Asymptotic dependence of reinsurance aggregate claim amounts

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Abstract

In this paper we study the effect of different dependence structures on the distribution of total losses when the reinsurer undertakes excess of loss for two or more dependent portfolios. We also study the asymptotic behaviour of the joint distribution of the reinsurance aggregate claim amounts for large values of the deductible for each dependence structure.

Keywords: Dependent risks, reinsurance layers, multivariate Panjer recursions, asymptotic independence.

1 Introduction

Recently the importance of modelling dependent insurance and reinsurance risks has attracted the attention of actuarial practitioners and scientists. Even though classical theories have been developed under the assumption of independence between risks, there are many situations in practice where this assumption is not valid.

In a recent paper Embrechts, McNeil and Straumann wrote:

"Although insurance has traditionally been built on the assumption of independence and the law of large numbers has governed the determination of premiums, the increasing complexity of insurance and reinsurance products has led recently to increased actuarial interest in the modelling of dependent risks..."

Although the literature on dependence between risks in insurance portfolios is increasing rapidly, very few authors have applied these development to practical problems.

In this paper we study the problem of dependence between risks from the reinsurer's point of view when he undertakes excess of loss for two dependent risks assuming different dependence structures. When the reinsurer undertakes excess of loss reinsurance for a portfolio, in particular for catastrophe excess of loss, usually the probability that a claim will affect the reinsurer is very close to zero. Therefore in many cases the correlation between aggregate claim amounts for the reinsurer in many cases becomes very small. We could then be tempted to think that in this case the dependence structure between portfolios also disapears and that therefore we could assume that the portfolios are independent. It has been largely discussed in the literature that the linear correlation is not a satisfactory measure of dependence in the non-normal case, see, for example, Embrechts, McNeil and Straumann (1999).

In this paper we study the effect of different dependence structures that can be used to model reinsurance risks and how the dependence structure affects the joint distribution of the aggregate claim amounts compared to the product of the marginals when the risks have a high probability of being zero, or when the retention or deductible is large. In Section 2 we describe different models that can be used to model insurance and reinsurance aggregate claim amounts that are subject to the same events. In Section 3 we discuss how to calculate the distribution of the sum of aggregate claim amounts under different dependence assumptions. Finally in Section 4 we give measures of asymptotic behaviour of the joint distribution of reinsurance aggregate claim amounts for large values of the retention level and we study the effect of different dependence structures on these measures of dependence.

2 Some models for dependent reinsurance aggregate claim amounts

In this section we describe in detail some models that have proposed in the actuarial literature to model insurance aggregate claim amounts that are subject to the same number of claims. These models have been proposed, for example, by Sundt (1999) and Ambagaspitiya (1999) where they develop multivariate recursions to calculate the joint distribution of the aggregate claim amounts. We will assume that there are only two portfolios, however the results can be generalised for any number of aggregate claim amounts.

MODEL 1: Two portfolios are affected by the same events, therefore they are subject to the same number of claims. This model is the general model described by Sundt (1999), and can be used in many practical situations. For example in catastrophe reinsurance the same event might affect different portfolios; or in fire insurance, where the same fire can cause damage to neighbouring buildings or properties insured under different policies, etc.

Assumptions:

1. Let N be the total number of claims in a fixed period of time. It is assumed N belongs to Panjer's class of counting distribution
2. Let $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$ be sequences of i.i.d. random variables representing the claim amounts for portfolios 1 and 2, respectively.

    Model 1a: In this model we assume that the individual claim amounts X_i and Y_i are dependent for each i and their joint probability density function is known $f(x, y)$. For example, there might be extra information about the dependence structure that can be incorporated as a mixing parameter, etc. X_i and Y_j are independent for $i \neq j$.

Model 1b: In this model we assume that the claim amounts X_i and Y_i are independent for each event, and the marginal probability density functions are given by $f_X(x)$ and $f_Y(y)$ respectively.

3. $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$ are independent of N.

Under these assumptions, the aggregate claim amounts for each risk at the end of the time period are

$$S_1 = \sum_{i=1}^{N} X_i \qquad \text{and} \qquad S_2 = \sum_{i=1}^{N} Y_i.$$

MODEL 2: We assume that an insurance company has a portfolio that receives claims in a certain period of time. This insurance company wants to buy excess of loss reinsurance for some layers of this portfolio.

Assumptions:

1. The total number of claims for the insurer in a fixed period of time is N.

2. Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables representing the individual claim amounts for the insurer.

3. The insurer passes to the reinsurer two layers for the intervals (m_1, m_2) and (m_2, m_3). Note that these layers are consecutive, which is a common situation in reinsurance practice. For each layer, the individual claim amounts to the reinsurer are:

$$Z_{i1} = \min(\max(0, X_i - m_1), m_2 - m_1) \qquad Z_{i2} = \min(\max(0, X_i - m_2), m_3 - m_2).$$

The total aggregate amounts for the reinsurer for each layer are

$$S_1 = \sum_{i=1}^{N} Z_{i1} \qquad \text{and} \qquad S_2 = \sum_{i=1}^{N} Z_{i2}.$$

Note that in this case both aggregate claim amounts are subject to the same number of claims, as in Model 1, and the claim amounts are dependent.

In the rest of this paper we will refer only to these three models. Sundt (1999) gives more examples where these models can be applied.

Our interest is to use these models when the reinsurer undertakes excess of loss for two dependent portfolios that can be modelled under the assumptions of Model 1 and Model 2.

In the case of excess of loss reinsurance, under the assumptions of Model 1, and each portfolio is reinsured by the same reinsurer under an excess of loss contract with retentions M_1 and M_2 (say), and therefore for each event the reinsurer's claim amounts will be

$$X_i^R(M_1) = \max(X_i - M_1, 0) \qquad Y_i^R(M_2) = \max(Y_i - M_2, 0).$$

and therefore in this case the aggregate claim amounts for the reinsurer at the end of the period are

$$S_1^R = \sum_{i=1}^{N} X_i^R \qquad \text{and} \qquad S_2^R = \sum_{i=1}^{N} Y_i^R. \tag{1}$$

We assume that the excess of loss contract has infinite upper limit, however we will see in the numerical examples that this is not a restriction, since we know the joint distribution of the individual claim amounts we are able to calculate the joint distribution of the individual claims for the reinsurer as well as the marginals. In the case of Model 2 we have directly assumed that the reinsurer takes two layers of the same risk.

3 Joint distribution of dependent aggregate claim amounts

Sundt (1999) and Ambagaspitiya (1999) developed multivariate recursions that allow us to calculate the joint distribution of the aggregate claim amounts under the assumptions of Model 1 and Model 2. The specification of these recursions in the bivariate case are detailed below.

1. There are two portfolios that are affected by the same events.

2. N, the number of claims, belongs to Panjer's class of counting distributions.

3. The individual claim amounts for each portfolio are not necessarily independent and are integer-valued random variables, whose joint distribution is known. The joint probability function is given by $f(x, y)$, for $x = 0, 1, 2, \ldots,$ $y = 0, 1, 2, \ldots.$

The aggregate claim amounts are as given in formula (1), and the recursion for the joint distribution of (S_1, S_2) is as follows:

$$g(s_1, s_2) = \sum_{u=0}^{s_1} \left(a + \frac{bu}{s_1}\right) \sum_{v=0}^{s_2} f(u, v) g(s_1 - u, s_2 - v), \tag{2}$$

for $s_1 = 1, 2, \ldots, s_2 = 0, 1, 2, \ldots$

$$g(s_1, s_2) = \sum_{v=0}^{s_2} \left(a + \frac{bv}{s_2}\right) \sum_{u=0}^{s_1} f(u, v) g(s_1 - u, s_2 - v), \tag{3}$$

for $s_1 = 0, 1, 2, \ldots, s_2 = 1, 2, \ldots$

It is clear that even when we use the models described above from the reinsurer's point of view this recusion can be used to calculate the joint distribution of (S_1^R, S_2^R). If the individual claim amounts are not integer-valued random variables we can discretise the corresponding distribution in appropriate units.

