Discrete Time Risk Models under Stochastic Forces of Interest

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Abstract/Resume

Discrete time risk models under stochastic forces of interest are discussed. Based on types of payments of premiums, annuity-due and annuity-immediate risk models are introduced. Recursive and integral equations are given for the ruin probabilities in the risk models. Inequalities for the ruin probabilities are derived by martingales and recursive techniques. The inequalities can be used to evaluate the ruin probabilities as upper bounds. Numerical examples are given to illustrate the applications of these results.

Keywords

Discrete time risk model; ruin probability; adjustment coefficient; force of interest; rate of discount; super-martingale; optional stopping theorem; NWU and NBU distribution functions.

1 Introduction

We consider a discrete time risk model. In this discrete time risk model, premiums, claims and surplus are recorded only at times $n = 0, 1, 2, \cdots$.

Assume that the initial surplus at time 0 is $u \geq 0$. Let U_n denote an insurer's surplus at time n with $U_0 = u$. The time of ruin is defined by $T = \inf\{n : U_n < 0\}$ with $T = \infty$ if $U_n \geq 0$ for all $n = 1, 2, \cdots$.

The ultimate ruin probability is defined by

$$\psi(u) = \Pr\{T < \infty\} = \Pr\{\bigcup_{k=1}^{\infty}(U_k < 0)\}$$

and the finite time ruin probability is denoted by

$$\psi_n(u) = \Pr\{T \leq n\} = \Pr\{\bigcup_{k=1}^{n}(U_k < 0)\}.$$

Furthermore, let Y_n denote the total claims over the nth period from time $n-1$ to time n and X_n represent the total premiums over the nth period. Assume that $\{Y_n, n \geq 1\}$ and $\{X_n, n \geq 1\}$ are two independent sequences of i.i.d. nonnegative random variables and the positive loading condition holds, namely $EX_1 > EY_1$.

If no investment is made on the surplus, then

$$U_n = u + \sum_{k=1}^{n}(X_k - Y_k) \tag{1.1}$$

is the surplus of the insurer at time n, and it is the classical discrete time risk model, which has been discussed by many authors, see, for example, Willmot (1996), Willmot and Lin (2001) and Yang (1998).

Let $R_0 > 0$ denote the adjustment coefficient in risk model (1.1) and satisfy

$$Ee^{-R_0(X_1 - Y_1)} = 1\,. \tag{1.2}$$

If $\psi_0(u)$ denotes the ruin probability in risk model (1.1), then the following well-known Lundberg's inequality gives an exponential upper bound for the ruin probability, namely

$$\psi_0(u) \;\leq\; e^{-uR_0}\,, \quad u \geq 0\,. \tag{1.3}$$

In this paper, we assume that an insurer would invest its surplus for each period and receive interest on its surplus.

Suppose that the force of interest over the nth period from time $n-1$ to time n is Δ_n, and $\{\Delta_n,\, n \geq 1\}$ are a sequence of i.i.d. nonnegative random variables. Assume that $\{\Delta_n,\, n \geq 1\}$ have a common distribution as that of $\Delta \geq 0$.

Denote $Z_n = e^{\Delta_n}$, then $Z_n \geq 1$ is the accumulation factor over the nth period and $0 < Z_n^{-1} = e^{-\Delta_n} \leq 1$ is the rate of discount over the nth period.

Assume that $\{Y_n,\, n \geq 1\}$, $\{X_n,\, n \geq 1\}$ and $\{Z_n,\, n \geq 1\}$ are independent, and they have common distributions F, H and G as those of $X \geq 0$, $Y \geq 0$ and $Z \geq 1$, respectively, where X, Y and Z are assumed to be independent and $F(0) = \Pr\{Y \leq 0\} = 0$. We denote the tail of a distribution function $B(x)$ by $\bar{B}(x) = 1 - B(x)$.

In addition, we suppose that the claims of each period are paid at the end of each period while the premiums can be received at the beginning or end of each period. According to the types of payments of premiums, we will introduce annuity-due and annuity-immediate risk models in Sections 2 and 3, respectively.

2 An annuity-due risk model

In this section, we assume that the premiums are collected at the beginning of each period. If $U_{n-1} \geq 0$, then the surplus is invested in the nth period from time $n-1$ to time n at the force of interest Δ_n and the surplus at time n is

$$U_n = (U_{n-1} + X_n)Z_n - Y_n = u\prod_{k=1}^{n} Z_k + \sum_{k=1}^{n}\left[(X_k Z_k - Y_k)\prod_{i=k+1}^{n} Z_i\right] \tag{2.1}$$

where $\prod_{i=n+1}^{n} Z_i = 1$, and if we denote the ultimate and finite time ruin probabilities in annuity-due risk model (2.1) respectively by $\psi^*(u)$ and $\psi_n^*(u)$, then

$$\psi^*(u) = \Pr\{\bigcup_{k=1}^{\infty}(U_k < 0)\} = \Pr\{\bigcup_{k=1}^{\infty}(V_k < 0)\}$$

and

$$\psi_n^*(u) = \Pr\{\bigcup_{k=1}^{n}(U_k < 0)\} = \Pr\{\bigcup_{k=1}^{n}(V_k < 0)\}$$

where

$$V_k = U_k \prod_{i=1}^{k} Z_i^{-1} = u + \sum_{j=1}^{k} (X_j Z_j - Y_j) \prod_{i=1}^{j} Z_i^{-1} \tag{2.2}$$

is the present value of the surplus at time k with $V_0 = u$.

We will derive probability inequalities for ruin probability $\psi^*(u)$ by using two different methods, which are martingales and recursive techniques, respectively.

