomes

$$\begin{aligned} L_t &= 1_{\{\tau>t\}}E^Q[e^{-\int_t^T r_u+\hat{\lambda}_u du}max(nS(T),g)|\Im_t] \\ &+ 1_{\{\tau>t\}}(1-\alpha)n\{S(t)-E^Q[e^{-\int_{u=t}^T r_u+\hat{\lambda}_u du}S(T)|\Im_t]\} \end{aligned}$$

These last two expressions are very simple ones that involve the calculation of the price of a call options with a modified term structure model $D_t^{\hat{\Gamma}} = D_t + \hat{\Gamma}_t$ or in the second case, with a modified instantaneous risk free interest rate $r_t^{\hat{\lambda}} = r_t + \hat{\lambda}_t$. It can be solved, in a number of cases, in closed form or in quasi-closed form by Fourier transform for example. Notice, we haven't made any assumptions on the form of the processes followed by $S(t)$, D_t or Γ_t, so that (9) is thus a general valuation formulas for single premium unit linked contract with stochastic interest rate and stochastic surrender.

3.2 Periodic Premium contract

Here, we assume premiums are paid at time t_i with $i = 0 \ldots N-1$ if the policyholder has not surrendered. We assume the value of these premiums is constant and equal to P. As we've already seen, the payoff is equal to :

$$\sum_{i=0}^{N-1} P1_{\tau>t_i}$$

The value of this payoff at date t, is given by :

$$A_t = E^Q[\sum_{i=0}^{N-1} e^{-(D_{t_i}-D_t)} P1_{\tau>t_i}|G_t]$$

Using the results of the preceding sections, we get :

$$\begin{aligned} A_t &= 1_{\{\tau>t\}} P \sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i}-D_t)} e^{-(\hat{\Gamma}_{t_i}-\hat{\Gamma}_t)}|\Im_t] \\ &= 1_{\{\tau>t\}} P \sum_{i=0}^{N-1} e^{(D_t+\hat{\Gamma}_t)} E^Q[e^{-(D_{t_i}+\hat{\Gamma}_{t_i})}|\Im_t] \end{aligned} \tag{10}$$

When an intensity process and an instantaneous risk free rate exist, this last expression becomes

$$A_t = 1_{\{\tau>t\}} P \sum_{i=0}^{N-1} E^Q[e^{-\int_t^{t_i} r_u+\hat{\lambda}_u du}|\Im_t] \tag{11}$$

As far as the value of the liabilities is concerned, we distinguish two cases. In the first one, the number of units the policyholder received, depends on the value of the units at the time of payment. In the second case, we assume the payment of a premium gives the right to the policyholder, to n units whatever the value of these units at the time of payment. We call the first kind of contract, a periodic premium contract of type I and the second one, a periodic premium contract of type II. We could also have distinguished as we've done in the single premium contracts, between continuous time and discrete time payments in case of surrender but since both of them lead to the same result we focus on the continuous time payment.

3.2.1 Periodic Premiums contract of type I

In this case, we assume a proportion $(1-\rho)$ of the premium P paid at time t_i is used to buy n_{t_i} units. This number of unit n_{t_i} is given by

$$n_{t_i} = \frac{(1-\rho)P}{S(t_i)}$$

The present value of the liabilities is given by

$$\begin{aligned} L_t &= E^Q[e^{-(D_T-D_t)} max(S(T) \sum_{i=0}^{N-1} n_{t_i}, g) 1_{\tau>T} \\ &+ e^{-(D_\tau-D_t)}(1-\alpha)S(\tau)(\sum_{i=0}^{\lfloor\tau\rfloor} n_{t_i}) 1_{\{t<\tau\leq T\}}|G_t] \end{aligned}$$

We show in appendix 1 we can rewrite this equation as :

$$
\begin{aligned}
L_t &= 1_{\{\tau>t\}} e^{(D_t+\hat{\Gamma}_t)} E^Q[e^{-(D_T+\hat{\Gamma}_T)} max((1-\rho)P \sum_{i=0}^{N-1} \frac{S(T)}{S(t_i)}, g) | \Im_t] \\
&+ 1_{\{\tau>t\}} (1-\alpha)(1-\rho)P \sum_{i=0}^{N-1} \{ e^{(D_t+\hat{\Gamma}_t)} E^Q[e^{-(D_{t_i}+\hat{\Gamma}_{t_i})} | \Im_t] \\
&- e^{(D_t+\hat{\Gamma}_t)} E^Q[e^{-(D_T+\hat{\Gamma}_T)} \frac{S(T)}{S(t_i)} | \Im_t] \} \qquad (12)
\end{aligned}
$$

