NON-HOMOGENEOUS SEMI-MARKOV REWARD PROCESS FOR THE MANAGEMENT OF HEALTH INSURANCE MODELS.

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Abstract

How it is simple and natural to apply NHSMRP to actuarial science is showed in the paper. Two models useful to solve Permanent Health Insurance (PHI) problems are proposed. The second one is a generalization of the first and permit to take in account not only the insured age, as is usually done in the literature, but also to follow the time evolution.
The reward structure permits to consider simultaneousely the financial development and the illness evolution of the health insurance contract. This gives the possibility to control the dynamic financial equilibrium.

Keywords
Stochastic processes, health insurance, semi-Markov reward processes, permanent health insurance

1. Introduction

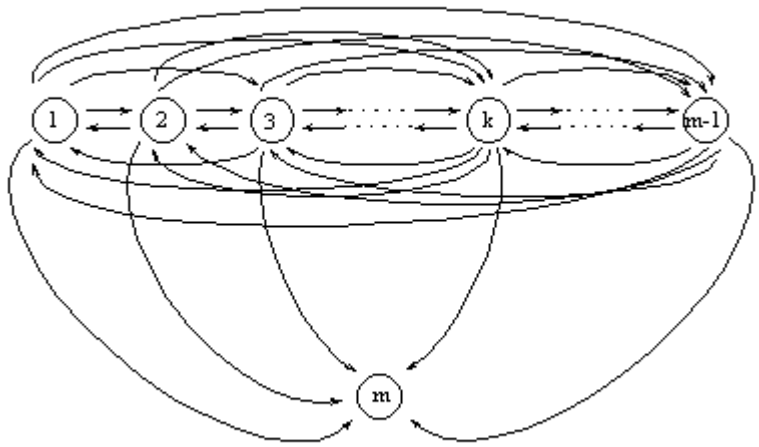
The first application of Semi-Markov Process (SMP) in actuarial field was given by J. Janssen [10]. Many authors successively used these processes and their generalizations for actuarial applications, (see Hoem, [6], Carravetta, De Dominicis, Manca, [1], Sahin, Balcer, [14]). In some books it is also shown how it is possible to use these processes in actuarial science, (see Pitacco, Olivieri, [13], CMIR12 [15]).
These processes can be generalised introducing a reward structure see for example Howard, [7]. in this way are defined the Homogeneous Semi-Markov Reward Processes (HSMRP). The Discrete Time Non-Homogeneous Semi-Markov Reward Processes (DTNHSMRP) were introduced in De Dominicis, Manca [4]. At the author knowing these processes in actuarial field were introduced only for the construction of theoretical models that were not yet applied, (see De Dominicis, Manca, Granata, [3], Janssen, Manca, [9]). The applications proposed in those papers were in pension field.
How it is simple and natural to apply NHSMRP to actuarial science will be shown. The paper will propose two models useful in Permanent Health Insurance (PHI) problems. The second one is a generalization of the first and permit to take in account not only the insured age, as is usually done in the literature, but also to follow the time evolution.
The reward structure permits to have directly the financial evolution of the health insurance contract. This gives the possibility to control the dynamic financial equilibrium.

2. A general actuarial model

It is posed that in the illness development are expected m states.. The first m-1 states are characterized by different illness level.. The state 1 corresponds to health state (no illness) the other ones correspond to different illness stages that give different insurance claims. The m-th state is an absorbing state, it represents the dead state. It is supposed that in the state 1 the insured people pays contributions, in the other m-2 states he get money. The money is paid in

form of an annuity that can be given daily, weekly or monthly, depending on the contract structure. The model is shown in the following graph.

Fig. 1. *m* states model for health insurance.



It is to precise that in this case the arcs are weighted and their weights are the change state probabilities, these probabilities changes in function of the evolution of the temporal variable that is considered, this variable can be the age or the time. Furthermore, also the nodes, that represent the model states, are weighted and their weights represent the reward paid or received in each state. These rewards can be fixed for each state or can change in the time evolution of the model.