2.1 Martingales and inequalities in the annuity-due risk model

We first derive a functional inequality for $\psi^*(u)$ in terms of NWU and NBU distribution functions. This idea was first introduced by Willmot (1994) and has been developed by many authors such as Cai and Garrido (1999), Grandell (1997), Lin (1996), Willmot (1996), Willmot and Lin (2001) and Yang (1998). Then, using the functional inequality, we give an exponential upper bound for $\psi^*(u)$.

A life distribution function $B(x)$ with $B(0) = 0$ is said to be new worse than used (NWU) if for any $x \geq 0$ and $y \geq 0$

$$\bar{B}(x+y) \ \geq \ \bar{B}(x)\bar{B}(y)\,. \tag{2.3}$$

If the reversed inequality holds in (2.3), then B is said to be new better than used (NBU).

Theorem 2.1 Let B_1 be an NWU distribution and B_2 be an NBU distribution. Assume that B_1 and B_2 satisfy for all $0 < \alpha \leq 1$,

$$E\left[\frac{\bar{B}_2(\alpha X)}{\bar{B}_1(\alpha Y Z^{-1})}\right] \ \leq \ 1\,. \tag{2.4}$$

Then, for any $u \geq 0$,

$$\psi^*(u) \ \leq \ \Lambda(u) \tag{2.5}$$

where

$$[\Lambda(x)]^{-1} = \inf_{y \geq 0} \frac{\bar{B}_2(y)}{\bar{B}_1(x+y)}\,, \qquad x \geq 0\,. \tag{2.6}$$

Proof. First, let

$$S_n = \frac{\bar{B}_2(P_n)}{\bar{B}_1(C_n)}$$

where

$$P_n = \sum_{k=1}^{n} X_k \prod_{i=1}^{k-1} Z_i^{-1} \qquad \text{and} \qquad C_n = \sum_{k=1}^{n} Y_k \prod_{i=1}^{k} Z_i^{-1} \tag{2.7}$$

with $P_0 = C_0 = 0$. In fact, P_n is the present value of the total premiums at time k, while C_n is the present value of the total claims at time k in annuity-due risk model (2.1). In addition, $V_n = u + P_n - C_n$ where V_n is defined in (2.2).

Define

$$\mathcal{F}_n = \sigma\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z_1, \cdots, Z_n\}\,. \tag{2.8}$$

Thus for any $n \geq 0$,

$$\begin{aligned} E(S_{n+1}\,|\,\mathcal{F}_n) &\leq S_n E\left[\frac{\bar{B}_2(X_{n+1}\prod_{k=1}^{n} Z_k^{-1})}{\bar{B}_1(Y_{n+1}\prod_{k=1}^{n+1} Z_k^{-1})}\,|\,\mathcal{F}_n\right] \\ &= S_n \int_0^1 E\left[\frac{\bar{B}_2(z\,X_{n+1})}{\bar{B}_1(z\,Y_{n+1}Z_{n+1}^{-1})}\right] d\,\mathrm{Pr}\{\prod_{k=1}^{n} Z_k^{-1} \leq z\} \\ &\leq S_n\,, \end{aligned} \tag{2.9}$$

which implies that $\{S_n,\, n \geq 0\}$ is a super-martingale, where the last equality holds since X_{n+1}, Y_{n+1} and Z_{n+1} are independent of $\mathcal{F}_n$, and (2.9) follows from (2.4).

However, we know that the time of ruin T is a stopping time. Hence, $n \wedge T = \min(n,\, T)$ is a finite stopping time. Thus, by the optional stopping theorem for super-martingales, see, for example, Taylor (1997), we get

$$ES_{n\wedge T} \;\leq\; ES_0 \;=\; 1\,. \tag{2.10}$$

Hence

$$\psi_n^*(u) \;\leq\; \Lambda(u)\,, \tag{2.11}$$

which follows from (2.10) and

$$ES_{n\wedge T} \quad \geq \quad E\,[S_{n\wedge T}\,I(T \leq n)] \;=\; E\,[S_T\,I(T \leq n)]$$

$$
\begin{aligned}
&= E\left[\frac{\bar{B}_2(P_T)}{\bar{B}_1(C_T)}\, I(T \le n)\right] \;\ge\; E\left[\frac{\bar{B}_2(P_T)}{\bar{B}_1(u+P_T)}\, I(T \le n)\right] && (2.12)\\
&\ge \Lambda^{-1}(u)\, E\left[I(T \le n)\right] && (2.13)\\
&= \Lambda^{-1}(u)\, \psi_n^*(u)
\end{aligned}
$$

where (2.12) follows from $C_T > u + P_T$ and (2.13) follows from (2.6). Thus, (2.5) follows from letting $n \to \infty$ in (2.11) and $\lim_{n\to\infty} \psi_n^*(u) = \psi^*(u)$. 2

Theorem 2.1 provides a functional inequality for $\psi^*(u)$. By the suitable choices of B_1 and B_2 in (2.4), we can get different upper bounds for $\psi^*(u)$. We give an application of Theorem 2.1 to an exponential upper bound for $\psi^*(u)$ in the following corollary.

Corollary 2.1 Suppose that $R_1 > 0$ is a constant and satisfies

$$
Ee^{-R_1(X-YZ^{-1})} = 1\,. \qquad (2.14)
$$

Then, for any $u \ge 0$,

$$
\psi^*(u) \;\le\; e^{-uR_1}\,. \qquad (2.15)
$$

Proof. Take $\bar{B}_1(x) = \bar{B}_2(x) = e^{-R_1 x}$ in Theorem 2.1. Thus, $\Lambda(x) = e^{-R_1 x}$. In addition, it can be seen by Jensen's inequality and (2.14) that for all $0 < \alpha \le 1$

$$
Ee^{-\alpha R_1(X-YZ^{-1})} \;=\; E\left(e^{-R_1(X-YZ^{-1})}\right)^{\alpha} \;\le\; \left(Ee^{-R_1(X-YZ^{-1})}\right)^{\alpha} \;=\; 1,
$$

which implies that condition (2.4) holds. Hence, (2.15) follows from (2.5). 2

Remark 2.1 A special annuity-due risk model has been studied by Yang (1998), in which the forces of interest are assumed to be constant. By using Doob's maximal inequality, Yang (1998) derives an NUW upper bound for $\psi^*(u)$ when the forces of interest are constant. The proof of Theorem 2.1 is from the optional stopping theorem and follows the arguments used by Gerber (1979) for Lundberg's inequality (1.3).