In many cases the insurer/reinsurer would only be interested in calculating the distribution of the sum of the total losses for both portfolios. For example, if we are interested in calculating how much capital we must allocate (under some criteria) to each portfolio or to the combined, then we would be interested in the distribution of the sum of the corresponding aggregate claim amounts. To calculate the distribution of the sum of dependent aggregate claim amounts under the assumptions of the models described above it is not necessary to calculate the joint distribution. In the next section we discuss how to do so.

3.1 Distribution of the sum of dependent aggregate claim amounts

MODEL 1. Under the assumptions of Model 1 both risks are subject to the same number of claims N. We are interested in calculating the probability density function of

$$S = S_1 + S_1 = \sum_{i=1}^{N} X_i + \sum_{i=1}^{N} Y_i = \sum_{i=1}^{N} (X_i + Y_i), \tag{4}$$

therefore we need the distribution of the sum $X_i + Y_i$ for $i \geq 1$. We denote $U_i = X_i + Y_i$. If we can calculate the distribution of U_i, then the distribution of S, defined in (4) can be calculated using Panjer's recursion for univariate compound random variables.

Under the assumptions of Model 1b, X_i and Y_i are independent, hence the distribution of $X_i + Y_i$ can be calculated using the convolution of the marginal distributions, and then it follows that we can calculate the distribution of S.

In the case of Model 1a, the individual claim amounts from the *ith* event have joint density $f(x, y)$, therefore in this case the probability density function of U_i can be calculated as follows

$$P(U_i = u) = P(X_i + Y_i = u) = \sum_{m=0}^{u} P(X_i = u - m, Y_i = m), \qquad \text{for } u = 0, 1, \ldots.$$

And since we know the joint probability function we can easily calculate the distribution of the sum of the individual claim amounts, and therefore the distribution of $S_1 + S_2$. If we are interested in doing the same from the reinsurer's point of view we can follow the same procedure but replacing X_i and Y_i by X_i^R and Y_i^R respectively.

MODEL 2: Under the assumptions of Model 2, the reinsurer is taking two layers of the same risk, and therefore would be interested in calculating the distribution of total losses for these dependent aggregate claim amounts. In this case the total losses are written as

$$S = S_1^R + S_2^R = \sum_{i=1}^{N} (Z_{i1} + Z_{i2}), \tag{5}$$

hence we require (as in Model 1) the distribution of $Z_{i1} + Z_{i2}$. It is clear that $Z_{i1} + Z_{i2} = \min(\max(0, X_i - m_1), m_3 - m_1) = Z_c$, which is a random variable representing the individual claim amount for the combined layer (m_1, m_3). Since we have the distribution of the $X_i s$ we know how to calculate the distribution of Z_c. Therefore, the distribution of S in (5) can be calculated using Panjer's recursion for univariate aggregate claim amounts.

However, Mata (2000) showed that when each layer is subject to different aggregate conditions such as reinstatement the distribution of the sum of total losses for each layers is not equivalent to the distribution of total losses for the combined layer, and therefore in this case the bivariate recursion given in formulae (2) and (3) must be used.

Example 1. Suppose a reinsurer is asked to underwrite an excess of loss contract for the following two layers 10 xs 20 and 10 xs 30 from any two risks. He is given the following information about the insurance portfolios:

1. Both portfolios are subject to the same number of events N that follows a Poisson distribution with parameter $\lambda = 1$.
2. The individual claims for each portfolio X_i and Y_i have the same marginals and follow a Pareto distribution with parameters $\alpha = 3$ and $\beta = 10$.

Therefore, for each event the reinsurer would pay for each portfolio the following amounts

$$X_i^R = \min(\max(0, X_i - 20), 10) \qquad \text{and} \qquad Y_i^R = \min(\max(0, X_i - 30), 10)$$

Since we do not have any extra information about the individual claim amounts for each portfolio there are many dependence structure that can be used in order to calculate the joint distribution of the individual claim amounts, or they could as well be independent. Even if we were given the marginals and the correlation coefficient there are many possibilities for the joint distribution of the individual claim amounts, see, for example, Embrechts, McNeil and Straumann (1999). Let us study the following three set ups:

(a) The individual claim amounts X_i and Y_i are independent.

(b) The individual claim amounts are dependent and their joint distribution follows a bivariate Pareto distribution with parameters $(\alpha, \beta_1, \beta_2)$ and the joint probability density function is given below, see, for example, Mardia *et al* (1979).

$$f(x, y) = \frac{\alpha(\alpha+1)}{\beta_1 \beta_2} \left(-1 + \frac{x + \beta_1}{\beta_1} + \frac{y + \beta_2}{\beta_2} \right)^{-(\alpha+2)}.$$

(c) The layers belong to the same risk in which case we observe that they are consecutive layers.

Under each of these set ups we are able to calculate the correlation coefficient between the aggregate claim amounts

(a) $\rho(S_1^R, S_2^R) = 0.019$

(b) $\rho(S_1^R, S_2^R) = 0.206$, and

(c) $\rho(S_1^R, S_2^R) = 0.761$

Figures 1 and 2 show the cumulative distribution of the sum of the aggregate claim amounts for these portfolios under the three different dependence structure and the cumulative distribution function if we assume that the portfolios are independent. We observe from Figure 1 that when we assume that the individual claims are independent as in case (a) the distribution of the sum of the aggregate claim amounts is very close to the distribution of the sum of the aggregate claim amounts when we ignore the whole dependence structure and we assume independent portfolios. However, under the assumptions of case (b) the distribution of the sum of the aggregate claim amounts is close to the distribution under the assumption of independence, but we observe that when the claim amounts are dependent with joint distribution a bivariate Pareto the distribution of the sum has heavier tail than in case (a).

However, in Figure 2 we notice that when the layers belong to the same risk as in case (c) there is a considerable difference between the distribution of the sum of the aggregate claim amounts as compare with the distribution under the assumption of independence. This is of course due to the fact that when the layers belong to the same risk there is a strong dependence structure between individual claim amounts since a claim for the second layer cannot be greater than zero unless the claim for the first layer has reached its maximum amount.

With this numerical example we have shown how different dependence assumptions affect the distribution of total losses for the reinsurer and that therefore more information is needed in order to have better measures of risk or even in the process of pricing such risks. Particularly, it is important to identify when two or more layers belong to the same risk. If we know that the layers belong to different risks, we should have appropriate models not only when there are common events but also when the claim amounts are dependent.