3. The Discrete Time non-Homogeneous Semi-Markov Process (DTNHSMP)

In this part will be described the DTNHSMP. These processes were introduced in De Dominicis, Janssen [2] previousely the continuous one case was defined in Iosifescu Manu [8].

First the stochastic process is defined. X_n, $n\in$ N, is a random variable (r.v.) with the set of states $E=\{1, 2, \ldots, m\}$ representing the state at the n-th transition. T_n, $n\in$ N, an other r.v. with set of states equal to N where T_n represents the time of the n-th transition,

$X_n : \Omega \to E \qquad T_n : \Omega \to \mathrm{N}$.

The process (X_n, T_n) is a non-homogeneous markovian renewal process. The kernel

$\mathbf{Q} = [Q_{ij}(\mathrm{s},t)]$

associated to the process is defined in the following way:

$Q_{ij}(\mathrm{s},t) = \mathrm{P}[\, X_{n+1} = j,\, T_{n+1} \le t \mid X_n = i,\, T_n = \mathrm{s}]$

and it results [8]:

$p_{ij}(\mathrm{s}) = \lim_{t \to \infty} Q_{ij}(\mathrm{s},t);\ i, j \in E,\ \mathrm{s}, t \in \mathrm{N}$

where $\mathbf{P}(s) = [p_{ij}(s)]$ is the transition matrix at time s of the embedded non-homogeneous Markov chain. Furthermore it is necessary to introduce the probability that process will leave the state i in a time t:

$S_i(\mathrm{s},t) = \mathrm{P}[\, T_{n+1} \le t \mid X_n = i,\, T_n = \mathrm{s}]$.

Obviously it results that:

(3.1) $$S_i(\mathrm{s},t) = \sum_{j \ne i}^{1,m} Q_{ij}(\mathrm{s},t).$$

Furthermore the following probabilities are considered:

$b_{ij}(\mathrm{s},t) = \mathrm{P}[\, X_{n+1} = j,\, T_{n+1} = t \mid X_n = i,\, T_n = \mathrm{s}]$

These probabilities can be given in function of the $Q_{ij}(\mathrm{s},t)$:

(3.2) $$b_{ij}(s,t)=\begin{cases} Q_{ij}(s,s)=0 \text{ if } t=s \\ Q_{ij}(s,t)-Q_{ij}(s,t-1) \text{ if } t=s+1,s+2,\ldots \end{cases}$$

Now it is possibile to define the probability distribution of the waiting time in each state *i*, given that the state successively occupied is known:

G_{ij} (s,t)=P[$T_{n+1} \leq t \mid X_n = i, X_{n+1} = j, T_n$=s].

Obviously the related probabilities can be obtained by means of the following formula:

(3.3) $$G_{ij}(s,t)=\begin{cases} Q_{ij}(s,t)/p_{ij}(s) & \text{if } p_{ij}(s)\neq 0 \\ 1 & \text{if } p_{ij}(s)=0 \end{cases}.$$

Now the DTNHSMP $Z = (Z_t, t\in \mathrm{N})$ can be defined.
It represents, for each waiting time, the state occupied by the process. The transition probabilities are defined in the following way:

p_{ij} (s,t) = P[$Z_t = j \mid Z_s = i$].

They are obtained solving the following evolution equations:

(3.4) $$p_{ij}(s,t)=\delta_{ij}(1-S_i(s,t))+\sum_{\beta\in E}\sum_{\vartheta=1}^{t} p_{\beta j}(\vartheta,t)b_{i\beta}(s,\vartheta)$$

where δ_{ij} represents the Kronecker symbol.
It is to precise that in our problem p_{ij} (s,t) represents the probability to go from the state *i* to the state *j*, given that at time *s* the system was in the state *i* and at time *t* is in the state *j*, where the states represent a illness degree in correspondence of which change the reward.