We point out that Theorem 3.1 of Yang (1998) is a special case of Corollary 2.1 when the forces of interest are constant, condition (2.4) is a generalization of condition (40) of Yang (1998), and condition (41) of Yang (1998) implies that $\Lambda(x) \le \bar{B}_1(x)$ in Theorem 2.1.

In Section 2.2, we will use a simple condition and a recursive technique to derive a different functional inequality for $\psi^*(u)$.

2.2 Recursive equations and inequalities in the annuity-due risk model

First, we have the following recursive and integral equations for $\psi_n^*(u)$ and $\psi^*(u)$, respectively.

Lemma 2.1 For all $u \ge 0$,

$$\psi_{n+1}^*(u) = \int_1^\infty \int_0^\infty \bar{F}((u+x)z)dH(x)dG(z) + \int_1^\infty \int_0^\infty \left[\int_0^{(u+x)z} \psi_n^*((u+x)z-y)dF(y)\right] dH(x)dG(z) \tag{2.16}$$

and

$$\psi^*(u) = \int_1^\infty \int_0^\infty \bar{F}((u+x)z)dH(x)dG(z) + \int_1^\infty \int_0^\infty \left[\int_0^{(u+x)z} \psi^*((u+x)z-y)dF(y)\right] dH(x)dG(z) \tag{2.17}$$

Proof. By conditioning on Y_1, X_1, and Z_1, we get

$$\begin{aligned}
\psi_{n+1}^*(u) &= E\left[\psi_n^*((u+X_1)Z_1 - Y_1)\right] \\
&= \int_1^\infty \int_0^\infty \left[\bar{F}((u+x)z) + \int_0^{(u+x)z} \psi_n^*((u+x)z-y)dF(y)\right] dH(x)dG(z)
\end{aligned}$$

where $\psi_n^*((u+x)z-y) = 1$ if $y > (u+x)z$.

Thus, (2.17) follows from letting $n \to \infty$ in (2.16), $\lim_{n\to\infty} \psi_n^*(u) = \psi^*(u)$, and Lebesgue dominated convergence theorem. 2

For a life distribution B_1 with $B_1(0) = 0$, we define

$$(\beta_1)^{-1} = \inf_{t\ge 0} \frac{\int_t^\infty [\bar{B}_1(y)]^{-1} dF(y)}{[\bar{B}_1(t)]^{-1}\bar{F}(t)}\,. \tag{2.18}$$

Hence, for any $x \geq 0$,

$$\bar{F}(x) \leq \beta_1 \bar{B}_1(x) \int_x^\infty [\bar{B}_1(y)]^{-1} dF(y) \tag{2.19}$$
$$\leq \beta_1 \bar{B}_1(x)\, E[\bar{B}_1(Y)]^{-1} \tag{2.20}$$

and

$$\left(E[\bar{B}_1(Y)]^{-1}\right)^{-1} \leq \beta_1 \leq 1\,. \tag{2.21}$$

Then, using a recursive technique, we derive the following result.

Theorem 2.2 Let B_1 be an NWU distribution and Λ_1 be a nonnegative function. Assume that B_1 and Λ_1 satisfy

$$E[\bar{B}_1(Y)]^{-1}\, E\Lambda_1(XZ) \leq 1\,, \tag{2.22}$$

and for all $y \geq 0$, $x \geq 0$,

$$\bar{B}_1(x+y) \leq \bar{B}_1(x)\,\Lambda_1(y)\,. \tag{2.23}$$

Then, for any $u \geq 0$,

$$\psi^*(u) \leq \beta_1 E[\bar{B}_1(Y)]^{-1}\, E\bar{B}_1((u+X)Z)\,. \tag{2.24}$$

Proof. First, we have

$$\psi_1^*(u) = \Pr\{Y_1 > (u+X_1)Z_1\} = \int_1^\infty \int_0^\infty \bar{F}((u+x)z) dH(x) dG(z). \tag{2.25}$$

Then by (2.25) and (2.20), we get

$$\psi_1^*(u) \leq \beta_1\, E[\bar{B}_1(Y)]^{-1}\, E\bar{B}_1((u+X)Z)\,.$$

By the inductive hypothesis, we get

$$\psi_n^*(u) \leq \beta_1\, E[\bar{B}_1(Y)]^{-1}\, E\bar{B}_1((u+X)Z)\,. \tag{2.26}$$

Since $(u+X)Z \geq u + XZ$, (2.26) implies

$$\psi_n^*(u) \leq \beta_1\, E[\bar{B}_1(Y)]^{-1}\, E\bar{B}_1(u+XZ)\,. \tag{2.27}$$

However, for $0 \le y \le (u+x)z$

$$\bar{B}_1((u+x)z - y + XZ) \le \bar{B}_1((u+x)z + XZ)\,[\bar{B}_1(y)]^{-1} \tag{2.28}$$
$$\le \Lambda_1(XZ)\,\bar{B}_1((u+x)z)\,[\bar{B}_1(y)]^{-1} \tag{2.29}$$

where (2.28) follows from the definition NWU distribution and (2.29) follows from (2.23).

