FURTHER ON LARGEST CLAIMS REINSURANCE

ERHARD KREMER
Fachbereich Mathematik
Universität Hamburg

Bundesstrasse 55, 20146 Hamburg, Germany
Tel. 040/428 38-4934 – Telefax 040/428 38-4924 – email kremer@math.uni-hamburg.de

NOTE 1:
LARGEST CLAIMS REINSURANCE PREMIUMS UNDER DISCRETE CLAIMS SIZES

ABSTRACT
The generalized largest claims reinsurance cover is reconsidered. Formulas for its net premium and loading are derived under assumption of an arbitrary discrete claims size distribution. The formula for the net premium is specialized to the assumption of Poisson–distributed claims number.

NOTE 2:
AN ELEMENTARY UPPER BOUND ON THE LOADING OF THE LARGEST CLAIMS REINSURANCE COVER

ABSTRACT
Again the largest claims reinsurance treaty is reconsidered. An upper bound for the loading of the treaty in case of using the standard deviation or variance principle is given by an elementary argument.

NOTE 3:
ON THE CHOICE OF THE PAREMETER p OF THE $LCR(p)$-TREATY

ABSTRACT
It is shown how one can determine the number of largest claims to be taken by the reinsurer, when having a priority for an excess-of-loss treaty.

NOTE 1:
LARGEST CLAIMS REINSURANCE PREMIUMS UNDER DISCRETE CLAIMS SIZES

1 INTRODUCTION

Nowadays reinsurance mathematics is one of the biggest fields of mathematical risk theory. One of its most important topics is the calculation of a risk adequate premium for a nonproportional reinsurance cover. Elegant premium theories were developed for all practical treaties. Especially for the so–called largest claims reinsurance cover a lot was published on its premium. More concretely finite and asymptotic premium formulas were derived (see Kremer (1982), (1984), (1985), (1989), (1998 a)). Furthermore recursions and premium bounds were deduced (see Kremer (1983), (1986), (1988), (1994), (1998 b)).
Quite general is the mentioned article Kremer (1985), where under fairly general conditions elegant results on the premiums of generalizations of the largest claims reinsurance cover are given.
Nevertheless the results do not hold under most general conditions, since it is basically assumed that the claims size distribution function is continuous. In case that the claims size distribution is discrete, the derived formulas are not any more valid. So the question arises what one can do in case that the claims size distribution function is discrete. This question is completely answered in the following paper.
Counterparts to the 1985–formulas are presented and proved in case of general discrete claims size distribution on the nonnegative integers.

2 THE TREATY

Let the random variables $X_1, X_2, X_3, \ldots$ on $(\Omega, \mathcal{A}, P)$ denote the claims of a collective of risks and let N on $(\Omega, \mathcal{A}, P)$ describe the number of claims. The claims ordered in nonincreasing size let be the random variables:

$$X_{N:1} \geq X_{N:2} \geq \cdots \geq X_{N:N}.$$

For given constants $c_1, c_2, \ldots$ such, that

$$\sum_{i=1}^{n} c_i \cdot y_i \in \left[0, \sum_{i=1}^{n} y_i\right]$$

holds for all $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$, define as claims amount taken over by a reinsurer:

$$S_N = \sum_{i=1}^{N} c_i \cdot X_{N:i}. \tag{2.1}$$

The reinsurance treaty defined by the family $(c_i, i \geq 1)$ shall be called <u>generalized largest claims reinsurance cover</u> (in short: GLCR). The name is obvious from the fact that (2.1)

reduces for the special choice

$$c_1 = c_2 = \cdots = c_p = 1$$
$$c_j = 0, \text{ for all } j > p$$

to the claims amount of the (classical) largest claims reinsurance cover, like defined e.g. by Ammeter (1964). Note that for the special choice

$$\begin{aligned} c_1 &= c_2 = \cdots = c_{p-1} = 1 \\ c_p &= 1 - p \\ c_j &= 0 \qquad \text{for all } j > p \end{aligned}$$

the so–called ECOMOR–treaty also comes out (see Thépaut (1950)). For the sequel assume that:

(A.1) $N, X_1, X_2, X_3, \ldots$ are independent

(A.2) $X_1, X_2, X_3, \ldots$ are identically distributed with discrete distribution function F on the nonnegative integers $\mathbb{N}_0$.

Denote $P(N = n)$ by p_n for all $n \in \mathbb{N}_0$.