4. The reward introduction.

Now a reward structure will be considered, see Mine, Osaki [10], this structure is connected with the *Z* process. In this way a DTNHSMRP will be considered.

This process considers, each time that the system passes for a given state, the "reward" that is received or paid in the state. ψ_i represents the reward received or paid in the *i*-th state. The following formula represents the evolution equation of the DTNHSMRP:

$$
\begin{aligned}
V_i(s,t) &= (1-S_i(s,t))\psi_i \cdot (t-s) + \\
&+ \sum_{\beta \in E} \sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)\psi_i \cdot (\vartheta - s) + \sum_{\beta \in E} \sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta) V_\beta(\vartheta,t)
\end{aligned}
\tag{4.1}
$$

In our case the meaning of the equation (4.1) is the following one.

The left member represents the sum of the payments (assets or liabilities) that were done from the time *s* up the time *t*, given that at time s the process was in the state *i*.

In the first element of the right side of (4.1) the term 1-S_i(s,*t*) represents the probability to remain in the state *i*. In this case a reward ψ_i was paid for *t*-*s* periods. The second element represents the rewards value received or paid in the state i up to the first change of state. At last the third element represents the sum the payments received or paid in the states visited after leaving i.

The following equation represents the discounted case of the DTNHSMRP. In this way (4.2) introduces the possibility to discount the rewards:

$$
\begin{aligned}
V_i(s,t) &= (1-S_i(s,t))\psi_i \cdot a_{t-s]r} + \\
&+ \sum_{\beta \in E} \sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)\psi_i \cdot a_{\vartheta-s]r} + \sum_{\beta \in E} \sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta) V_\beta(\vartheta,t)(1+r)^{s-\vartheta}
\end{aligned}
\tag{4.2}
$$

The only difference with the (4.1) is that the rewards are discounted at time s. The interest rate *r* is supposed, in a first approach, to be constant. Each time that the system is in the state *i*, is given a reward ψ_i, that should be discounted for each epoch at time s. This can be done by means of $a_{t-s]r}$ (present value of an unitary annuity). Clearly in the actuarial applications the non discounting case has no relevance.

Obviously it is possibile to consider rewards that change in the time. In this case (4.1) and (4.2) can be respectively written in the following way:

$$
(4.3)\quad \begin{aligned} V_i(s,t) &= (1-S_i(s,t))\sum_{\nu=s+1}^{t}\psi_i(\nu) + \\ &+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)\sum_{\nu=s}^{\vartheta}\psi_i(\nu) + \sum_{\beta\in E}\sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)V_\beta(\vartheta,t) \end{aligned}
$$

$$
(4.4)\quad \begin{aligned} V_i(s,t) &= (1-S_i(s,t))\sum_{\nu=s+1}^{t}\psi_i(\nu)\cdot(1+r)^{s-\nu} + \\ &+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)\sum_{\nu=1}^{\vartheta}\psi_i(\nu)\cdot(1+r)^{s-\nu} + \\ &+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)V_\beta(\vartheta,t)(1+r)^{s-\vartheta}. \end{aligned}
$$

(4.4) can be furtherly generalized supposing to have the following term structure of implied forward rates $r_1, r_2, \ldots, r_t$. Denoting by:

$$
\gamma_{s,h} = \begin{cases} 1 & if \quad s = h \\ \prod_{\mu=s+1}^{h}(1+r_\mu)^{-1} & if \quad s < h \end{cases}
$$

the discounted factors related to the implied forward rate structure, the evolution equation of the DTNHSMRP can be written:

$$
(4.5)\quad \begin{aligned} V_i(s,t) &= (1-S_i(s,t))\sum_{\nu=s+1}^{t}\psi_i(\nu)\cdot\gamma_{s\nu} + \\ &+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)\sum_{\nu=s}^{\vartheta}\psi_i(\nu)\cdot\gamma_{s\nu} + \\ &+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)V_\beta(\vartheta,t)\cdot\gamma_{s\vartheta}. \end{aligned}
$$