Thus, by (2.16), (2.19) and (2.27), we get

$$\begin{aligned}\psi^*_{n+1}(u) &\le \beta_1 \int_1^\infty \int_0^\infty \left[\bar{B}_1((u+x)z) \int_{(u+x)z}^\infty [\bar{B}_1(y)]^{-1} dF(y)\right] dH(x)dG(z)\\ &+ \beta_1 \int_1^\infty \int_0^\infty \left[\bar{B}_1((u+x)z) \int_0^{(u+x)z} [\bar{B}_1(y)]^{-1} dF(y)\right] dH(x)dG(z)\\ &= \beta_1\, E[\bar{B}_1(Y)]^{-1}\, E\bar{B}_1((u+X)Z).\end{aligned}$$

Hence, for any $n \ge 1$, (2.26) holds, and (2.24) follows from letting $n \to \infty$ in (2.26) and $\lim_{n\to\infty} \psi^*_n(u) = \psi^*(u)$. 2

We notice that the distribution of $X - YZ^{-1}$ in (2.14) is the one of the discounted value of the surplus gained in one period while the distribution of $XZ - Y$ is the one of the accumulated value of the surplus gained in one period. By comparison with R_1 defined in (2.14) and Corollary 2.1, we derive the following upper bound by using Theorem 2.2.

Corollary 2.2 Suppose that $R_2 > 0$ is a constant and satisfies

$$Ee^{-R_2(XZ-Y)} = 1\,. \tag{2.30}$$

Then for any $u \ge 0$,

$$\psi^*(u) \le \beta_2\, Ee^{R_2 Y}\, Ee^{-R_2(u+X)Z} \tag{2.31}$$

where

$$(\beta_2)^{-1} = \inf_{t\ge 0} \frac{\int_t^\infty e^{R_2 y} dF(y)}{e^{R_2 t}\bar{F}(t)}\,. \tag{2.32}$$

In particular, if F is new worse than used in convex ordering (NWUC), then for any $u \ge 0$,

$$\psi^*(u) \le Ee^{-R_2(u+X)Z}\,. \tag{2.33}$$

Proof. (2.31) follows from taking $\bar{B}_1(x) = \Lambda_1(x) = e^{-R_2 x}$ in Theorem 2.2. In addition, it has been proved that if F is NWUC, which includes the class of life distributions with a decreasing failure rate (DFR), then

$$\beta_2 = (Ee^{R_2 Y})^{-1}, \tag{2.34}$$

see, for example, Proposition 6.1.1 of Willmot and Lin (2001). Thus (2.34) and (2.31) yield (2.33). 2

Furthermore, if the forces of interest are a constant δ, then $Z = e^{\delta}$. Thus (2.31) and (2.33) yield the following exponential upper bound for $\psi^*(u)$.

Corollary 2.3 Suppose that $\{\Delta_n, n \geq 1\}$ are a constant δ. Then, for any $u \geq 0$,

$$\psi^*(u) \ \leq \ \beta_2 \, e^{-uR_2 e^{\delta}}. \tag{2.35}$$

In particular, if F is NWUC, then

$$\psi^*(u) \ \leq \ (Ee^{R_2 Y})^{-1} \, e^{-uR_2 e^{\delta}}. \tag{2.36}$$

Remark 2.2 The conditions in Theorem 2.2 are simpler than the conditions in Theorem 2.1. Condition (2.23) is weaker than (41) of Yang (1998), in which $\Lambda_1(x)$ is assumed to be the tail of an NBU distribution.

In addition, if the forces of interest are a constant δ, i.e. $Z = e^{\delta}$, then (2.14) and (2.30) are reduced to $Ee^{-R_1(X - Ye^{-\delta})} = Ee^{-R_2 e^{\delta}(X - Ye^{-\delta})} = 1$, which implies that $R_1 = R_2 e^{\delta}$. Thus, the upper bounds in Corollary 2.3 are the refinements of that in (2.15) when the forces are constant.

Furthermore, it will be shown in Section 4 by numerical examples that the upper bound in (2.31) can be tighter than that in (2.15) when the forces are random variables.

Also, we would like to point out that it appears to be difficult to derive Theorem 2.2 by using martingales.

3 An annuity-immediate risk model

In this section, we assume that the premiums are received at the end of each period. If $U_{n-1} \geq 0$, the surplus is invested in the nth period form time $n-1$ to time n at the force of interest Δ_n. In this case, the surplus at time n is

$$U_n = U_{n-1} Z_n + X_n - Y_n = u \prod_{k=1}^{n} Z_k + \sum_{k=1}^{n} \left[(X_k - Y_k) \prod_{i=k+1}^{n} Z_i \right] \tag{3.1}$$

and denote the ultimate and finite time ruin probabilities in annuity-immediate risk model (3.1) respectively by $\psi^\star(u)$ and $\psi_n^\star(u)$, then

$$\psi^\star(u) = \Pr\{\bigcup_{k=1}^{\infty} (U_k < 0)\} = \Pr\{\bigcup_{k=1}^{\infty} (V_k < 0)\}$$

and

$$\psi_n^\star(u) = \Pr\{\bigcup_{k=1}^{n} (U_k < 0)\} = \Pr\{\bigcup_{k=1}^{n} (U_k < 0)\}$$

where

$$V_k = U_k \prod_{i=1}^{k} Z_i^{-1} = u + \sum_{j=1}^{k} (X_j - Y_j) \prod_{i=1}^{j} Z_i^{-1} \tag{3.2}$$

is the present value of the surplus at time n with $V_0 = 0$.

For annuity-immediate risk model (3.1), we first get the following functional inequality for $\psi^\star(u)$, which is similar to Theorem 2.1.

Theorem 3.1 Let B_1 be an NWU distribution and B_2 be an NBU distribution. Assume that for all $0 < \alpha \leq 1$,

$$E\left[\frac{\bar{B}_2(\alpha X Z^{-1})}{\bar{B}_1(\alpha Y Z^{-1})}\right] \leq 1\,. \tag{3.3}$$

Then, for any $u \geq 0$,

$$\psi^\star(u) \ \leq \ \Lambda(u) \tag{3.4}$$

where $\Lambda(u)$ is defined in (2.6).