3 THE GENERAL PREMIUM FORMULA

In risk theory various principles for premium calculation were defined and analyzed (see e.g. De Vylder et.al. (1984) or Kremer (1999)). In nonproportional reinsurance it is most adequate to take the so–called standard deviation principle (see e.g. Reich (1985)), giving as risk premium of the GLCR just:

$$\pi = \mu + \Lambda \cdot \sigma$$

with the mean and standard deviation

$$\mu = E(S_N), \ \sigma = (Var(S_N))^{1/2}$$

and a suitable loading factor $\Lambda > 0$. Here no care shall be taken about Λ, so that premium rating of the GLCR reduces to giving formulas for μ and σ^2.

On both one has as general result:

Theorem

Denote with $M(t)$ the generating function of the claims number distribution, i.e.:

$$M(t) = \sum_{n=0}^{\infty} p_n \cdot t^n,$$

and with $M^{(i)}(t)$ its i–th derivate. Let

$$\Gamma_\lambda(s) = \int_0^\lambda t^{s-1} \cdot \exp(-t)dt$$

be the incomplete Γ–function and

$$\Gamma(s) = \Gamma_\infty(s)$$

be the complete Γ–function (note $\Gamma(i) = (i-1)!$).

With that notation one has:

a)

$$\mu = \sum_{i=1}^{\infty} \frac{c_i}{\Gamma(i)} \cdot \sum_{n=0}^{\infty} \left[P(N \geq i)\Gamma(i) - H_{F(n)}(i)\right]$$

where $H_p(\cdot)$ is defined as:

$$H_p(i) = \int_0^p (1-t)^{i-1} \cdot M^{(i)}(t)dt,$$

b)

$$\begin{aligned}\sigma^2 &= \sum_{i=1}^{\infty} \frac{c_i^2}{\Gamma(i)} \cdot \sum_{n=0}^{\infty} (2n+1) \left[P(N \geq i)\ \Gamma(i) - H_{F(n)}(i)\right] \\ &+2 \cdot \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} \frac{c_i \cdot c_j}{\Gamma(j-i)\ \Gamma(i)} \cdot \sum_{m=0}^{\infty} \sum_{n=0}^{m} n \cdot m \cdot I_{nm}(i,j) - \mu^2\end{aligned}$$

where $I_{nm}(\cdot,\cdot)$ is defined as:

$$I_{nm}(i,j) = \int_{F(m-1)}^{F(m)} \int_{F(n-1)}^{min(v,F(n))} (v-w)^{j-i-1} \cdot (1-v)^{i-1} M^{(j)}(w)\ dw\, dv$$

□

Proof

a) According to page 45 in David (1981) one has:

$$E(X_{N:i}|N=n) = \sum_{m=0}^{\infty} (1 - F_{ni}(m))$$

with:

$$F_{ni}(m) = \frac{n!}{(i-1)!(n-i)!} \cdot \int_0^{F(m)} t^{n-i} \cdot (1-t)^{i-1} dt,$$

and consequently:

$$\begin{aligned}
\mu &= E(\sum_{i=1}^{N} c_i \cdot E(X_{N:i}|N)) = \\
&= \sum_{n=1}^{\infty} p_n \cdot \sum_{i=1}^{n} c_i \cdot E(X_{N:i}|N=n) = \\
&= \sum_{i=1}^{\infty} c_i \cdot \sum_{m=0}^{\infty} \left[\sum_{n=i}^{\infty} p_n \cdot (1 - F_{ni}(m)) \right] = \\
&= \sum_{i=1}^{\infty} c_i \cdot \sum_{m=0}^{\infty} \left[P(N \geq i) - \frac{1}{(i-1)!} \cdot \right. \\
&\quad \left. \cdot \int_0^{F(m)} (1-t)^{i-1} \left(\sum_{n=i}^{\infty} \frac{n!}{(n-i)!} \cdot t^{n-i} \cdot p_n \right) dt \right] \\
&= \sum_{i=1}^{\infty} \frac{c_i}{\Gamma(i)} \cdot \sum_{m=0}^{\infty} \left[P(N \geq i)\Gamma(i) - H_{F(m)}(i) \right]
\end{aligned}$$

since:

$$M^{(i)}(t) = \sum_{n=i}^{\infty} \frac{n!}{(n-i)!} \cdot t^{n-i} \cdot p_n \ . \tag{3.1}$$

b) One has:

$$\begin{aligned}
E(S_N^2) &= E\left(\sum_{i=1}^{N} c_i^2 \cdot E(X_{N:i}^2|N) \right) + \\
&\quad + 2 \cdot E\left(\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_i c_j \cdot E(X_{N:i} \cdot X_{N:j}|N) \right)
\end{aligned} \tag{3.2}$$