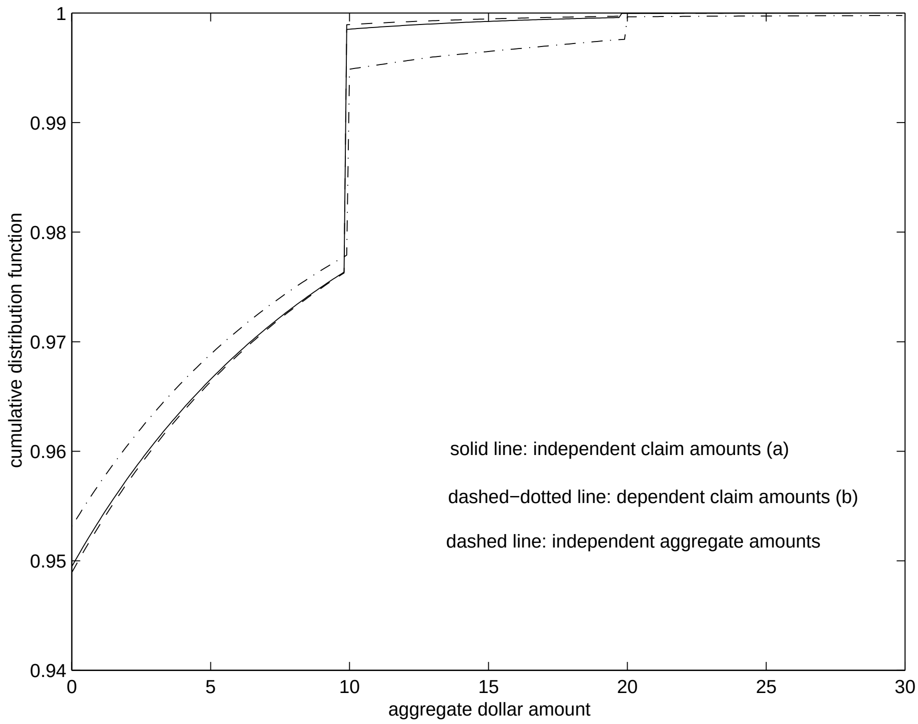

Figure 1: Distribution of total losses for reinsurance aggregate claim amounts

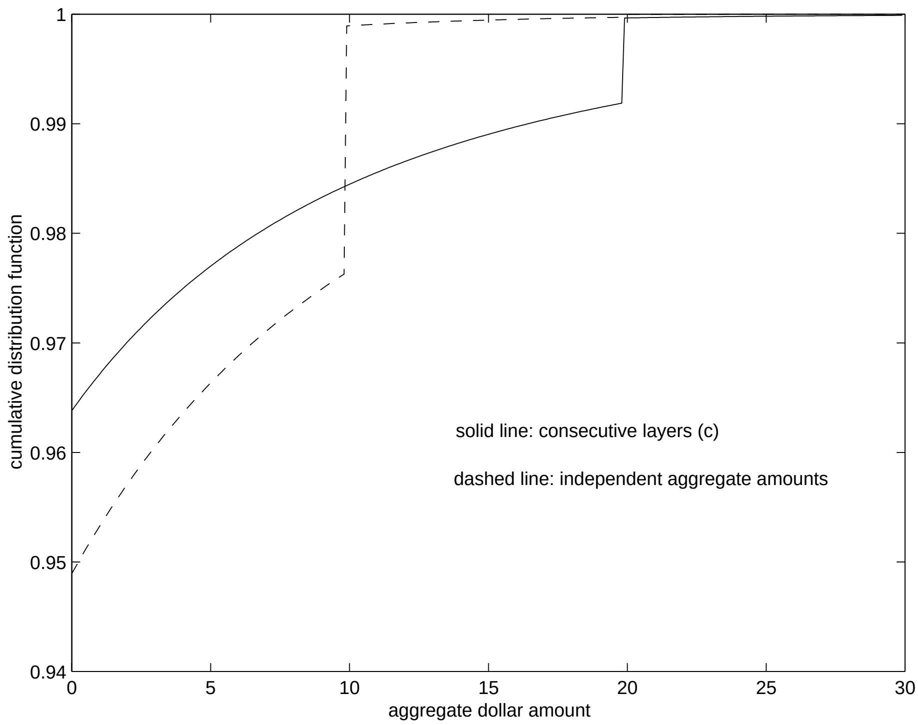

Figure 2: Distribution of total losses for reinsurance aggregate claim amounts from the same risk

4 Asymptotic behaviour of dependent reinsurance aggregate claim amounts

In the previous section we studied the distribution of the sum of dependent aggregate claim amounts from the reinsurer's point of view. In Figures 1 and 2 we observe the effect of different dependence structures that might be used to model reinsurance aggregate claim amounts.

Clearly, given the distribution of the individual claim amounts for the insurance portfolios and the distribution of the number of claims, the choice of retention level M completely determines the distribution of the aggregate claim amounts for the reinsurer. For large values of the retention level, the reinsurer has a high probability of having zero losses, and the probability of having a positive claim becomes very small.

From all the above and based on the results shown in Figures 1 and 2,it is our objective to give some answers to the following question:

Are the reinsurer's aggregate claim amounts from separate but dependent risks approximately independent for large values of the retention levels?

In the next section we give some theoretical insight into the asymptotic behaviour of the aggregate claim amounts for large values of the retention levels under different dependence assumptions.

4.1 On measures of asymptotic independence/dependence for reinsurance aggregate claim amounts

In order to provide some answers to the question outlined above we start by giving the definition of asymptotic independence we will use in our case. Since for large values of the retention levels the probability that the reinsurer makes a positive payment tends to zero we use the following definition of asymptotic independence.

Definition 1 *Suppose two sequences of random variables $\{X_n\}$ and $\{Y_n\}$ are dependent for each n and their joint distribution and their marginals are known. If these random variables satisfy*

$$\lim_{n\to\infty} \frac{P(X_n \in A, Y_n \in B)}{P(X_n \in A)\, P(Y_n \in B)} = 1, \tag{6}$$

for all sets A and B that have positive probability, then it is said that X_n and Y_n are asymptotically independent.

We prove below that under certain assumptions about the reinsurer's aggregate claim amounts, the joint distribution satisfies the condition given in (6) for some sets A and B but not *for all* sets. We set out below the assumptions we require to prove this result.

Assumptions and notation for Proposition 1:

1. The insurance portfolios satisfy the assumptions of Model 1b, where the dependence structure arises only through the common number of events N.

2. The individual claim amounts for the insurer X_i and Y_i are integer-valued independent random variables and can take values $x = 0, 1, 2, \ldots$ and $y = 0, 1, 2, \ldots$. We assume that the probability functions of the individual claim amounts do not have finite upper limit. We denote $f_1(x)$ and $f_2(y)$ the probability functions of the individual claim amounts for each portfolio.

3. The common number of claims N belongs to Panjer's class of counting distributions. Thus, a and b will represent the constants of Panjer's class.

4. We denote by $pgf_N(s)$ the probability generating function of N which is defined as

$$pgf_N(s) = E[s^N]$$

5. Let $\{M_{1,n}\}_{n\geq 1}$ and $\{M_{2,n}\}_{n\geq 1}$ be sequences of integer numbers representing the retention levels for the excess of loss reinsurance for each portfolio. These sequences satisfy $M_{i,n} > M_{i,n-1}$ for $i = 1, 2$. For a given value of the retention the reinsurer would pay for each claim the following amounts

$$X_i^R(M_{1,n}) = \max(X_i - M_{1,n}, 0) \qquad Y_i^R(M_{2,n}) = \max(Y_i - M_{2,n}, 0).$$

Therefore the aggregate claim amounts for the reinsurer are

$$S_1^R(M_{1,n}) = \sum_{i=1}^{N} X_i^R(M_{1,n}) \qquad \text{and} \qquad S_2^R(M_{2,n}) = \sum_{i=1}^{N} Y_i^R(M_{2,n}), \tag{7}$$

6. The retention levels are such that the reinsurer's aggregate claim amounts satisfy

$$\lim_{n\to\infty} P\left(S_i^R(M_{i,n}) = 0\right) = 1, \qquad \text{for } i = 1, 2. \tag{8}$$

7. The probability functions for the individual claim amounts for the reinsurer are $f_{1,n}(x) = P\left(X_i^R(M_{1,n}) = x\right)$ and $f_{2,n}(y) = P\left(Y_i^R(M_{2,n}) = y\right)$, for $x, y = 0, 1, 2 \ldots$

8. We assume that the probability functions for the individual claim amounts satisfy

$$\lim_{n\to\infty} \frac{f_i(x + M_{i,n})}{f_i(y + M_{i,n})} = C(x, y) \qquad \text{for } x = 1, 2, \ldots, y$$

for $y = 1, 2, \ldots,$ where $C(x, y)$ is a constant that depends on x and y and $0 \leq C(x, y) < \infty$.