Proof. The proof is similar to that of Theorem 2.1. 2

We notice that in this annuity-immediate risk model, the distribution of $(X - Y)Z^{-1}$ is the one of the discounted value of the surplus gained in one period. By comparison with R_1 in (2.14) and Corollary 2.1, we obtain the following exponential upper bound for $\psi^\star(u)$ by using Theorem 3.1.

Corollary 3.1 Suppose that $R_3 > 0$ is a constant and satisfies

$$Ee^{-R_3(X-Y)Z^{-1}} = 1\,. \tag{3.5}$$

Then for any $u \geq 0$,

$$\psi^\star(u) \;\leq\; e^{-uR_3}\,. \tag{3.6}$$

In particular, if the forces of interest $\{\Delta_n,\, n \geq 1\}$are a constant δ, then for any $u \geq 0$,

$$\psi^\star(u) \;\leq\; e^{-uR_0e^{\delta}}, \quad u \geq 0\,. \tag{3.7}$$

Proof. Take $\bar{B}_1(x) = \bar{B}_2(x) = e^{-R_3x}$ in Theorem 3.1. Thus, condition (3.3) holds by Jensen's inequality and (3.5). Hence, (3.6) follows from $\Lambda(x) = e^{-R_3x}$ and (3.4).

If the forces of interest are a constant δ, i.e. $Z = e^{\delta}$, then (3.5) and (1.2) imply that $R_3e^{-\delta} = R_0$ or $R_3 = R_0e^{\delta}$. Thus (3.6) yields (3.7). 2

Remark 3.1 It is interesting to note that the upper bound in (3.7) satisfies for $\delta > 0$ and $u > 0$,

$$e^{-uR_0e^{\delta}} \;<\; e^{-uR_0} \quad \text{and} \quad e^{-uR_0e^{\delta}} = o(e^{-uR_0})\,. \tag{3.8}$$

Moreover, (3.8) implies that the upper bound in (3.7) for $\psi^\star(u)$ and Lundberg's upper bound for $\psi_0(u)$ are consistent with the relationship $\psi^\star(u) \;\leq\; \psi_0(u),\; u \geq 0\,.$

Furthermore, we can derive a refinement of (3.7) by the recursive technique. First, we have the following recursive and integral equations for $\psi_n^\star(u)$ and $\psi^\star(u)$, respectively.

Lemma 3.1 Foe any $u \geq 0$,

$$\psi^\star_{n+1}(u) = \int_1^\infty \int_0^\infty \left[\bar{F}(uz+x) + \int_0^{uz+x} \psi^\star_n(uz+x-y)dF(y)\right] dH(x)dG(z) \quad (3.9)$$

and

$$\psi^\star(u) = \int_1^\infty \int_0^\infty \left[\bar{F}(uz+x) + \int_0^{uz+x} \psi^\star(uz+x-y)dF(y)\right] dH(x)dG(z). \quad (3.10)$$

Proof. The proof is similar to that of Lemma 2.1. 2

Then, we can get the following result by using the recursive technique.

Theorem 3.2 Let B_1 be an NWU distribution and Λ_2 be a nonnegative function. Suppose that B_1 and Λ_2 satisfy

$$E[\bar{B}_1(Y)]^{-1}\, E\Lambda_2(X) \;\leq\; 1 \quad (3.11)$$

and for all $y \geq 0$, $x \geq 0$,

$$\bar{B}_1(x+y) \;\leq\; \bar{B}_1(x)\,\Lambda_2(y)\,. \quad (3.12)$$

Then, for any $u \geq 0$,

$$\psi^\star(u) \;\leq\; \beta_1 E[\bar{B}_1(Y)]^{-1}\, E\bar{B}_1(uZ+X) \quad (3.13)$$

where β_1 is defined in (2.18).

Proof. The proof is similar to that of Theorem 2.2. 2

We notice that in annuity-immediate risk model (3.1), the distribution of $X-Y$ is the one of the accumulated value of the surplus gained in one period. So, by comparison with R_2 in (2.30) and Corollary 2.2, we expect an upper bound in terms of R_0, which is defined in (1.2). That can be obtained in the following corollary by Theorem 3.2.

Corollary 3.2 For any $u \geq 0$,

$$\psi^\star(u) \;\leq\; \beta_0\, Ee^{-uR_0 Z} \quad (3.14)$$

where

$$(\beta_0)^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{R_0 y} dF(y)}{e^{R_0 t} \bar{F}(t)}\,. \tag{3.15}$$

In particular, if F is NWUC, then for any $u \geq 0$,

$$\psi^\star(u) \;\leq\; (Ee^{R_0 Y})^{-1}\, Ee^{-uR_0 Z}\,. \tag{3.16}$$

Proof. (3.14) follows from taking $\bar{B}_1(x) = \Lambda_2(x) = e^{-R_0 x}$ in Theorem 3.2. Similarly to β_2, if F is NWUC, then $\beta_0 = (Ee^{R_0 Y})^{-1}$, which leads to (3.16) by (3.14). 2

Furthermore, if the forces of interest is a constant δ, i.e. $Z = e^\delta$, then (3.14) and (3.16) yield the following exponential upper bound for $\psi^\star(u)$.

Corollary 3.3 If the forces of interest $\{\Delta_n, n \geq 1\}$ are a constant δ, then for any $u \geq 0$,

$$\psi^\star(u) \;\leq\; \beta_0\, e^{-uR_0 e^\delta}\,. \tag{3.17}$$

In particular, if F is NWUC, then for any $u \geq 0$

$$\psi^\star(u) \;\leq\; (Ee^{R_0 Y})^{-1}\, e^{-uR_0 e^\delta}\,. \tag{3.18}$$

Remark 3.2 The upper bounds in Corollary 3.3 are the refinements of that in (3.7). For the case when the forces of interest are random variables, the upper bound in (3.14) can be tighter than that in (3.6), which will be shown in Section 4 by numerical examples.