Again according to David (1981), page 45 one has:

$$E(X_{N:i}^2|N=n) = \sum_{m=0}^{\infty} (2m+1) \cdot (1 - F_{ni}(m))$$

what implies similar to part a):

$$E\left(\sum_{i=1}^{N} c_i^2 \cdot E(X_{N:i}^2|N)\right) =$$
$$= \sum_{i=1}^{\infty} \frac{c_i^2}{\Gamma(i)} \cdot \sum_{m=0}^{\infty} (2m+1)\left[P(N \geq i)\Gamma(i) - H_{F(m)}(i)\right] .$$

Furthermore one has according to David (1981), page 46, for $i < j$:

$$E(X_{N:i} \cdot X_{N:j}|N = n) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \cdot$$
$$\cdot \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} x \cdot y \cdot \int_{F(y-1)}^{F(y)} \int_{F(x-1)}^{min(v,F(x))} w^{n-j} \cdot (v-w)^{j-i-1} \cdot (1-v)^{i-1} dw\, dv$$

This implies for the second term in (3.2) as result:

$$\sum_{n=2}^{\infty} p_n \cdot \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} c_i c_j \cdot \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \cdot$$
$$\cdot \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} x \cdot y \cdot \int_{F(y-1)}^{F(y)} \int_{F(x-1)}^{min(v,F(x))} w^{n-j} \cdot (v-w)^{j-i-1} \cdot (1-v)^{i-1} dw\, dv =$$
$$= \sum_{j=2}^{\infty}\sum_{i=1}^{j-1} \frac{c_i c_j}{\Gamma(j-i)\Gamma(i)} \cdot \sum_{y=0}^{\infty}\sum_{x=0}^{y} x \cdot y \cdot$$
$$\cdot \int_{F(y-1)}^{F(y)} \int_{F(x-1)}^{min(v,F(x))} (v-w)^{j-i-1}(1-v)^{i-1} \cdot \left[\sum_{n=j}^{\infty} \frac{n!}{(n-j)!} w^{n-j} p_n\right] dw\, dv =$$

$$= \sum_{j=2}^{\infty}\sum_{i=1}^{j-1} \frac{c_i c_j}{\Gamma(j-i)\Gamma(i)} \cdot \sum_{y=0}^{\infty}\sum_{x=0}^{y} x \cdot y \cdot I_{xy}(i,j)$$

with the last equality because of (3.1). □

4 MORE SPECIAL NET PREMIUM

For the net premium μ one can derive a more handy formula under the additional assumption that the N is Poisson–distributed with mean λ, i.e.:

(A.3) $$P(N = n) = \left(\frac{\lambda^n}{n!}\right) \cdot \exp(-\lambda),\ n = 0, 1, 2 \ldots$$

One has the nice result:

Corollary

Under (A.1)–(A.3) one gets for the net premium:

$$\mu = \sum_{i=1}^{\infty} \left(\frac{c_i}{\Gamma(i)}\right) \cdot \left[\sum_{n=0}^{\infty} \Gamma_{\lambda_n}(i)\right]$$

with

$$\lambda_n = \lambda \cdot (1 - F(n)).$$

□

Proof

One knows from elementary probability theory that under (A.3):

$$P(N \geq i) \cdot \Gamma(i) = \Gamma_\lambda(i).$$

Into the formula for H_p one inserts the formula:

$$M^{(i)}(t) = \lambda^i \cdot \exp(\lambda(t-1)),$$

giving after some routine calculations with $\lambda_p = \lambda \cdot (1-p)$:

$$H_p(i) = \Gamma_\lambda(i) - \Gamma_{\lambda_p}(i).$$

Part a) of the theorem gives the statement. □

REFERENCES

Ammeter, H. (1964): *The rating of "'Largest claims"' reinsurance covers. Quartely letter. Jubilee number 2.*

David, H.A. (1981): *Order statistics. John Wiley & Sons, New York.*

De Vylder, F., M.J. Goovaerts and H. Haezendonck (1984): *Insurance Premiums. North Holland, Amsterdam.*