When an intensity process and an instantaneous risk free rate exist, we get

$$
\begin{aligned}
L_t &= 1_{\{\tau>t\}} E^Q[e^{-\int_t^T r_u+\hat{\lambda}_u du} max((1-\rho)P \sum_{i=0}^{N-1} \frac{S(T)}{S(t_i)}, g) | \Im_t] \\
&+ 1_{\{\tau>t\}} (1-\alpha)(1-\rho)P \sum_{i=0}^{N-1} \{ E^Q[e^{-\int_t^{t_i} r_u+\hat{\lambda}_u du} | \Im_t] \\
&- E^Q[e^{-\int_t^T r_u+\hat{\lambda}_u du} \frac{S(T)}{S(t_i)} | \Im_t] \}
\end{aligned}
$$

Notice if we take $N = 1$ and pose $(1-\rho)P = nS(t)$, we get the same result than the one we found for the single premium contract.

The last two expectations can often be found in closed form or in quasi-closed form. Unfortunately, the first expectation is much harder to calculate. No closed form solution is available for this one, even in the simplest case. However, we can still rely on the extensive literature on the numerical techniques used for the pricing of Asian options in finance.

Once again, we'd like to stress out (12) is a general valuation formula for periodic premiums unit linked contract since weak assumptions have been made concerning the specific distributions of the different processes.

3.2.2 Periodic Premiums contract of Type II

we assume the payment of the premium gives the right to the policyholder, to n units whatever the value of these units at the time of payment. We assume there are N dates of payment. In this case, we have

$$
\begin{aligned}
R^c(\tau) &= (1-\alpha) \sum_{i=0}^{\lfloor\tau\rfloor} nS(\tau) \\
&= (1-\alpha) \lfloor\tau\rfloor nS(\tau) \\
g(T,\omega) &= max(\sum_{i=0}^{N-1} nS(T), g) \\
&= max(NnS(T), g)
\end{aligned}
$$

The present value of the liabilities is given by

$$\begin{aligned} L_t &= E^Q[e^{-(D_T-D_t)}max(NnS(T),g)1_{\tau>T} \\ &+ e^{-(D_\tau-D_t)}(1-\alpha)\lfloor\tau\rfloor nS(\tau)1_{\{t<\tau\leq T\}}|G_t] \end{aligned}$$

We show in appendix 2, we can rewrite this equation as :

$$\begin{aligned} L_t &= 1_{\{\tau>t\}}e^{(D_t+\hat{\Gamma}_t)}E^Q[e^{-(D_T+\hat{\Gamma}_T)}max(NnS(T),g)|\Im_t] \\ &+ 1_{\{\tau>t\}}(1-\alpha)n\{e^{(D_t+\hat{\Gamma}_t)}\sum_{i=0}^{N-1}E^Q[e^{-(D_{t_i}+\hat{\Gamma}_{t_i})}S(t_i)|\Im_t] \\ &- Ne^{(D_t+\hat{\Gamma}_t)}E^Q[e^{-(D_T+\hat{\Gamma}_T)}S(T)|\Im_t]\} \end{aligned} \quad (13)$$

When an intensity process and an instantaneous risk free rate exist, we have

$$\begin{aligned} L_t &= 1_{\{\tau>t\}}E^Q[e^{-\int_t^T r_u+\hat{\lambda}_u du}max(NnS(T),g)|\Im_t] \\ &+ 1_{\{\tau>t\}}(1-\alpha)n\{\sum_{i=0}^{N-1}E^Q[e^{-\int_t^{t_i} r_u+\hat{\lambda}_u du}S(t_i)|\Im_t] \\ &- NE^Q[e^{-\int_t^T r_u+\hat{\lambda}_u du}S(T)|\Im_t]\} \end{aligned} \quad (14)$$

4 Conclusion.

In this paper, we presented an alternative model of the surrender time in life insurance. Technicalities set aside, the main difference with the traditional model of the surrender time, is the way we model the insurer's and policyholder's information. In the traditional model, the insurer and the policyholder have exactly the same set of information $\mathbb{F}$ and all their decisions, in particular the surrender decision, are based on this set. Accordingly, the surrender time has to be a $\mathbb{F}$-stopping time. We argue this is not realistic and should be avoided. On the contrary, in our alternative model, we assume the policyholder takes his surrender decision on a larger set of information. In this situation, the surrender time cannot be a $\mathbb{F}$-stopping time. In this paper, we showed that essentially all such random times can be characterized by their so-called $\mathbb{F}$-hazard processes.