4 Numerical Examples

In this section, we consider three examples of claim sizes to illustrate the applications of the exponential upper bounds derived in Sections 2 and 3. The first one is a gamma distribution with a decreasing failure rate. The second one is also a gamma distribution but with an increasing failure rate. The third one is a truncated normal distribution, which has an increasing failure rate. Moreover, the forces of interest are assumed to be constant or stochastic.

We let $X = 1$. That is to assume that the premiums of each period is one unit. The discrete time risk model without forces of interest has been considered by many authors, see, for example, Bowers et al (1997), Dickson (1994), Gerber (1988), Shiu (1989), and Willmot (1993).

We denote the moment generating function of Y by $M_Y(t) = \int_0^\infty e^{ty} dF(y)$. Thus it follows from (2.14) that R_1 satisfies

$$EM_Y(R_1 Z^{-1}) = e^{R_1}. \tag{4.1}$$

In addition, we know by (2.30) that R_2 satisfies

$$M_Y(R_2) = \left(Ee^{-R_2 Z}\right)^{-1}, \tag{4.2}$$

by (3.5) that R_3 satisfies

$$E\left[e^{-R_3 Z^{-1}} M_Y(R_3 Z^{-1})\right] = 1\,, \tag{4.3}$$

and by (1.2) that R_0 satisfies

$$M_Y(R_0) = e^{R_0}\,. \tag{4.4}$$

Adjustment coefficients R_0, R_1, R_2, and R_3 are keys to the calculation of the exponential upper bounds for the ruin probabilities in the annuity-due and annuity-immediate risk models.

Example 4.1 Suppose that the claim size Y has a gamma density with

$$g(y) = \frac{\lambda^\alpha y^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda y}, \quad y \ge 0 \tag{4.5}$$

where $\alpha > 0$ and $\lambda > 0$.

Then

$$M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, \quad t < \lambda\,. \tag{4.6}$$

Let $\alpha = 0.5$ and $\lambda = 1$. Thus $EY = \alpha/\lambda = 0.5$ and Y has a decreasing failure rate (DFR) since $0 < \alpha < 1$. By (4.4), we get that $R_0 = 0.7968121216$, which, together with the following calculations in this section, is simply obtained by Mathematica.

(1) Let the interest force Δ be a constant $\delta = 0.05$. In this case, $R_1 = R_2 e^{\delta}$, $R_3 = R_0 e^{\delta}$. By (4.2), we get $R_2 = 0.8226574018$. Hence $R_1 = 0.8648359487$ and $R_3 = 0.8376655527$. Since DFR $\subset$ NWUC, (2.36) applies to $\psi^*(u)$, and (3.18) applies to $\psi^\star(u)$. The numerical results of these upper bounds and Lundberg's upper bound (1.3) for $\psi_0(u)$ are given in Table 1.

(2) Let Δ have a uniform distribution on $[0.04, 0.06]$. By (4.1), (4.2) and (4.3), we get in this case that $R_1 = 0.8646531059$, $R_2 = 0.8226597883$ and $R_3 = 0.8375431475$. Then, (2.15) and (2.33) apply to $\psi^*(u)$, and (3.6) and (3.16) apply to $\psi^\star(u)$. The numerical results of the upper bounds are given in Table 2, which shows that the upper bound in (2.33) is tighter than that in (2.15) for $\psi^*(u)$; the upper bound in (3.16) is tighter than that in (3.6) for $\psi^\star(u)$.

Example 4.2 Suppose that Y has a gamma density of (4.5) with $\alpha = 1.5$ and $\lambda = 3$. Thus, $EY = \alpha/\lambda = 0.5$ as the case in Example 4.1. But, in this example, Y has an increasing failure rate (IFR) since $\alpha > 1$. Furthermore, we get that $R_0 = 2.3904363901$.

(1) Let Δ be a constant $\delta = 0.06$. Similarly to what we get in Example 4.1, we get $R_2 = 2.4824848546$, $R_1 = R_2 e^{\delta} = 2.6359931448$ and $R_3 = R_0 e^{\delta} = 2.5382527219$. In this case, (2.15) applies to $\psi^*(u)$, and (3.7) applies to $\psi^\star(u)$, whose numerical results are given in Table 3.

(2) Let Δ have a uniform distribution on $[0.05, 0.07]$. We get in this case that $R_1 = 2.6350933465$, $R_2 = 2.4824457160$, and $R_3 = 2.5377829534$. Then, (2.15) and (2.31) apply to $\psi^*(u)$, and (3.6) and (3.14) apply to $\psi^\star(u)$ in Table 4, where $\beta_2 = 1$ in (2.31) and $\beta_0 = 1$ in (3.14). It can be seen from Table 4 that the upper bound in (2.31) is tighter than that in (2.15) for $\psi^*(u)$ while the upper bound in (3.14) is tighter than (3.6) for $\psi^\star(u)$. Tables 3 and 4 also show that the upper bounds for $\psi^*(u)$ and $\psi^\star(u)$ and Lundberg's upper bound are consistent with the relationships:

$$\psi^*(u) \ \le \ \psi^\star(u) \ \le \ \psi_0(u), \qquad u \ge 0\,, \tag{4.7}$$

which follows from the definitions of the annuity-due and annuity-immediate risk models.

Example 4.3 Suppose that Y has a truncated normal density function

$$g(t) = \frac{1}{\Phi(\mu/\sigma)\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}, \quad t \geq 0 \tag{4.8}$$

where $-\infty < \mu < \infty$, $\sigma > 0$ and

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \int_{-\infty}^{x} \phi(y) dy$$

is the standard normal distribution.