Kremer, E. (1982): *Rating of largest claims and ECOMOR reinsurance treaties for large portfolios. Astin Bulletin.*

Kremer, E. (1983): *Distribution-free upper bounds on the premiums of the LCR and ECOMOR treaties. Insurance: Mathematics and Economics.*

Kremer, E. (1984): *An asymptotic formula for the net premium of some reinsurance treaties. Scandinavian Actuarial Journal.*

Kremer, E. (1985): *Finite formulae for the premium of the general reinsurance treaty based on ordered claims. Insurance: Mathematics and Economics.*

Kremer, E. (1986): *Recursive calculation of the net premium for largest claims reinsurance covers. Astin Bulletin.*

Kremer, E. (1988): *A general bound for the net premium of largest claims reinsurance treaties. Astin Bulletin.*

Kremer, E. (1989): *Largest claims reinsurance premiums under a generalized claim size model. Bulletin de l'Association Royale des Actuaries Belges.*

Kremer, E. (1994): *Recursive largest claims reinsurance rating, revisited. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*

Kremer E. (1998 a): *Largest claims reinsurance premiums for the Weibull model. Blätter der deutschen Gesellschaft für Versicherungsmathematik.*

Kremer E. (1998 b): *Largest claims reinsurance premiums under possible claims dependence. Astin Bulletin.*

Kremer, E. (1999): *Applied Risk Theory. Shaker, Aachen*

Reich, A. (1985): *Eine Charakterisierung des Standardabweichungsprinzips. Blätter der deutschen Gesellschaft für Versicherungsmathematik.*

Thépaut, A. (1950): *Le traité d'excédent du coût moyen relatif. Bulletin Trimestriel de l'institut des Actuaries Français, N₀ 192.*

NOTE 2:
AN ELEMENTARY UPPER BOUND ON THE LOADING OF THE LARGEST CLAIMS REINSURANCE COVER

1 INTRODUCTION

Since the begin of the eighties the author published a lot on the theory of the largest claims reinsurance treaty. Mentioned can be his papers on the efficiency and total claims amount (see e.g. Kremer (1990 a), (1990 b), (1992)), furthermore his important work on the premium (see e.g. Kremer (1984), (1985), (1986), (1988), (1994), (1998), (2000)). Many comparably handy results were given for the net premium, whereas results on the loading, when using the standard deviation or variance principle, were more rare and quite unhandy. So in Kremer (1985), (2000) general formulas on the loading are given that seem to be mainly of theoretical importance since their structures are not practical enough. Something more handy is liked to be known. In the following note a first attempt is made to give a quite simple upper bound on the loading.

2 THE TREATY

Let the random variables $X_1, X_2, X_3, \ldots$ on $(\Omega, \mathcal{A}, P)$ denote the claims of a collective of risks and let N on $(\Omega, \mathcal{A}, P)$ describe the number of claims. The random variables:

$$X_{N:1} \geq X_{N:2} \geq \cdots \geq X_{N:N}$$

let be the claims ordered in nonincreasing size. For given constants $c_1, c_2 \ldots$ such that:

$$\sum_{i=1}^{n} c_i \, y_i \in \left[0, \sum_{i=1}^{n} y_i\right]$$

holds for all $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$ and all n, define as claims amount taken over by a reinsurer:

$$S_n = \sum_{i=1}^{N} c_i \cdot X_{N:i}. \tag{2.1}$$

The reinsurance treaty defined by the family $(c_i, i \geq 1)$ shall be called <u>generalized largest claims reinsurance cover</u> (in short: GLCR). The name is obvious from the fact that (2.1) reduces for the special choice:

$$c_1 = c_2 = \cdots = c_p = 1$$

$$c_j = 0, \text{ for all } j > p$$

to the claims amount of the classical largest claims reinsurance cover, like defined e.g. in Ammeter (1964). For the sequel assume that:

(A.1) $N, X_1, X_2, X_3, \ldots$ are independent,

(A.2) $X_1, X_2, X_3, \ldots$ are identically distributed with moments $\mu = E(X_i)$, $\sigma^2 = Var(X_i)$, $\nu^2 = E(X_i - \mu)^4$.