This equation in matrix form can be written:

$$
(4.6)\quad \begin{aligned} &\mathbf{V}(s,t) - \sum_{\vartheta=1}^{t}\mathbf{B}(s,\vartheta)\mathbf{V}(\vartheta,t)\cdot\gamma_{s,\vartheta} = \\ &= \mathbf{D}(s,t)\cdot\left(\mathbf{\Psi}(s,t)\mathbf{R}(s,t)\right) + \sum_{\vartheta=1}^{t}\left(\mathbf{B}(s,\vartheta)\mathbf{1}\right)\cdot\left(\mathbf{\Psi}(s,\vartheta)\mathbf{R}(s,\vartheta)\right) \end{aligned}
$$

or equivalently:

$$
\begin{bmatrix}
\mathbf{I} & -\mathbf{B}(0,1)\cdot\gamma_{0,1} & -\mathbf{B}(0,2)\cdot\gamma_{0,2} & -\mathbf{B}(0,3)\cdot\gamma_{0,3} & \cdots \\
\mathbf{0} & \mathbf{I} & -\mathbf{B}(1,2)\cdot\gamma_{1,2} & -\mathbf{B}(1,3)\cdot\gamma_{1,3} & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{B}(2,3)\cdot\gamma_{2,3} & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} *
$$

$$
* \begin{bmatrix}
\mathbf{V}(0,0) & \mathbf{V}(0,1) & \mathbf{V}(0,3) & \mathbf{V}(0,4) & \cdots \\
\mathbf{0} & \mathbf{V}(1,1) & \mathbf{V}(1,2) & \mathbf{V}(1,3) & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{V}(2,2) & \mathbf{V}(2,2) & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{V}(3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} =
$$

$$
= \begin{bmatrix}
\mathbf{D}(0,0) & \mathbf{D}(0,1) & \mathbf{D}(0,2) & \mathbf{D}(0,3) & \cdots \\
\mathbf{0} & \mathbf{D}(1,1) & \mathbf{D}(1,2) & \mathbf{D}(1,3) & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{D}(2,2) & \mathbf{D}(2,3) & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}(3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \cdot
$$

$$
\cdot \begin{bmatrix}
\mathbf{\Psi}(0,0) & \mathbf{\Psi}(0,1)\mathbf{R}(0,1) & \mathbf{\Psi}(0,2)\mathbf{R}(0,2) & \mathbf{\Psi}(0,3)\mathbf{R}(0,3) & \cdots \\
\mathbf{0} & \mathbf{\Psi}(1,1) & \mathbf{\Psi}(1,2)\mathbf{R}(1,2) & \mathbf{\Psi}(1,3)\mathbf{R}(1,3) & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{\Psi}(2,2) & \mathbf{\Psi}(2,3)\mathbf{R}(2,3) & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Psi}(3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} + \qquad (4.7)
$$

$$+\begin{bmatrix} \mathbf{B}(0,0)\mathbf{1} & \mathbf{B}(0,1)\mathbf{1} & \mathbf{B}(0,2)\mathbf{1} & \mathbf{B}(0,3)\mathbf{1} & \cdots \\ \mathbf{0} & \mathbf{B}(1,1)\mathbf{1} & \mathbf{B}(1,2)\mathbf{1} & \mathbf{B}(1,3)\mathbf{1} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{B}(2,2)\mathbf{1} & \mathbf{B}(2.3)\mathbf{1} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}(3,3)\mathbf{1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot$$