It can be shown that

$$M_Y(t) = \frac{\Phi(\mu/\sigma + \sigma t)}{\Phi(\mu/\sigma)} e^{\frac{t^2\sigma^2}{2} + \mu t},$$

and

$$E(Y) = \mu + \frac{\sigma\phi(\mu/\sigma)}{\Phi(\mu/\sigma)}.$$

We know that Y has an increasing failure rate, see, for example, Barlow and Proschan (1981). Bowers, et al (1997) and Yang (1998) use a normal distribution as an example of Y. Using the normal distribution, the exact and simple expressions of R_0, R_1, R_2, and R_3 are easy to be obtained when the forces of interest are constant. But the disadvantage of the normal distribution is that it is not a nonnegative random variable. However, the truncated normal random variable is nonnegative. Moreover, if $\mu \gg \sigma$, it is similar to the normal distribution $N(\mu, \sigma^2)$. But, in general, they have different properties. In this example, we let $\mu = 0.1$ and $\sigma = 0.6$. Then $EY = 0.5207$ and $R_0 = 4.2628728967$.

(1) Let Δ be a constant $\delta = 0.07$. We get that $R_2 = 4.7372669852$. Hence $R_1 = R_2 e^{\delta} = 5.0807575985$; $R_3 = R_0 e^{\delta} = 4.5719660574$. In this case, (2.15) applies to $\psi^*(u)$, and (3.7) applies to $\psi^\star(u)$. The numerical results of the upper bounds are given in Table 5.

(2) Let Δ have a uniform distribution on [0.06, 0.08]. Thus, we get $R_1 = 5.0785748383$, $R_2 = 4.7367949264$, and $R_3 = 4.5715041898$. Then (2.15) and (2.31) apply to $\psi^*(u)$, and (3.6) and (3.14) apply to $\psi^\star(u)$. We get Table 6, where $\beta_2 = 1$ in (2.31) and $\beta_0 = 1$ in (3.14). It can be found from Table 6 that the upper bound in (2.31) is tighter than that in (2.15) while the upper bound in (3.14) is tighter that in (3.6). Moreover, in this example, the upper bounds for $\psi^*(u)$ and $\psi^\star(u)$ and Lundberg's upper bound are still consistent with (4.7).

5 Concluding Remarks

We introduce annuity-due and annuity-immediate risk models, which are the generalizations of the classical discrete time risk model. The functional inequalities for the ruin probabilities in such models are derived by two different methods. The applications of the functional inequalities to exponential upper bounds are discussed in detail. All these exponential upper bounds are the generalizations of Lundberg's upper bound. In general, it is very difficult to derive the exact ruin probabilities in the annuity-due and annuity-immediate risk models. However, these upper bounds can be used to estimate the ruin probabilities.

We point out that the results of Theorems 2.1, 2.2, 3.1, and 3.2 can be also applied to non-exponential upper bounds when the adjustment coefficients do not exist by choosing non-exponential distributions for B_1. For example, when the adjustment coefficients do not exist, we can choose $B_1(x) = (1 + \kappa x)^{-m}$ or $B_1(x) = (1 + \kappa x)^{-m} e^{-\mu x}$, and thus we obtain non-exponential upper bounds for the ruin probabilities in the annuity-due and annuity-immediate risk models. We refer to Willmot (1994, 1996), Willmot and Lin (2001), and Yang (1998) for the details of such choices, which are however omitted in this paper.

Furthermore, we point out that $B_1(x)$ and $B_2(x)$ in Theorems 2.1, 2.2, 3.1, and 3.2 are not necessarily distribution functions. In fact, they only need to be nonnegative increasing functions, which are bounded between 0 and 1 with $B_1(0) = B_2(0) = 0$, and to satisfy (2.3) or its reversed inequality.

References

[1] Barlow, R.E. and Proschan, F. (1981) *Statistical Theory of Reliability and Life Testing.* To Begin With, Silver Spring, MD.

[2] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J. (1997) *Actuarial Mathematics.* The Society of Actuaries, Schaumburg, IL.

[3] Cai, J. and Garrido, J. (1999) A unified approach to the study of tail probabilities of compound distributions. *J. Appl. Prob.* **36**, 1058-1073.

[4] Dickson, D.C.M. (1994) Some comments on the compound binomial model. ASTIN Bulletin. **24**, 33-45.

[5] Gerber, H.U. (1979) *An Introduction to Mathematical Risk Theory.* S.S. Heubner Foundation Monograph Series 8, Philadelphia.

[6] Gerber, H.U. (1988) Mathematical fun with the compound binomial process. ASTIN Bulletin. **18**, 161-168.

[7] Grandell, J. (1997) *Mixed Poisson Processes.* Chapman & Hall, London.

[8] Lin, X. (1996) Tail of compound distributions and excess time. J. Appl. Prob. **33**, 184-195.

[9] Shiu, E.S.W. (1989) The probability of eventual ruin in the compound binomial model. ASTIN Bulletin. **19**, 179-190.

[10] Taylor, J.C. (1997) *An Introduction to Measure and Probability.* Spring-Verlag, New York.

[11] Willmot, G.E. (1993) Ruin probabilities in the compound binomial model. Insurance: Math. Econom. **12**, 133-142.

[12] Willmot, G.E. (1994) Refinements and distributional generalizations of Lundberg's inequality. Insurance: Math. Econom. **15**, 49-63.

[13] Willmot, G.E. (1996) A non-exponential generalization of an inequality arising in queueing and insurance risk. J. Appl. Prob. **33**, 176-183.

[14] Willmot, G.E. and Lin, X.S. (2001) *Lundberg Approximations for Compound Distributions with Insurance Applications.* Springer-Verlag, New York.