9. The probability functions for the aggregate claim amounts are $g_{1,n}(s_1) = P\left(S_1^R(M_{1,n}) = s_1\right)$ and $g_{2,n}(s_2) = P\left(S_2^R(M_{2,n}) = s_2\right)$, for $s_1, s_2 = 0, 1, 2, \ldots.$ We assume that $g_{i,n}(s_i) > 0$ for $i = 1, 2$ and for $s_1, s_2 = 0, 1, 2, \ldots$

10. The joint probability function for the aggregate claim amounts is defined as $g_n(s_1, s_2) = P\left(S_1^R(M_{1,n}) = s_1, S_2^R(M_{2,n}) = s_2\right)$, for $s_1, s_2 = 0, 1, 2, \ldots$

Proposition 1 *Under the assumptions outlined above the aggregate claim amounts for the reinsurer defined in (7) satisfy:*

a) $\displaystyle\lim_{n\to\infty} \frac{g_n(0, 0)}{g_{1,n}(0) g_{2,n}(0)} = 1$

b) $\displaystyle\lim_{n\to\infty} \frac{P\left(S_1^R(M_{1,n}) \leq s_1, S_2^R(M_{2,n}) \leq s_2\right)}{P\left(S_1^R(M_{1,n}) \leq s_1\right) P\left(S_2^R(M_{2,n}) \leq s_2\right)} = 1 \qquad$ *for all* $s_1, s_2 = 0, 1, 2, \ldots$

c) $\lim_{n\to\infty} \dfrac{g_n(s_1,0)}{g_{1,n}(s_1)g_{2,n}(0)} = 1 \qquad \textit{for } s_1 = 1,2,\ldots$

d) $\lim_{n\to\infty} \dfrac{g_n(0,s_2)}{g_{1,n}(0)g_{2,n}(s_2)} = 1 \qquad \textit{for } s_2 = 1,2,\ldots$

e) $\lim_{n\to\infty} \dfrac{g_n(s_1,s_2)}{g_{1,n}(s_1)g_{2,n}(s_2)} = 1 + \dfrac{1}{a+b} \qquad \textit{for } \ s_1, s_2 = 1,2,\ldots$

Proof.

a) Since the aggregate claim amounts satisfy the condition (8) in assumption 6, we have that $\lim_{n\to\infty} g_{i,n}(0) = 1$ for $i = 1,2$. This also implies that $\lim_{n\to\infty} f_{i,n}(0) = 1$ for $i = 1,2$. The joint probability of being zero is given by $g_n(0,0) = pgf_N(f_{1,n}(0)f_{2,n}(0))$, therefore from assumption 6 it also holds that $\lim_{n\to\infty} g_n(0,0) = 1$. Then we directly obtain the result in a).

b) From a) we have that the joint probability of being zero tends to one as well as the probability of each aggregate claim amount being zero. Hence, the result in b) follows directly since we are considering cumulative probabilities which include the point zero.

c) Since we have assumed that the random variables are integer-valued we can evaluate the joint distribution of the aggregate claim amounts using the bivariate recursion proposed by Sundt (1999) defined in formulae (2) and (3). We will prove the statement in c) by induction and we do the basic step for $s_1 = 1$.

$$\lim_{n\to\infty} \frac{g_n(1,0)}{g_{1,n}(1)g_{2,n}(0)} = \lim_{n\to\infty} \frac{\frac{1}{1-af_{1,n}(0)f_{2,n}(0)}(a+b)f_{1,n}(1)f_{2,n}(0)g_n(0,0)}{\left(\frac{1}{1-af_{1,n}(0)}(a+b)f_{1,n}(1)g_{1,n}(0)\right) g_{2,n}(0)} = \frac{1-a}{1-a},$$

where the last inequality is due to the result in a). The limit above is equal to 1, only when $a \neq 1$. For the Poisson, Negative Binomial and Binomial distributions a takes the values: $a = 0$, $a = (1-p)$ and $a = -p/(1-p)$, respectively. Hence $a \neq 1$ for the three distributions that belong to Panjer's class. Therefore the limit above is equal to 1.

We now state the inductive hypothesis: Let us assume that the result in c) holds for all the pairs $(s_1, 0)$ such that $s_1 = 1, 2, \ldots, X$, then we have to prove that it is also true for $(X+1, 0)$. From the recursion in (2) we have

$$g_n(X+1,0) = \frac{f_{2,n}(0)}{1-af_{1,n}(0)f_{2,n}(0)} \sum_{u=1}^{X+1} \left(a + \frac{bu}{X+1}\right) f_{1,n}(u) g_n(X+1-u,0)$$

Using Panjer's univariate recursion for $g_{1,n}(X+1)$, we have

$$\frac{g_n(X+1,0)}{g_{1,n}(X+1)g_{2,n}(0)} =$$

$$\frac{\frac{f_{2,n}(0)g_{2,n}(0)[1-af_{1,n}(0)]}{1-af_{1,n}(0)f_{2,n}(0)} \sum_{u=1}^{X+1} \left(a + \frac{bu}{X+1}\right) f_{1,n}(u) g_{1,n}(X+1-u) \frac{g_n(X+1-u,0)}{g_{1,n}(X+1-u)g_{2,n}(0)}}{g_{2,n}(0) \sum_{u=1}^{X+1} \left(a + \frac{bu}{X+1}\right) f_{1,n}(u) g_{1,n}(X+1-u)}$$

Since $\lim_{n\to\infty} f_{1,n}(0) = \lim_{n\to\infty} f_{2,n}(0) = 1$, and using the inductive hypothesis, for any $\epsilon > 0$ there is a N such that for $n \geq N$

$$1 - \epsilon < \frac{f_{2,n}(0)[1-af_{1,n}(0)]}{1-af_{1,n}(0)f_{2,n}(0)} \frac{g_n(X+1-u,0)}{g_{1,n}(X+1-u)g_{2,n}(0)} < 1 + \epsilon,$$

for $u = 1, 2, \ldots, X+1$. Hence for $n \geq N$

$$1 - \epsilon < \frac{g_n(X+1,0)}{g_{1,n}(X+1)g_{2,n}(0)} < 1 + \epsilon,$$

which proves c).

d) The prove of the statement in d) follows the same argument as c) but using the recursion in formula (3) instead of (2).

e) To prove the statement in e) we will use the results in a), c) and d). We start by proving the result for $s_1 = 1$ and $s_2 = 1$. Using the recursion in formula (2) we have

$$\lim_{n\to\infty} \frac{g_n(1,1)}{g_{1,n}(1)g_{2,n}(1)} = \lim_{n\to\infty} \frac{1}{1-af_{1,n}(0)f_{2,n}(0)} \left[\frac{af_{1,n}(0)f_{2,n}(1)g_n(1,0)}{g_{1,n}(1)g_{2,n}(1)} \right.$$
$$\left. + \frac{(a+b)f_{1,n}(1)\sum_{v=0}^{1} f_{2,n}(v)g_n(0,1-v)}{g_{1,n}(1)g_{2,n}(1)} \right]$$