We also studied the impact of this framework on the fair valuation of life insurance contract. In particular, we derived fair valuation formulas for single premium and periodic premium unit linked contracts with stochastic term structure and stochastic surrender in a general semimartingale market model.

We would like to stress out, once again, two key advantages of this framework, its versatility and its tractability. As far as the versatility is concerned, we showed by a few examples, that the surrender time can be constructed in a great variety of ways by explicitly or implicitly modeling the asymmetry of information between the insurer and the policyholder. It also reconciles exogenously and endogenously defined surrender time. As far as the tractability is concerned, we saw, in a number of cases, the valuation of an insurance contract with stochastic surrender basically comes down to the valuation of this insurance contract with a modified stochastic term structure. Accordingly, all the tools used in stochastic interest rates models could be transposed here. For examples, affine term structure models that have been extensively studied in the financial literature, could also be a useful tool in life insurance to model the surrender decisions.

To conclude, this paper can be extended in a number of ways. We illustrated our model with the valuation of unit linked contracts but obviously, the same framework can be apply to model the surrender time in traditional participating life insurance products. The same mathematical framework could also be used to model stochastic mortality. A very special case of this framework is described in [7]. It would also be interesting to study the simultaneous effect of the stochastic surrender and the stochastic mortality on the valuation of life insurance contracts. But more importantly, in our framework, it is even more difficult than in the traditional model, to deal with endogenously defined surrender times. Most of the time, we will probably define it exogenously and this is the tricky part. Actually, finding a tractable and realistic definition of the surrender time is probably the most challenging task remaining.

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Appendix 1 *The value of the liabilities of the periodic premium unit linked contract of Type I is given by :*

$$\begin{aligned} L_t &= E^Q[e^{-(D_T-D_t)} max(S(T)\sum_{i=0}^{N-1} n_{t_i}, g) 1_{\tau>T} \\ &+ e^{-(D_\tau-D_t)}(1-\alpha)S(\tau)(\sum_{i=0}^{\lfloor\tau\rfloor} n_{t_i}) 1_{\{t<\tau\leq T\}}|G_t] \end{aligned}$$

We can rewrite this equation as :

$$
\begin{aligned}
L_t &= E^Q[e^{-(D_T-D_t)}max(S(T)\sum_{i=0}^{N-1} n_{t_i}, g)1_{\tau>T} \\
&+ e^{-(D_\tau-D_t)}(1-\alpha)S(\tau)(\sum_{i=0}^{\lfloor\tau\rfloor} n_{t_i})(1_{\{\tau>t\}} - 1_{\{\tau>T\}})|G_t] \\
L_t &= \underbrace{E^Q[e^{-(D_T-D_t)}max(S(T)\sum_{i=0}^{N-1} n_{t_i}, g)1_{\tau>T}|G_t]}_{I} \\
&+ \underbrace{E^Q[e^{-(D_\tau-D_t)}(1-\alpha)S(\tau)(\sum_{i=0}^{\lfloor\tau\rfloor} n_{t_i})1_{\{\tau>t\}}|G_t]}_{II} \\
&- \underbrace{E^Q[e^{-(D_\tau-D_t)}(1-\alpha)S(\tau)(\sum_{i=0}^{\lfloor\tau\rfloor} n_{t_i})1_{\{\tau>T\}}|G_t]}_{III}
\end{aligned}
$$

where $\lfloor\tau\rfloor = sup\{i|t_i \leq \tau\}$

Simplifying these terms, we get :

$$
\begin{aligned}
I &= E^Q[e^{-(D_T-D_t)}max(S(T)\sum_{i=0}^{N-1} n_{t_i}, g)1_{\tau>T}|G_t] \\
&= 1_{\{\tau>t\}}E^Q[e^{-(D_T-D_t)}e^{-(\hat\Gamma_T-\hat\Gamma_t)}max(S(T)\sum_{i=0}^{N-1} n_{t_i}, g)|\Im_t] \\
&= 1_{\{\tau>t\}}E^Q[e^{-(D_T-D_t)}e^{-(\hat\Gamma_T-\hat\Gamma_t)}max((1-\rho)P\sum_{i=0}^{N-1} \frac{S(T)}{S(t_i)}, g)|\Im_t]
\end{aligned}
$$