$$\cdot\begin{bmatrix} \mathbf{\Psi}(0,0) & \mathbf{\Psi}(0,1)\mathbf{R}(0,1) & \mathbf{\Psi}(0,2)\mathbf{R}(0,2) & \mathbf{\Psi}(0,3)\mathbf{R}(0,3) & \cdots \\ \mathbf{0} & \mathbf{\Psi}(1,1) & \mathbf{\Psi}(1,2)\mathbf{R}(1,2) & \mathbf{\Psi}(1,3)\mathbf{R}(1,3) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{\Psi}(2,2) & \mathbf{\Psi}(2,3)\mathbf{R}(2,3) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Psi}(3,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where:

$$\mathbf{R}(s,t) = \begin{bmatrix} 1 \\ \gamma_{s,s+1} \\ \gamma_{s,s+2} \\ \vdots \\ \gamma_{s,t} \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

$$\mathbf{\Psi}(s,s) = \begin{bmatrix} \psi_1(s) \\ \psi_2(s) \\ \vdots \\ \psi_m(s) \end{bmatrix}, \mathbf{\Psi}(s,s+1) = \begin{bmatrix} \psi_1(s) & \psi_1(s+1) \\ \psi_2(s) & \psi_2(s+1) \\ \vdots & \vdots \\ \psi_m(s) & \psi_m(s+1) \end{bmatrix}, \cdots, \mathbf{\Psi}(s,t) = \begin{bmatrix} \psi_1(s) & \cdots & \psi_1(t) \\ \psi_2(s) & \cdots & \psi_2(t) \\ \vdots & \ddots & \vdots \\ \psi_m(s) & \cdots & \psi_m(t) \end{bmatrix}$$

and:

$$\mathbf{D}(s,t) = \begin{bmatrix} 1 - S_1(s,t) \\ 1 - S_2(s,t) \\ 1 - S_3(s,t) \\ \vdots \\ 1 - S_m(s,t) \end{bmatrix}.$$

Furthermore the $*$ represents the usual row column matrix product and $\cdot$ the element for element product.

5. A non-homogeneous semi-Markov stochastic interest rate approach.

A stochastic term structure of implied forward rates is introduced in this part.
The structure will be constructed by means of DTNHSMP.
In this case the state of the process will be:

$$F = \{\sigma_1, \sigma_2, \ldots, \sigma_k\},$$

where the σ_i represents all the possible implied stochastic interest rates and k gives the number of the implied interest rates.
The evolution equation of the DTHSMP will be the following one:

$$\phi_{ij}(s,t) = \delta_{ij}(1 - S_i(s,t)) + \sum_{\beta \in E} \sum_{\vartheta=1}^{t} \phi_{\beta j}(\nu, t) b_{i\beta}(s, \vartheta)$$

where $\phi_{ij}(s,t)$ represent the probability that at time t the implied interest rate will be σ_j, given that the implied interest rate was σ_i at time s. The related mean discount structure at time h will be constructed in the following way:

$$\gamma_i(s,h) = \prod_{\mu=s+1}^{h} \left(1 + \left(\sum_{j=1}^{k} \phi_{ij}(s,\mu)\sigma_j\right)\right)^{-1} \tag{5.1}$$

Where $\gamma_i(s,h)$ represents the mean discounting factor for a time h given that at time s the interest rate was σ_i, and the sum inside the parenthesis in (5.1) gives the mean interest rate at epoch μ, given that at epoch s the interest rate was σ_i.
At last the evolution equation (3.7) becomes:

$$V_i^{\varepsilon}(s,t)=(1-S_i(s,t))\sum_{\nu=s+1}^{t}\psi_i(\nu)\cdot\gamma_{\varepsilon}(s,\nu)+$$

(5.2)

$$+\sum_{\beta\in E}\sum_{\vartheta=s}^{t}b_{i\beta}(s,\vartheta)\sum_{\nu=s}^{\vartheta}\psi_i(\nu)\cdot\gamma_{\varepsilon}(s,\nu)+$$

$$+\sum_{\beta\in E}\sum_{\vartheta=s}^{t}b_{i\beta}(s,\vartheta)V_{\beta}(\vartheta,t)\cdot\gamma_{\varepsilon}(s,\vartheta)$$

supposed that σ_{ε} will be the implied interest rate at time s.