In the limit above we have three terms, we will analyse each term separately. Using Panjer's univariate recursion for $g_{2,n}(1)$ the limit for the first term can be calculated as follows:

$$\lim_{n\to\infty} \frac{af_{1,n}(0)f_{2,n}(1)g_{2,n}(0)}{\frac{1}{1-af_{2,n}(0)}(a+b)f_{2,n}(1)g_{2,n}(0)} \frac{g_n(1,0)}{g_{1,n}(1)g_{2,n}(0)} = \frac{a(1-a)}{a+b},$$

since from part c) we know that $\lim_{n\to\infty} \frac{g_n(1,0)}{g_{1,n}(1)g_{2,n}(0)} = 1$. For the second term we use the result in d) and Panjer's univariate recursion for $g_{1,n}(1)$, therefore the limit for the second term is

$$\lim_{n\to\infty} \frac{(a+b)f_{1,n}(1)f_{2,n}(0)g_{1,n}(0)}{\frac{1}{1-af_{1,n}(0)}(a+b)f_{1,n}(1)g_{1,n}(0)} \frac{g_n(0,1)}{g_{1,n}(0)g_{2,n}(1)} = 1-a.$$

And finally for the third term we use the result in a) together with Panjer's univariate algorithms for $g_{1,n}(1)$ and $g_{2,n}(1)$, hence the limit is

$$\lim_{n\to\infty} \frac{(a+b)f_{1,n}(1)g_{1,n}(0)f_{2,n}(1)g_{2,n}(0)}{g_{1,n}(1)g_{2,n}(1)} \frac{g_n(0,0)}{g_{1,n}(0)g_{2,n}(0)} = (1-a)\left(\frac{1-a}{a+b}\right).$$

Putting these three results together we obtain

$$\lim_{n\to\infty} \frac{g_n(1,1)}{g_{1,n}(1)g_{2,n}(1)} = \frac{1}{(1-a)}\left[\frac{a(1-a)}{a+b} + (1-a) + \frac{(1-a)^2}{a+b}\right] = 1 + \frac{1}{a+b}.$$

We have to use a bivariate induction to prove the result in e). We state the inductive hypothesis as follows: assume that

$$\lim_{n\to\infty} \frac{g_n(s_1,s_2)}{g_{1,n}(s_1)g_{2,n}(s_2)} = 1 + \frac{1}{a+b}$$

for all (s_1, s_2) such that $s_1 = 1, 2, \ldots, X$ and $s_2 = 1, 2, \ldots, Y$, together with the results in a), c) and d). Therefore using this hypothesis we need to prove that the result in statement e) holds in the following three cases:

i) For all $(X+1, y)$ such that $y = 1, 2, \ldots, Y$

ii) For all $(x, Y+1)$ such that $x = 1, 2, \ldots, X$

iii) For $(X+1, Y+1)$.

In each case the argument is similar except that in i) we use the recursion in (2) whereas in ii) we use the recursion in formula (3). We will prove the result only in case i), the other cases follow. Let us fix y such that $y = 2, 3, \ldots, Y$. Together with the inductive hypothesis and a), c) and d) we also assume that the statement in e) holds for all the pairs $(X+1, s_2)$ such that $s_2 = 1, \ldots, y-1$, then we want to prove the result for $(X+1, y)$.

Using the recursion in (2) to evaluate $g_n(X+1, y)$ we have

$$\lim_{n\to\infty} \frac{g_n(X+1,y)}{g_{1,n}(X+1)g_{2,n}(y)} =$$

$$\lim_{n\to\infty} \frac{1}{1 - af_{1,n}(0)f_{2,n}(0)} \left[\frac{af_{1,n}(0)\sum_{v=1}^{y} f_{2,n}(v)g_n(X+1, y-v)}{g_{1,n}(X+1)g_{2,n}(y)} \right.$$

$$\left. + \frac{\sum_{u=1}^{X+1}\left(a + b\frac{u}{X+1}\right) f_{1,n}(u)\sum_{v=0}^{y} f_{2,n}(v)g_n(X+1-u, y-v)}{g_{1,n}(X+1)g_{2,n}(y)} \right]$$

To be able to use the results in a), c) and d) we must separate those terms for which one of the entries is zero in the evaluation of g_n from the terms where both entries are greater than zero. Doing so we obtain the following result

$$\lim_{n\to\infty} \frac{1}{1 - af_{1,n}(0)f_{2,n}(0)} \times$$

$$\left[\frac{af_{1,n}(0)\left(\sum_{v=1}^{y-1} f_{2,n}(v)g_n(X+1, y-v) + f_{2,n}(y)g_n(X+1, 0)\right)}{g_{1,n}(X+1)g_{2,n}(y)} + \right.$$

$$\frac{\sum_{u=1}^{X}\left(a + b\frac{u}{X+1}\right) f_{1,n}(u)\left(\sum_{v=0}^{y-1} f_{2,n}(v)g_n(X+1-u, y-v)\right)}{g_{1,n}(X+1)g_{2,n}(y)} +$$

$$\frac{\sum_{u=1}^{X}\left(a + b\frac{u}{X+1}\right) f_{1,n}(u)f_{2,n}(Y)g_n(X+1-u, 0)}{g_{1,n}(X+1)g_{2,n}(y)} +$$

$$\left. \frac{(a+b)f_{1,n}(X+1)\left(\sum_{v=0}^{y-1} f_{2,n}(v)g_n(0, y-v) + f_{2,n}(y)g_n(0,0)\right)}{g_{1,n}(X+1)g_{2,n}(y)} \right]$$

Now we can use the same method of multiplying and dividing each term that contains g_n by the corresponding product of the marginals. Then for those terms where one or both entries is zero the ratio tends to one, and for the ratios where both entries are greater than zero the ratio tends to $1 + 1/(a+b)$ due to the inductive hypothesis. Therefore, the limit above can be written as follows

$$\lim_{n\to\infty}\frac{g_n(X+1,y)}{g_{1,n}(X+1)g_{2,n}(y)} = \lim_{n\to\infty}\frac{1}{1-af_{1,n}(0)f_{2,n}(0)}\times$$

$$\left[\left(\frac{af_{1,n}(0)g_{1,n}(X+1)}{g_{1,n}(X+1)}\right)\left(\frac{\left(1+\frac{1}{a+b}\right)\sum_{v=1}^{y-1}f_{2,n}(v)g_{2,n}(y-v)+f_{2,n}(y)g_{2,n}(0)}{g_{2,n}(y)}\right)+\right.$$

$$+\left(\frac{\sum_{u=1}^{X}\left(a+b\frac{u}{X+1}\right)f_{1,n}(u)g_{1,n}(X+1-u)}{g_{1,n}(X+1)}\right)\times$$

$$\left(\frac{\left(1+\frac{1}{a+b}\right)\sum_{v=0}^{y-1}f_{2,n}(v)g_{2,n}(y-v)+f_{2,n}(y)g_{2,n}(0)}{g_{2,n}(y)}\right)+$$

$$\left.+\left(\frac{(a+b)f_{1,n}(X+1)g_{1,n}(0)}{g_{1,n}(X+1)}\right)\left(\frac{f_{2,n}(0)g_{2,n}(y)+\sum_{v=1}^{y}f_{2,n}(v)g_{2,n}(y-v)}{g_{2,n}(y)}\right)\right]. \quad (9)$$