3 THE LOADING BOUND

In risk theory various premium principles were defined and analyzed (see e.g. De Vylder et.al. (1984) or Kremer (1999)). For the reinsurance treaty introduced in section 2 it is most adequate to take the so–called standard deviation principle, giving as risk premium:

$$\pi = m + \Lambda \cdot s$$

with the net premium:

$$m = E(S_N)$$

and the standard deviation:

$$s = [Var(S_N)]^{1/2}.$$

The loading factor $\Lambda > 0$ shall be given, so that just m and s^2 are of interest. Handy formulas on m are given in Kremer (1985), (2000), so that because of:

$$Var(S_N) = E(S_N^2) - m^2,$$

only the $E(S_N^2)$ remains to be of further interest. Results on $E(S_N^2)$ can also be found in Kremer (1985), (2000), but they turn out to be too unhandy. A handy upper bound on $E(S_N^2)$ is given in the paper's result:

Theorem

Define new coefficients c_i^* according:

$$c_i^* = \left(c_i + 2 \cdot \sum_{j=i+1}^{\infty} c_j \right) \cdot c_i$$

and with:

$$S_N^* = \sum_{i=1}^{N} c_i^* \cdot X_{N:i}$$

the mean:

$$m_* = E(S_N^*).$$

Suppose that $c_i \geq 0$ holds for all i.

Then:

$$E(S_N^2) \leq [E(N \cdot (N-1) \cdot v_N^*)]^{1/2} \cdot \nu + 2 \cdot \mu \cdot m_* - E(N \cdot \overline{c_N^*}) \cdot [\sigma^2 + \mu^2]$$

where:

$$\overline{c_{n\cdot}^*} = \frac{1}{n} \cdot \sum_{i=1}^{n} c_i^*$$

$$v_n^* = \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(c_i^* - \overline{c_{n\cdot}^*}\right)^2 .$$

□

Proof

One takes the splitting up:

$$S_N^2 = \sum_{i=1}^{N} c_i^2 \cdot X_{N:i}^2 + 2 \cdot \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_i c_j X_{N:i} \cdot X_{N:j}$$

where obviously the double sum is bounded by:

$$\sum_{i=1}^{N-1} c_i \cdot \left(\sum_{j=i+1}^{N} c_j \right) \cdot X_{N:i}^2 ,$$

since for $j > i$ holds $X_{N:i} \geq X_{N:j}$. Consequently:

$$E(S_N^2) \leq E(T_N)$$

for:

$$T_N = \sum_{i=1}^{N} c_i^* \cdot X_{N:i}^2 .$$

One has by the Schwarz–inequality of linear algebra:

$$\begin{aligned}
&\sum_{i=1}^{N} \left(c_i^* - \overline{c_{N\cdot}^*}\right) (X_{N:i} - \mu)^2 \\
&\leq \left[\sum_{i=1}^{N} (c_i^* - \overline{c_{N\cdot}^*})^2 \cdot \sum_{i=1}^{N} (X_{N:i} - \mu)^4 \right]^{1/2} \\
&= \left[(N-1) \cdot v_N^* \cdot \sum_{i=1}^{N} (X_i - \mu)^4 \right]^{1/2}
\end{aligned}$$

On both sides one first takes $E(\cdot \mid N)$ and then $E(\cdot)$. By Jensen's inequality the right hand side turns out to be bounded just by:

$$[E(N(N-1)\cdot v_N^*)]^{1/2}\cdot \nu$$

The left hand side gives:

$$E(T_N) - 2\cdot\mu\cdot m_* + E(N\cdot \overline{c_{N\cdot}^*})\cdot(\sigma^2+\mu^2)\ .$$

□

The m_* has to be determined similar like the m. The bound can be expected to be quite crude in general. In case of the (classical) largest claims reinsurance treaty it should be useful when p is comparably small (i.e. $P(N \leq p)$ is very small), what one usually would have in practice. Note that for this treaty one simply has:

$$\begin{aligned} c_i^* &= 1 + 2\cdot(p-i), \qquad \text{for } i \leq p \\ c_i^* &= 0, \qquad \text{for } i > p \end{aligned}$$

Remark

The result can be regarded as being the counterpart to the former bound of the author for the stoploss treaty (see Kremer (1990 c)). □

REFERENCES

Ammeter, H. (1964): *The rating of "'Largest claims"' reinsurance covers. Quartely letter. Jubilee number 2.*

De Vylder, F., M.J. Goovaerts and H. Haezendonck (1984): *Insurance Premiums. North Holland, Amsterdam.*

Kremer, E. (1984): *An asymptotic formula for the net premium of some reinsurance treaties. Scandinavian Actuarial Journal.*