$$
\begin{aligned}
II &= E^Q[e^{-(D_\tau - D_t)}(1-\alpha)S(\tau)(\sum_{i=0}^{\lfloor\tau\rfloor} n_{t_i})1_{\{\tau>t\}}|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)E^Q[e^{-(D_\tau - D_t)}S(\tau)(\sum_{i=0}^{\lfloor\tau\rfloor} \frac{P(1-\rho)}{S(t_i)})|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)(1-\rho)PE^Q[e^{-(D_\tau - D_t)}S(\tau)(\sum_{i=0}^{N-1} \frac{1}{S(t_i)}1_{\{\tau>t_i\}})|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_\tau - D_t)}S(\tau)\frac{1}{S(t_i)}1_{\{\tau>t_i\}}|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i} - D_t)}\frac{1}{S(t_i)}1_{\{\tau>t_i\}}E^Q[S(\tau)e^{-(D_\tau - D_{t_i})}|G_{t_i}]|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i} - D_t)}\frac{1}{S(t_i)}1_{\{\tau>t_i\}}S(t_i)|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i} - D_t)}1_{\{\tau>t_i\}}|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i} - D_t)}e^{-(\hat\Gamma_{t_i}-\hat\Gamma_t)}|\Im_t]
\end{aligned}
$$

$$
\begin{aligned}
III &= E^Q[e^{-(D_\tau - D_t)}(1-\alpha)S(\tau)(\sum_{i=0}^{\lfloor\tau\rfloor} n_{t_i})1_{\{\tau>T\}}|G_t] \\
&= (1-\alpha)E^Q[e^{-(D_\tau - D_t)}S(\tau)(\sum_{i=0}^{N-1} n_{t_i})1_{\{\tau>T\}}|G_t] \\
&= (1-\alpha)(1-\rho)PE^Q[e^{-(D_\tau - D_t)}S(\tau)(\sum_{i=0}^{N-1} \frac{1}{S(t_i)})1_{\{\tau>T\}}|G_t] \\
&= (1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_\tau - D_t)}S(\tau)\frac{1}{S(t_i)}1_{\{\tau>T\}}|G_t] \\
&= (1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_T - D_t)}\frac{1}{S(t_i)}1_{\{\tau>T\}}E^Q[S(\tau)e^{-(D_\tau - D_T)}|G_T]|G_t] \\
&= (1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_T - D_t)}\frac{1}{S(t_i)}1_{\{\tau>T\}}S(T)|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)(1-\rho)P\sum_{i=0}^{N-1} E^Q[e^{-(D_T - D_t)}e^{-(\hat\Gamma_T-\hat\Gamma_t)}\frac{S(T)}{S(t_i)}|\Im_t]
\end{aligned}
$$

In the above equations, we have used the fact that discounted price of the financial asset is, under Q, a $\mathbb{G}$-martingale. Eventually, we have

$$
\begin{aligned}
L_t &= 1_{\{\tau>t\}} e^{(D_t+\hat{\Gamma}_t)} E^Q[e^{-(D_T+\hat{\Gamma}_T)} max((1-\rho)P\sum_{i=0}^{N-1}\frac{S(T)}{S(t_i)}, g)|\Im_t] \\
&+ 1_{\{\tau>t\}}(1-\alpha)(1-\rho)P\sum_{i=0}^{N-1}\{e^{(D_t+\hat{\Gamma}_t)} E^Q[e^{-(D_{t_i}+\hat{\Gamma}_{t_i})}|\Im_t] \\
&- e^{(D_t+\hat{\Gamma}_t)} E^Q[e^{-(D_T+\hat{\Gamma}_T)}\frac{S(T)}{S(t_i)}|\Im_t]\}
\end{aligned}
$$

Appendix 2 *The present value of the liabilities of unit linked contract of type II is given by*

$$
\begin{aligned}
L_t &= E^Q[e^{-(D_T-D_t)} max(NnS(T), g)1_{\tau>T} \\
&+ e^{-(D_\tau-D_t)}(1-\alpha)\lfloor\tau\rfloor nS(\tau)1_{\{t<\tau\leq T\}}|G_t]
\end{aligned}
$$