From condition (8) in assumption 6 it follows that $\lim_{n\to\infty} g_{i,n}(x) = 0$ for all $x = 1, 2, 3, \ldots$ and for $i = 1, 2$. The same result holds for $f_{i,n}(x)$. Using these results we have

$$\lim_{n\to\infty}\sum_{v=1}^{y} f_{2,n}(v)g_{2,n}(y-v) = 0,$$

since each term tends to zero. However, when we divide the above sum by $g_{2,n}(y)$ we obtain the following result

$$\lim_{n\to\infty}\frac{\sum_{v=1}^{y} f_{2,n}(v)g_{2,n}(y-v)}{\frac{1}{1-af_{2,n}(0)}\sum_{v=1}^{y}\left(a+b\frac{v}{y}\right)f_{2,n}(v)g_{2,n}(y-v)} =$$

$$\lim_{n\to\infty}\frac{f_{2,n}(1)g_{2,n}(y-1)+\cdots+f_{2,n}(y-1)g_{2,n}(1)+f_{2,n}(y)g_{2,n}(0)}{\frac{1}{1-af_{2,n}(0)}\left(\left(a+\frac{b}{y}\right)f_{2,n}(1)g_{2,n}(y-1)+\cdots+(a+b)f_{2,n}(y)g_{2,n}(0)\right)}$$

We observe that in the limit above each term tends to zero, however the last term contains $g_{2,n}(0)$ which tends to 1 as n tends to infinity. Therefore we divide each term by $f_{2,n}(y)$ and we obtain

$$\lim_{n\to\infty}\frac{\frac{f_{2,n}(1)g_{2,n}(y-1)}{f_{2,n}(y)}+\cdots+g_{2,n}(0)}{\frac{1}{1-af_{2,n}(0)}\left(\left(a+\frac{b}{y}\right)\frac{f_{2,n}(1)g_{2,n}(y-1)}{f_{2,n}(y)}+\cdots+(a+b)g_{2,n}(0)\right)} = \frac{1-a}{a+b}.$$

The last equality is due to the result in assumption 8 where we assumed that $\lim_{n\to\infty} \frac{f_{i,n}(x)}{f_{i,n}(y)} = C(x,y)$ which is a constant for $i = 1, 2$, and $\lim_{n\to\infty} g_{i,n}(x) = 0$ for all $x = 1, 2, \ldots$. From the discussion above we obtain directly the following results

$$\lim_{n\to\infty} \frac{\sum_{v=1}^{y-1} f_{2,n}(v) g_{2,n}(y-v)}{g_{2,n}(y)} = 0$$

$$\lim_{n\to\infty} \frac{\sum_{u=1}^{X} \left(a + b\frac{u}{X+1}\right) f_{1,n}(u) g_{1,n}(X+1-u)}{g_{1,n}(X+1)} = 0$$

$$\lim_{n\to\infty} \frac{(a+b) f_{1,n}(X+1) g_{1,n}(0)}{g_{1,n}(X+1)} = 1-a$$

Hence from all the above, we can evaluate the limit in formula (9) and we obtain the following result

$$\lim_{n\to\infty} \frac{g_n(X+1,Y)}{g_{1,n}(X+1) g_{2,n}(Y)} =$$

$$\frac{1}{1-a}\left[a\left(\frac{1-a}{a+b}\right) + (1-a)\left(1 + \frac{1-a}{a+b}\right)\right] = 1 + \frac{1}{a+b},$$

which is the result shown in f). □

Remark: Note that $1 + \frac{1}{a+b} > 1$ since $a + b$ is always positive.

From the results shown in Proposition 1 we make the following remarks:

1. The statement in b) implies that when we consider cumulative distributions we are including the value of zero which has a high probability for large values of the retention level. Therefore in this case the joint cumulative distribution is very close to the product of the marginal cumulative distributions, which is the cumulative distribution in the case of independent risks. This result explains the behaviour observed in Figures 1 and 2 where we considered the cumulative distribution function of the sum of the reinsurer's aggregate claim amounts.

2. The key point for the proof of Proposition 1 was that the aggregate claim amounts are dependent only through the common number of events and therefore the individual claim amounts are independent. This is one of the simplest cases of

dependent insurance/reinsurance risks. Hence, under more complicated assumptions of dependence between risks the asymptotic behaviour of the aggregate claim amounts could be different. In the next section we illustrate this fact by comparing numerically the asymptotic behaviour under the assumptions of Proposition 1 with the asymptotic behaviour when the layers belong to the same risk as in Model 3.

3. Statement e) shows cases where the rate between the joint distribution and the product of the marginals does not tend to 1. However, if $a+b$ takes large values the limit would be close to 1. For example, when N follows a Poisson distribution with parameter λ, $a+b=\lambda$ which is the expected number of common events per unit of time. If we increase λ we are increasing the dependence parameter as the expected number of common events becomes large. Nevertheless, by increasing λ the limit in e) would be closer to 1 which implies that the joint distribution is closer to the independent case.

The following result shows another case when the result in e) also holds.

Proposition 2 *Suppose that two risks follow assumptions 1 and 3 of Proposition 1. Let $\{M_{1,n}\}$ and $\{M_{2,n}\}$ be sequences of real numbers representing the reinsurance retention levels for each portfolio. These sequences are such that the reinsurer's aggregate claim amounts satisfy the condition in (8) and that for each n*

$$P(S_1^R(M_{1,n})=0)=P(S_2^R(M_{2,n})=0)=\alpha_n.$$

Then, the reinsurance aggregate claim amounts defined in (7) satisfy

$$\lim_{n\to\infty}\frac{P\left(S_1^R(M_{1,n})>0,S_2^R(M_{2,n})>0\right)}{P\left(S_1^R(M_{1,n})>0\right)P\left(S_2^R(M_{2,n})>0\right)}=1+\frac{1}{a+b}.$$

Proof. Note that in this case we do not require that the individual claim amounts are integer-valued random variables. We also do not need that the retention levels are integer numbers. We denote by $f_{i,n}(0)$ the probability that an individual claim amount for the reinsurer is zero and by $g_{i,n}(0)$ the probability that the aggregate claim amount for the reinsurer is zero for retention level $M_{i,n}$ for $i=1,2$.

If $n\to\infty$ then $\alpha_n\to 1^-$. We can write the probabilities of being zero in terms of the probability generating function as

$$g_{i,n}(0) = P(S_i^R(M_{i,n}) = 0) = pgf_N(f_{i,n}(0)) = \alpha_n \qquad \text{for } i = 1, 2.$$

Hence, $f_{i,n}(0) = pgf_N^{-1}(\alpha_n)$ for $i = 1, 2$, provided that the inverse of the probability generating function exists. For the Poisson, the Negative Binomial and the Binomial distributions the inverse of the probability generating function can be written explicitly.