Kremer, E. (1985): *Finite formulae for the premium of the general reinsurance treaty based on ordered claims. Insurance: Mathematics and Economics.*

Kremer, E. (1986): *Recursive calculation of the net premium for largest claims reinsurance covers. Astin bulletin.*

Kremer, E. (1988): *A general bound for the net premium of largest claims reinsurance treaties. Astin bulletin.*

Kremer, E. (1990 a): *The asymptotic efficiency of largest claims reinsurance treaties. Astin bulletin.*

Kremer, E. (1990 b): *On a generalized total claims amount. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*

Kremer, E. (1990 c): *An elementary upper bound for the loading of a stoploss cover. Scandinavian Actuarial Journal.*

Kremer, E. (1992): *The total claims amount of largest claims reinsurance treaties, revisited. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*

Kremer, E. (1994): *Recursive largest claims reinsurance rating, revisited. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*

Kremer, E. (1998): *Largest claims reinsurance premiums under possible claims dependence. Astin bulletin.*

Kremer, E. (1999): *Applied Risk Theory. Shaker, Aachen*

Kremer, E. (2000): *Largest claims reinsurance premiums under discrete claims sizes. To appear.*

NOTE 3:
ON THE CHOICE OF THE PARAMETER p OF THE $LCR(p)$-TREATY

a) During the past 20 years a lot of theory was published on the largest claims reinsurance (see e.g. Kremer (1984), (1985), (1986), (1988), (1990a), (1990b), (1992a), (1994), (1998b)). According to a note of the author (see Kremer (1992a)) the largest claims reinsurance treaty turned out to be nearly as good as the classical excess-of-loss treaty under practical conditions. But surprisingly nothing was published on how to choose the number of largest claims taken over by the reinsurer under the largest claims reinsurance treaty. In the present note a method on how to choose adequately that number is deduced.

b) Let N be the claims number of a collective of risks and $X_1, X_2, \ldots$ the corresponding claims amounts. Denote the distribution function of the X_i with F. In the (classical) **largest claims reinsurance cover** with parameter p (short: $LCR(p)$) the reinsurer takes over the p largest claims $X_{N:1} \geq X_{N:2} \geq \ldots \geq X_{N:p}$, whereas in the (classical) **excess-of-loss cover** with priority P (short: $XL(P)$) each claim excess P, i.e. the $(X_i - P)_+$. Denote the net premium of the $LCR(p)$ with μ_p^{LC} and the net premium of the $XL(P)$ with μ_P^{XL}. In a certain asymptotic model the author proved the equivalence (see Kremer (1982)):

$$\mu_p^{LC} \sim \mu_{P_p}^{XL} + p \cdot P_p \tag{3.1}$$

with priority

$$P_p = F^{-1}(1 - p/E(N)) \ .$$

c) Suppose the reinsurer knows an adequate priority P for an $XL(P)$ for the collective (perhaps determined by ruin-theoretical investigations), but likes to offer instead of the $XL(P)$ an in some sense equivalent $LCR(p)$. The problem obviously consists in determining an adequate p from the P. Because of (1), a nearlying solution to the problem is the following:
Determine a $p \in \mathbb{N}$ such that:

$$\mu_{P_p}^{XL} + p \cdot P_p \stackrel{!}{=} \mu_P^{XL} \ . \tag{3.2}$$

d) In practice the reinsurer does not have data for estimating the F, he has only claims sizes exceeding a certain threshold $a > 0$. In theory this means that one does not have the whole F but only the conditional distribution function F_a defined as:

$$\begin{aligned} F_a(x) &= Prob(X_i \leq x | X_i \geq a) \\ &= \left(\frac{F(x) - F(a)}{1 - F(a)} \right) \ , \quad \text{for } x \geq a \ . \end{aligned} \tag{3.3}$$

Furthermore one does not have the $E(N)$ but just the n_a defined as:

$$n_a = E(N) \cdot (1 - F(a)) \ ,$$

the mean number of claims exceeding the threshold $a > 0$.