We can rewrite this equation as :

$$
\begin{aligned}
L_t &= E^Q[e^{-(D_T-D_t)} max(NnS(T), g)1_{\tau>T} \\
&+ e^{-(D_\tau-D_t)}(1-\alpha)\lfloor\tau\rfloor nS(\tau)(1_{\{\tau>t\}} - 1_{\{\tau>T\}})|G_t] \\
&= \underbrace{E^Q[e^{-(D_T-D_t)} max(NnS(T), g)1_{\tau>T}|G_t]}_{I} \\
&+ \underbrace{E^Q[e^{-(D_\tau-D_t)}(1-\alpha)\lfloor\tau\rfloor nS(\tau)1_{\{\tau>t\}}|G_t]}_{II} \\
&- \underbrace{E^Q[e^{-(D_\tau-D_t)}(1-\alpha)\lfloor\tau\rfloor nS(\tau)1_{\{\tau>T\}})|G_t]}_{III}
\end{aligned}
$$

These terms can be simplified

$$
\begin{aligned}
I &= E^Q[e^{-(D_T-D_t)} max(NnS(T), g)1_{\tau>T}|G_t] \\
&= 1_{\{\tau>t\}} E^Q[e^{-(D_T-D_t)} e^{-(\hat{\Gamma}_T-\hat{\Gamma}_t)} max(NnS(T), g)|\Im_t]
\end{aligned}
$$

$$
\begin{aligned}
II &= E^Q[e^{-(D_\tau - D_t)}(1-\alpha)\lfloor\tau\rfloor nS(\tau)1_{\{\tau>t\}}|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)nE^Q[e^{-(D_\tau - D_t)}\sum_{i=0}^{N-1} 1_{\{\tau>t_i\}}S(\tau)|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)n\sum_{i=0}^{N-1} E^Q[e^{-(D_\tau - D_t)}1_{\{\tau>t_i\}}S(\tau)|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)n\sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i} - D_t)}1_{\{\tau>t_i\}}E^Q[e^{-(D_\tau - D_{t_i})}S(\tau)|G_{t_i}]|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)n\sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i} - D_t)}1_{\{\tau>t_i\}}S(t_i)|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)n\sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i} - D_t)}e^{-(\hat{\Gamma}_{t_i} - \hat{\Gamma}_t)}S(t_i)|\Im_t]
\end{aligned}
$$

$$
\begin{aligned}
III &= E^Q[e^{-(D_\tau - D_t)}(1-\alpha)\lfloor\tau\rfloor nS(\tau)1_{\{\tau>T\}})|G_t] \\
&= (1-\alpha)nE^Q[e^{-(D_\tau - D_t)}NS(\tau)1_{\{\tau>T\}})|G_t] \\
&= (1-\alpha)nNE^Q[e^{-(D_\tau - D_t)}S(\tau)1_{\{\tau>T\}})|G_t] \\
&= (1-\alpha)nNE^Q[e^{-(D_T - D_t)}1_{\{\tau>T\}}E^Q[e^{-(D_\tau - D_T)}S(\tau)|G_T]|G_t] \\
&= (1-\alpha)nNE^Q[e^{-(D_T - D_t)}1_{\{\tau>T\}}S(T)|G_t] \\
&= 1_{\{\tau>t\}}(1-\alpha)nNE^Q[e^{-(D_T - D_t)}e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t)}S(T)|\Im_t]
\end{aligned}
$$

In the above equations, we have used the fact that the discounted price of the financial asset is, under Q, a $\mathbb{G}$*-martingale. Eventually, we have*

$$
\begin{aligned}
L_t &= 1_{\{\tau>t\}}e^{(D_t+\hat{\Gamma}_t)}E^Q[e^{-(D_T+\hat{\Gamma}_T)}max(NnS(T),g)|\Im_t] \\
&+ 1_{\{\tau>t\}}(1-\alpha)n\{e^{(D_t+\hat{\Gamma}_t)}\sum_{i=0}^{N-1} E^Q[e^{-(D_{t_i}+\hat{\Gamma}_{t_i})}S(t_i)|\Im_t] \\
&- Ne^{(D_t+\hat{\Gamma}_t)}E^Q[e^{-(D_T+\hat{\Gamma}_T)}S(T)|\Im_t]\}
\end{aligned}
$$