As in part a) of Proposition 1 we can write the joint probability of the aggregate claim amounts being zero as follows

$$g_n(0,0) = pgf_N\left(f_{1,n}(0) f_{2,n}(0)\right) = pgf_N\left(\left(pgf_N^{-1}(\alpha_n)\right)^2\right).$$

Therefore,

$$\lim_{n\to\infty} \frac{P\left(S_1^R(M_{1,n}) > 0, S_2^R(M_{2,n}) > 0\right)}{P\left(S_1^R(M_{1,n}) > 0\right) P\left(S_2^R(M_{2,n}) > 0\right)} = \lim_{n\to\infty} \frac{1 - g_{1,n}(0) - g_{2,n}(0) + g_n(0,0)}{(1 - g_{1,n}(0))(1 - g_{2,n}(0))}$$

$$= \lim_{\alpha_n \to 1^-} \frac{1 - 2\alpha_n + pgf_N\left(\left(pgf_N^{-1}(\alpha_n)\right)^2\right)}{(1-\alpha_n)^2} = \frac{0}{0}.$$

Applying L'Hospital rule twice the limit above can be calculated as follows

$$\lim_{\alpha_n \to 1^-} \left[2\frac{d^2}{d\alpha_n^2} pgf_N\left(\left(pgf_N^{-1}(\alpha_n)\right)^2\right) \frac{\left(pgf_N^{-1}(\alpha_n)\right)^2}{\left(\frac{d}{d\alpha_n} pgf_N(pgf_N^{-1}(\alpha_n))\right)^2} + \right.$$

$$\left. \frac{1}{\frac{d}{d\alpha_n} pgf_N(pgf_N^{-1}(\alpha_n))} - \frac{d^2}{d\alpha_n^2} pgf_N((pgf_N^{-1}(\alpha_n)) \frac{(pgf_N^{-1}(\alpha_n))}{\left(\frac{d}{d\alpha_n} pgf_N(pgf_N^{-1}(\alpha_n))\right)^2} \right]$$

From the properties of the probability generating function we have $pgf_N(1) = 1$ and therefore $pgf_N^{-1}(1) = 1$. Moreover, the probability generating function satisfies $\frac{d}{ds} pgf_N(s) = E[N]$. Therefore, the limit above is given by

$$\lim_{n\to\infty} \frac{P\left(S_1^R(M_{1,n}) > 0, S_2^R(M_{2,n}) > 0\right)}{P\left(S_1^R(M_{1,n}) > 0\right) P\left(S_2^R(M_{2,n}) > 0\right)} = \frac{\frac{d^2}{ds^2} pgf_N(1)}{(E[N])^2} + \frac{1}{E[N]}.$$

Therefore, by evaluating the limit above with the corresponding values for each of the distributions that belong to Panjer's class it follows that

$$\lim_{n\to\infty} \frac{P\left(S_1^R(M_{1,n}) > 0, S_2^R(M_{2,n}) > 0\right)}{P\left(S_1^R(M_{1,n}) > 0\right) P\left(S_2^R(M_{2,n}) > 0\right)} = 1 + \frac{1}{a+b}. \qquad \square$$

For Proposition 2 we assumed that both aggregate claim amounts for the reinsurer have the same probability of being zero; nevertheless, in Example 2 we show that this is not a necessary assumption. It seem to be sufficient that when n tends to infinity both probabilities of being zero tend to one. We discuss this in more detail in Example 2

4.2 Numerical illustrations

It is our objective in this section to illustrate numerically the results shown in the previous section. We compare numerically the asymptotic behaviour of reinsurance dependent aggregate claim amounts under the assumptions of Propositions 1 and 2 with the asymptotic behaviour when we consider layers of the same risk as in Model 3.

Example 2. Suppose that two insurance portfolios follow the same distributional assumptions as in Example 1. For each risk the reinsurer takes a layer of size 10 with deductibles $M_{1,n} = n$ and $M_{2,n} = 10 + n$. Therefore, for each event the reinsurer receives claims for the following amounts

$$X_i^R(M_{1,n}) = \min(\max(0, X_i - n), 10) \quad \text{and} \quad Y_i^R(M_{2,n}) = \min(\max(0, Y_i - 10 - n), 10).$$

We consider the same three set ups as described in Example 1 and we want to study the rate of asymptotic dependence as in Definition 1. In this case N follows a Poisson distribution with parameter $\lambda = 1$, therefore $1 + \frac{1}{a+b} = 2$.

Figure 3 shows the asymptotic behaviour as $n \to \infty$ of

$$\frac{P\left(S_1^R(M_{1,n}) > 0, S_2^R(M_{2,n}) > 0\right)}{P\left(S_1^R(M_{1,n}) > 0\right) P\left(S_2^R(M_{2,n}) > 0\right)}$$

From this figures we observe that the asymptotic behaviour of the joint probability of being greater than zero is very different under the three dependence structure. In the case of independent claim amounts the rate of dependence converge to $1 + \frac{1}{a+b}$ as shown in Proposition 1. However, when the claim amounts are dependent as in cases (b) and (c) the rate of dependence tends to infinity. We also notice that in the case of layers of the same risk the rate of dependence goes to infinity faster since in this case the probability of both aggregate claim amounts being greater than zero is equal to the probability of the second aggregate claim amount being greater than zero, therefore

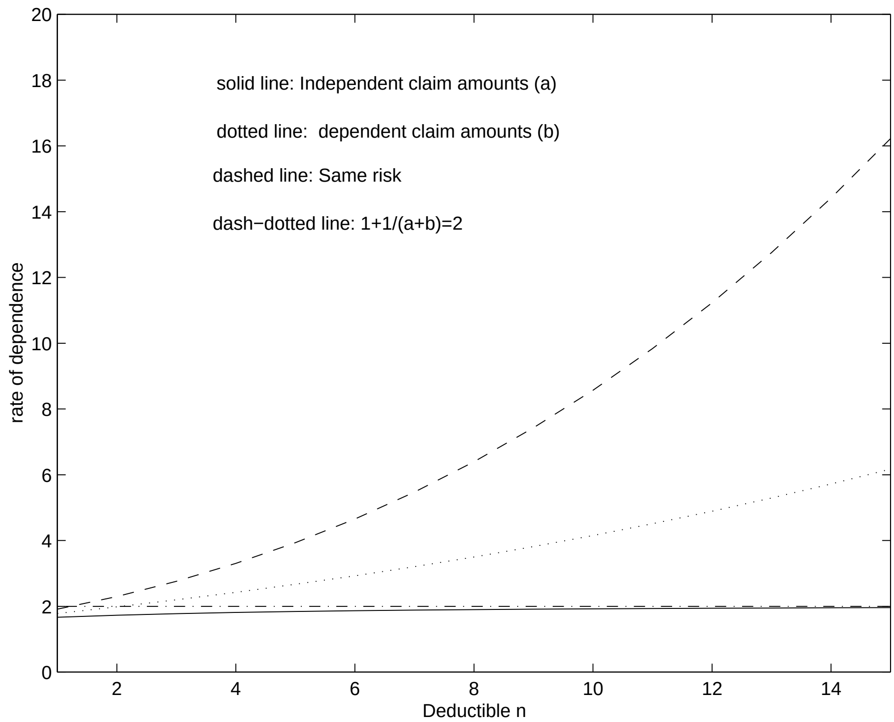

Figure 3: Asymptotic behaviour of $\frac{P\left(S_1^R(M_{1,n})>0, S_2^R(M_{2,n})>0\right)}{P\left(S_1^R(M_{1,n})>0\right)P\left(S_2^R(M_{2,n})>0\right)}$, for Poisson $(\lambda = 1)$

$$\lim_{n\to\infty} \frac{P\left(S_1^R(M_{1,n}) > 0, S_2^R(M_{2,n}) > 0\right)}{P\left(S_1^R(M_{1,n}) > 0\right) P\left(S_2^R(M_{2,n}) > 0\right)} = \lim_{n\to\infty} = \frac{1}{\left(S_1^R(M_{1,n}) > 0\right)} = \infty.$$

Figure 4 shows the asymptotic beahviour of $\frac{g_n(0,1)}{g_{1,n}(0)g_{2,n}(1)}$. We observe that for layers of the same risk the rate has a constant value of zero since it is not possible that the second layer takes a positive value if the first layer is zero. However, for layers of separate risks the rate tends to 1. We observe than in the case of layers from separate risks with dependent claim amounts the convergence to one is much slower than in the case of independent claims amounts as assumed for Proposition 1.