From (3) one gets:

$$F(x) = F_a(x)(1 - F(a)) + F(a) \ , \quad \text{for } x \geq a \ ,$$

what implies:

$$F^{-1}(t) = F_a^{-1}\left(\frac{t - F(a)}{1 - F(a)} \right) \ , \text{ for } t \geq F(a) \ .$$

Consequently one has with

$$\pi_p := (p/n_a)$$

the formula:

$$P_p = F_a^{-1}(1 - \pi_p) \tag{3.4}$$

Remember that (under the usual assumptions):

$$\mu_P^{XL}/n_a = \int_P^\infty (x - P) F_a(dx) \ , \text{ for } P \geq a \ ,$$

what shall be denoted by ρ_P^{XL}. So (2) and (4) give as equation for determining π_p:

$$\boxed{\rho_{F_a^{-1}(1-\pi_p)}^{XL} + \pi_p \cdot F_a^{-1}(1 - \pi_p) - \rho_P^{XL} \stackrel{!}{=} 0} \tag{3.5}$$

what has a solution $\pi_p \in (0, 1)$. With that π_p one would choose as **adequate p of the LCR(p)** the integer nearest to $(\pi_p \cdot n_a)$.

e) In serveral papers the author recommended the use of the so-called **generalized Pareto-distribution** in reinsurance (for a survey see Kremer (1998a)). Its distribution function is defined as:

$$F_a(x) = 1 - (1 + (x - a) \cdot (g/s))^{-1/g} \ , \quad \text{for} \quad x > a$$

with parameters $0 < g < 1, s > 0$. Under that model one has:

$$\begin{aligned} F_a^{-1}(t) &= a + ((1-t)^{-g} - 1) \cdot (s/g) \ , \quad t \in (0,1) \ , \\ \rho_P^{XL} &= (\tfrac{s}{1-g}) \cdot (1 + (P - a) \cdot (g/s))^{1-1/g} \ , \quad P \geq a \ . \end{aligned}$$

This implies for (5) the specialized equation

$$\boxed{\begin{aligned} &\pi_p^{1-g} + \pi_p \cdot (a - (s/g)) \cdot (g/s) \cdot (1-g) - \\ &\qquad -g \cdot (1 + (P-a)(g/s))^{1-1/g} \stackrel{!}{=} 0 \end{aligned}} \tag{3.6}$$

what has a (unique) solution $\pi_p \in (0,1)$.

For the special case of the (classical) **Pareto-model** ($s = a \cdot g, \alpha = 1/g$) the equation (6) simplifies considerably and can be solved analytically. One gets then:

$$\boxed{\pi_p = \alpha^{\frac{\alpha}{1-\alpha}} \cdot [r(P,a)]^{-\alpha}} \tag{3.7}$$

with the ratio:

$$r(P,a) = (P/a) \ .$$

f) For illustration take the example:

$$\begin{aligned} &a = 100\ 000 \ , \quad P = 200\ 000 \ , \quad n_a = 100 \\ &s = a \cdot g \ , \quad \alpha = 1/g \quad \text{(classical Pareto)} \end{aligned}$$

From formula (7) one gets:

$\alpha =$	1.5	2.0	2.5	3.0
π_p	0.1048	0.0625	0.0384	0.0241
p	10	6	4	2

For practical work it might be useful to plot for different r-values the π_p-curves.

REFERENCES

Kremer, E. (1982): *Rating of largest claims and ECOMOR-reinsurance treaties for large portfolios. ASTIN Bulletin.*

Kremer, E. (1984): *An asymptotic formula for the net premium of some reinsurance treaties. Scandinavian Actuarial Journal.*

Kremer, E. (1985): *Finite formulae for the premium of the general reinsurance treaty based on ordered claims. Insurance: Mathematics & Economics.*

Kremer, E. (1986): *Recursive calculation of the net premium for largest claims reinsurance covers. ASTIN Bulletin.*

Kremer, E. (1988): *A general bound for the net premiums of largest claims reinsurance treaties. ASTIN Bulletin.*

Kremer, E. (1990a): *The asymptotic efficiency of largest claims reinsurance covers. ASTIN Bulletin.*

Kremer, E. (1990b): *On a generalized total claims amount. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*

Kremer, E. (1992a): *The limit-equivalence of the excess-of-loss and largest claims reinsurance treaty. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*

Kremer, E. (1992b): *The total claims amount of largest claims reinsurance, revisited. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*

Kremer, E. (1994): *Recursive largest claims reinsurance rating, revisited. Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*

Kremer, E. (1998a): *Die verallgemeinerte Pareto-Verteilung in der Rückversicherung. Transactions of the international congress of actuaries, Birmingham.*

Kremer, E. (1998b): *Largest claims reinsurance premiums under possible claims dependence. ASTIN Bulletin.*