At last considering the (4.3) the following results are obtained:

(5.3) $$V_i(s,t)=\sum_{\varepsilon=1}^{k}V_i^{\varepsilon}(s,t)\phi_{\eta\varepsilon}(0,s)$$

where σ_{η} was the known rate of interest at time 0.

It is to outline that to obtain the stochastic term structure of implied forward rates it is necessary to solve the evolution equation of DTNHSMP given in (4.1).

6. A health insurance DTNHSMRP.

The concept of DTNHSMP and DTNHSMRP were introduced in the paragraphs 3 and 4. How to apply these models in a health insurance problem will be given in the next two parts. It was showed that a SMP follows the evolution of the r.v. couple (X_n, T_n). More precisely X_n represents the state of the system at the n-th transition and T_n the time of the n-th transition.

To apply the model in the health insurance environment it will posed that T_n represents the age of the insured person at the n-th transition. In a first approach, the time development will be ignored. For this reason all the rewards and the interest rate will be supposed constant in the time. In this light the equation (4.2) can be used to describe the evolution of our model. In that equation we have that:

$V_i(s,t)$

represents the present value at time 0 of all the sums paid and/or received in *t-s* periods by a person that at time 0 had age *s* and was in the state i (the age is measured in time periods).

$$(1-S_i(s,t))\psi_i \cdot a_{t-s]r}$$

represents the present value of the rewards paid in the case in which there wasn't change state from the age *s* up to the age *t*.

$$\sum_{\beta\in E}\sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)\psi_i \cdot a_{\vartheta-s]r}$$

represents the present value of the rewards paid in the state i to a person that was in this state from the age *s* up to the age ϑ and at age ϑ went in the state β.

$$\sum_{\beta\in E}\sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta)V_\beta(\vartheta,t)(1+r)^{s-\vartheta}$$

represents the present value of the rewards paid after the first change of the state to a person that was from the age *s* up to the age ϑ in the state i and at age ϑ went in the state β.

To apply the model it s necessary to know the $Q_{ij}(s,t)$, in fact $b_{ij}(s,t)$ and $S_i(s,t)$ as it results from the (3.1) and (3.2) can be obtained by means of $Q_{ij}(s,t)$, and from (3.3) it results:

$$Q_{ij}(s,t) = G_{ij}(s,t)\,p_{ij}(s).$$

So it is necessary to evaluate the increasing d.f. $G_{ij}(s,*)$ and the non-homogeneous Markov chain $\mathbf{P}(s)$. This can be done by means of raw data in which is reported, for each person of the considered population, the age of entrance in each state during his/her life.

The models that are usually used to manage these problems are continuous time models instead the proposed model is a discrete time model. The evolution of a illness is a continuous phenomenon, but in PHI real applications the phenomenon is considered discrete in time (the most frequent discrete time step is the week). For this reason the model doesn't implies

simplification for the consideration of discrete time. Furthermore the consideration of continuous time model involves the computation of transition intensities. These probabilities are generally evaluated by means of integro-differential equations that involves numerical difficulties to get the solutions. The only difficulty that the application of a DTNHSMRP model involves is the great size of data that are necessary to evaluate the $Q_{ij}(s,t)$ and the difficulties to get the data in the right way. In this case the two difficulties are less relevant because in the healt insurance models the number of states is small, furthermore usually the evolution of the illness for each person is known. This fact involves that the data are naturally saved in the way that are useful for the application of semi-Markov process.

7. Generalized health insurance DTNHSMRP model.

In the previous paragraph was presented a DTNHSMRP model in which was not considered the time and T_n represented the age during the illness evolution. This simplification is usual in the construction of health insurance model. In this part it will be presented a model that will consider both the evolution of time and of the age of the insured person. In this way it will be possible to take in account time dependent rewards and term structures of implied forward rates. To obtain this kind of models it is necessary to generalize the ones given in the previous paragraphs. A similar generalization was given in [6] for the application of DTNHSMRP in pension field.