Figure 5 shows the asymptotic dependence of

$$\frac{P\left(S_1^R(M_{1,n}) > s_1, S_2^R(M_{2,n}) > s_2\right)}{P\left(S_1^R(M_{1,n}) > s_1\right) P\left(S_2^R(M_{2,n}) > s_2\right)}, \qquad \text{for } s_1, s_2 = 1, 2, \ldots$$

We notice from Figure 5 that the asymptotic behaviour of joint survival functions for $s_1, s_2 > 0$ is very similar to the asymptotic behaviour of the joint survival functions for $s_1 = s_2 = 0$. In the case of independent claim amounts the rate of dependence for the joint survival function also converges to $1+\frac{1}{a+b}$ as shown in Proposition 2 for $s_1 = s_2 = 0$.

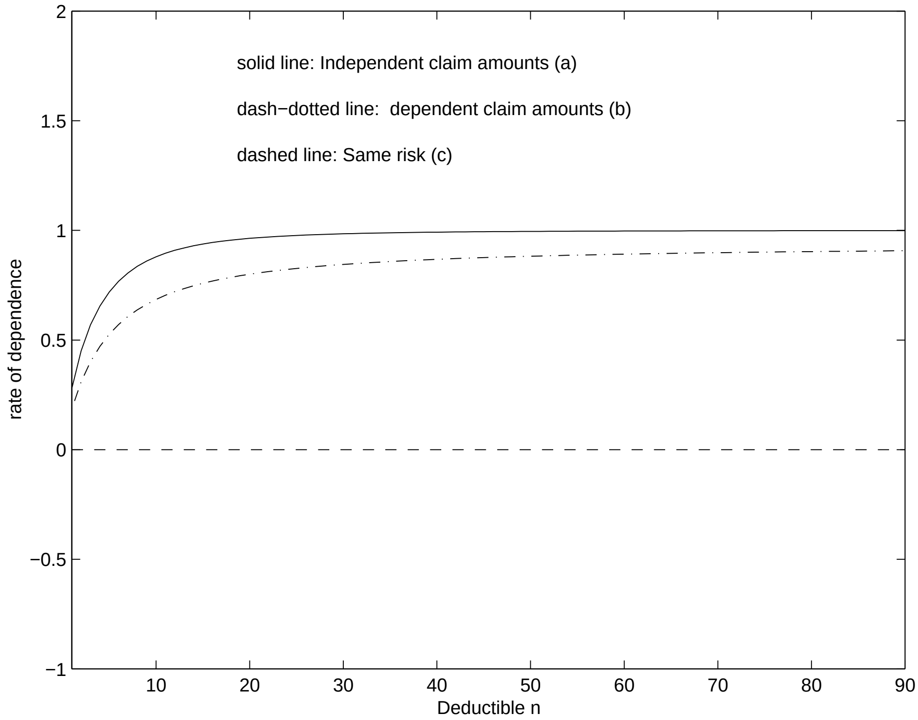

Figure 4: Asymptotic behaviour of $\frac{g_n(0,1)}{g_{1,n}(0)g_{2,n}(1)}$, for Poisson ($\lambda = 1$)

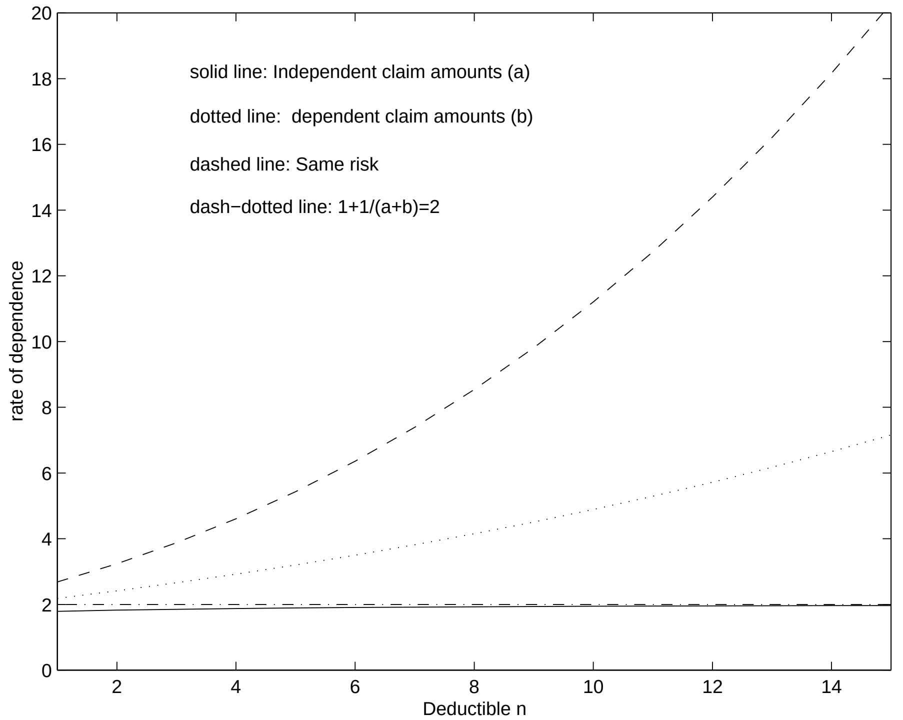

Figure 5: Asymptotic behaviour of $\frac{P\left(S_1^R(M_{1,n})>2, S_2^R(M_{2,n})>2\right)}{P\left(S_1^R(M_{1,n})>2\right)P\left(S_2^R(M_{2,n})>2\right)}$, for Poisson ($\lambda = 1$)

4.2.1 Comments about the assumptions in Section 4.1 in practical applications

1. Assumption 5: Note that in Example 2 we have assumed that the excess of loss layers for the reinsurer have a finite upper limit equal to 10. Therefore assumption 5 seems not to be a restriction in practical terms. However, the fact that we assumed that the reinsurer takes excess of loss layers with infinite limit was to facilitate the analytical proof.

2. Assumption 6: It is not unreasonable to assume that for non proportional reinsurance the retention or deductible is such that the probability of claims affecting the reinsurer has a very small probability. In practice typically the probability of the reinsurer having zero losses is very high (close to 1). However, once a claim occurs the reinsurer would have large losses.

3. Assumption 8: We assumed that the probability function is such that $\lim_{M\to\infty} \frac{f(x+M)}{f(y+M)} = C(x,y)$ for $0 < x \leq y$, where $0 \leq C(x,y) < \infty$. This property is satisfied by most of the continuous loss distributions used to model insurance/reinsurance losses, such as: Exponential, Gamma, Log-normal, Pareto and Generalised Pareto. The Normal distribution does not satisfy this property, but in practical terms the loss distribution for insurace claims are usually skewed and heavy-tailed.

4. Proposition 2: In this proposition we assumed that both aggregate claim amounts have the same probability of being zero. Example 2 shows that this seems not to be a restriction. In fact in Example 2 the layers are such that for each n we have

$$P\left(S_1^R(M_{1,n}) = 0\right) < P\left(S_2^R(M_{2,n}) = 0\right).$$

However, for any $\epsilon > 0$ there is N such that for $n \geq N$

$$P\left(S_2^R(M_{2,n}) = 0\right) - P\left(S_1^R(M_{1,n}) = 0\right) < \epsilon,$$

and as n tends to infinite both probabilities tend to 1.

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