[15] Yang, H. (1998) Non-exponential bounds for ruin probability with interest effect included. Scand. Actuarial J. **1**, 66-79.

Table 1: Upper bounds in Example 4.1 with constant forces of interest

u	(2.36) for $\psi^*(u)$	(3.18) for $\psi^\star(u)$	Lundberg
0	0.421121	0.450764	1.000000
0.5	0.273281	0.296519	0.671389
1.0	0.177343	0.195054	0.450764
1.5	0.115084	0.128309	0.302638
2.0	0.074683	0.084404	0.203188
2.5	0.048464	0.055522	0.136418
3.0	0.031450	0.036523	0.091590
3.5	0.020409	0.024025	0.061492
4.0	0.013244	0.015804	0.041285
4.5	0.008595	0.010396	0.027719
5.0	0.005577	0.006839	0.018610
5.5	0.003619	0.004499	0.012495

Table 2: Upper bounds in Example 4.1 with stochastic forces of interest

u	(2.15) for $\psi^*(u)$	(2.33) for $\psi^*(u)$	(3.6) for $\psi^\star(u)$	(3.16) for $\psi^\star(u)$	Lundberg
0	1.000000	0.421119	1.000000	0.450764	1.000000
0.5	0.648997	0.273282	0.657854	0.296518	0.671389
1.0	0.421198	0.177345	0.432772	0.195054	0.450764
1.5	0.273356	0.115088	0.284701	0.128310	0.302638
2.0	0.177407	0.074687	0.187292	0.084405	0.203188
2.5	0.115137	0.048469	0.123211	0.055524	0.136418
3.0	0.074724	0.031455	0.081055	0.036525	0.091590
3.5	0.048495	0.020413	0.053322	0.024028	0.061492
4.0	0.031473	0.013247	0.035078	0.015806	0.041285
4.5	0.020426	0.008597	0.023076	0.010398	0.027719
5.0	0.013257	0.005579	0.015181	0.006840	0.018610
5.5	0.008603	0.003621	0.009987	0.004500	0.012495

Table 3: Upper bounds in Example 4.2 with constant forces of interest

u	(2.15) for $\psi^*(u)$	(3.7) for $\psi^\star(u)$	Lundberg
0.15	0.673411	0.683357	0.698678
0.30	0.453483	0.466977	0.488151
0.45	0.305380	0.319112	0.341060
0.60	0.205647	0.218067	0.238291
0.75	0.138485	0.149018	0.166489
0.90	0.093257	0.101832	0.116322
1.05	0.062800	0.069588	0.081272
1.20	0.042291	0.047553	0.056783
1.35	0.028479	0.032496	0.039673
1.50	0.019178	0.022206	0.027719
1.65	0.012915	0.015175	0.019366
1.80	0.008697	0.010370	0.013531

Table 4: Upper bounds in Example 4.2 with stochastic forces of interest

u	(2.15) for $\psi^*(u)$	(2.31) for $\psi^*(u)$	(3.6) for $\psi^\star(u)$	(3.14) for $\psi^\star(u)$	Lundberg
0.15	0.673502	0.673436	0.683405	0.683354	0.698678
0.30	0.453605	0.453519	0.467043	0.466975	0.488151
0.45	0.305504	0.305419	0.319179	0.319113	0.341060
0.60	0.205758	0.205684	0.218129	0.218070	0.238291
0.75	0.138578	0.138518	0.149070	0.149022	0.166489
0.90	0.093333	0.093285	0.101875	0.101837	0.116322
1.05	0.062860	0.062824	0.069622	0.069593	0.081272
1.20	0.042336	0.042309	0.047580	0.047558	0.056783
1.35	0.028514	0.028494	0.032517	0.032500	0.039673
1.50	0.019204	0.019190	0.022222	0.022210	0.027719
1.65	0.012934	0.012924	0.015187	0.015178	0.019366
1.80	0.008711	0.008704	0.010379	0.010373	0.013531

Table 5: Upper bounds in Example 4.3 with constant forces of interest

u	(2.15) for $\psi^*(u)$	(3.7) for $\psi^\star(u)$	Lundberg
0.1	0.601652	0.633056	0.652929
0.2	0.361985	0.400760	0.426316
0.3	0.217789	0.253703	0.278354
0.4	0.131033	0.160608	0.181745
0.5	0.078837	0.101674	0.118668
0.6	0.047432	0.064365	0.077481
0.7	0.028538	0.040747	0.050590
0.8	0.017170	0.025795	0.033031
0.9	0.010330	0.016330	0.021567
1.0	0.006215	0.010338	0.014082
1.1	0.003739	0.006544	0.009194
1.2	0.002250	0.004143	0.006003

Table 6: Upper bounds in Example 4.3 with stochastic forces of interest

u	(2.15) for $\psi^*(u)$	(2.31) for $\psi^*(u)$	(3.6) for $\psi^\star(u)$	(3.14) for $\psi^\star(u)$	Lundberg
0.1	0.601784	0.601731	0.633085	0.633053	0.652929
0.2	0.362143	0.362084	0.400797	0.400759	0.426316
0.3	0.217932	0.217881	0.253738	0.253705	0.278354
0.4	0.131148	0.131109	0.160638	0.160612	0.181745
0.5	0.078923	0.078895	0.101698	0.101679	0.118667
0.6	0.047494	0.047476	0.064383	0.064370	0.077481
0.7	0.028581	0.028569	0.040760	0.040752	0.050590
0.8	0.017200	0.017192	0.025805	0.025799	0.033031
0.9	0.010351	0.010346	0.016336	0.016333	0.021567
1.0	0.006229	0.006226	0.010342	0.010340	0.014082
1.1	0.003748	0.003747	0.006548	0.006546	0.009194
1.2	0.002256	0.002255	0.004145	0.004145	0.006003