Taking in account this generalization, the formula (4.5) can be written in the following way:

$$
\begin{aligned}
{}^{\mu}V_i(s,t) = (1-{}^{\mu}S_i(s,t)) \sum_{\nu=s+1}^{t} \psi_i(\nu)\cdot\gamma_{s\nu} + \\
+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} {}^{\mu}b_{i\beta}(s,\vartheta) \sum_{\nu=s}^{\vartheta} \psi_i(\nu)\cdot\gamma_{s\nu} + \\
+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} {}^{\mu}b_{i\beta}(s,\vartheta)\,{}^{\mu}V_{\beta}(\vartheta,t)\cdot\gamma_{s\vartheta}
\end{aligned}
\tag{7.1}
$$

In this case s and t represent the time and μ represent the age of the insured person at time s. So at time $t>s$ he will have $\mu+t-s$ age.

In this case too there will not be difficulties to get right data because the date of the visits are known with the evolution of the illness and the age of the population persons.

In the case of stochastic interest rate the (7.1) becomes:

$$
\begin{aligned}
{}^{\mu}V_i^{\varepsilon}(s,t) &= (1-{}^{\mu}S_i^{\varepsilon}(s,t)) \sum_{\nu=s+1}^{t} \psi_i(\nu)\cdot\gamma_{\varepsilon}(s,\nu) + \\
&+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} {}^{\mu}b_{i\beta}^{\varepsilon}(s,\vartheta)\sum_{\nu=s}^{\vartheta}\psi_i(\nu)\cdot\gamma_{\varepsilon}(s,\nu) + \\
&+ \sum_{\beta\in E}\sum_{\vartheta=s}^{t} {}^{\mu}b_{i\beta}^{\varepsilon}(s,\vartheta)\,{}^{\mu}V_{\beta}^{\varepsilon}(\vartheta,t)\cdot\gamma_{\varepsilon}(s,\vartheta)
\end{aligned}
$$

Supposed that σ_{ε} was the interest rate at time *s*.

At last as for (5.2) knowing that σ_{η} was the rate of interest at time 0 is obtained:

$$
(7.2) \qquad {}^{\mu}V_i(s,t) = \sum_{\varepsilon=1}^{k} {}^{\mu}V_i^{\varepsilon}(s,t)\phi_{\eta\varepsilon}(0,s)\,.
$$

8. Conclusions

Applying the DTNHSMRP model for each year at the proposed actuarial problem, we are able, in the first approach, to compute the elements of the **V**(s,*t*) or V_i(s,*t*) , $s,t\in$N. They represent the present value of the payments (assets or liabilities) that were done from the age *s* to the age *t* for a person that was in the state *i* at age *s*. In this way it is possible to get the exposition of the insurer company respect each given insured position. In this approach it is not possible to consider the time evolution of the system.

The formulas useful to take in account the time evolution were given in the previous paragraph. The (7.1) consider the time evolution of interest rate and of the reward. In (7.2) an implied stochastic interest rate structure is considered. In this way the time evolution considers both the demographic and the financial risk in a natural way. To get the solution it is necessary to solve the evolution equation of the DTNHSMP that follows the stochastic interest rate development and after a generalized DTNHSMRP evolution equation that solves in a complete way the problem that we are facing.

It is to outline that also when the authors solved the problem of the time development of a pension “fund” by means of a DTNHSMRP they got similar results [6]. In their opinion the

NHSMRP is a very important tool for the construction of actuarial models. Now they are working to give a general model that could be applied to measure the any kind of actuarial risk. An other direction in which the authors are working is the construction of a the algorithms and the related computer program useful for the resolution of DTHSMRP and DTNHSMRP in the way to apply to real problems their semi-Markov models.

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