AN APPLICATION OF THE RISK THEORY FOR THE MANAGEMENT OF A DREAD DISEASE PORTFOLIO (°)

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ABSTRACT
The present study deals with a group of insurance policies complementary to life insurance ones, concerning Dread Disease risk in additional form as to term insurance policies or in early form as to upgraded endowment ones, provided by the Italian insurance market.
They deal with random benefits of parts with reference to disease and death occurrences as well as a first and a second moments, in order to get probability for the portfolio here considered, having a coverage fund but no reinsurance.
The problem is faced also in case of a specific correlation among risks and the correlation is determined among the various components of the random gain on single policies.
In the end a numerical application of schemes involved is also worked out.

KEYWORDS
Correlation - Dread Disease - Risk

1. INTRODUCTION

The Italian insurance market includes life policies offering an additional coverage on "Dread Disease" (briefly: DD) risk.
This study deals with two specific insurance forms, included in the ANIA "Note tecniche attuariali", namely

- DD coverage, linked in additional form to term insurance policies, with capital and annual premiums to be constant;
- DD coverage, linked in early form to endowment policies, with capital and annual premiums to be upgraded.

As to them we give a random variables' "Company benefits" formulation.
In portfolios of that kind the hypothesis of independence among risks allows for a direct application of well known models concerning risk theory, meant at calculating "ruin probability" bounds. However a point of special interest concerns the consequences of a positive correlation likely to issue from a concurrence of events acting simultaneously on risks insured.

We shall therefore consider those changes affecting ruin probability and strategies to counterbalance its increase due to such a correlation.

2. DISTRIBUTION FUNCTION OF BENEFITS ASSURED

In connection with purposes specified in §1, it is necessary to determine the random present value (r.p.v.) on drawing up of the contract (taken as initial time) for the Company's obligations towards a policy-holder aged x, distinguishing the two policy forms indicated above.

a) *Additional form as to a term insurance with capital and annual constant premiums in advance, n term like that of principal insurance, and exemption from its remaining annual premiums after DD rise.*

The r.p.v. of this insurance form is the sum of the two components' r.p.v.$_s$, the additional value and the exemptive one. Let integer T be the starting instant of DD rise random year. Put T=∞ in case of non-claim. We shall deal with the case T<n .

Assuming an uniform distribution of claims' arrivals, we put DD rise and payment of indemnity C at T+0.5. Therefore, given rate i, we obtain

$$(2.1)\qquad B_{AGG} = C(1+i)^{-T-0.5} \quad , \qquad T < n$$

If $T \geq n$, the outcome is $B_{AGG} = 0$.

Let's now introduce the following probabilities concerning a policy-holder aged x, DD free

$_h p_x^{(H)}$ = probability of persisting h years in state H (= Healthy)

$q_x^{(HD)}$ = transition probability to state D (= Dead)

$q_x^{(HS)}$ = transition probability to state S (= DD)

Assuming here no return from S to H, transition after h years from H to S is ruled by

$$(2.2)\qquad \text{Prob}\,(T=h) = {}_{h/1}q_x^{(HS)} = {}_h p_x^{(H)}\, q_{x+h}^{(HS)} \quad , \quad (h = 0,1,\dots,n-1)$$

which identifies also the distribution function of (2.1), whereas the one from H to D is ruled by

$$(2.2')\qquad {}_{h/1}q_x^{(HD)} = {}_h p_x^{(H)}\, q_{x+h}^{(HD)} \quad , \qquad (h = 0,1,\dots,n-1)$$

With reference to the r.p.v.'s latter component, indicating by Y≥0 the random integer number of years from DD rise to death (both supposed to occur at half year) and by Q the exempted annual premium, we obtain

$$(2.3)\qquad B_{ESO1} = Q(1+i)^{-T} R_{T,Y} \quad , \qquad T < n-1$$

when

$$(2.4)\qquad R_{T,Y} = a_{\overline{M}|i} \quad , \qquad M = \min(Y, n-T-1)$$

and $a_{\overline{r}|i}$ represents the present value at rate *i* of the immediate annuity with *r* years term, being $a_{\overline{0}|i} = 0$. Obviously $B_{ESO1} = 0$ if $Y = 0$ or if $T \geq n-1$.

Put again, for a policy-holder in state S, aged x

$_h p_x^{(S)}$ = probability of persisting h years in S;

$_{k/1}q_x^{(SD)}$ = transition probability to D after k years;

v = 1/(1+i) = discount factor

the probabilities concerning Y, conditioned to the event "T = h", are

$$(2.5)\quad \mathrm{Prob}(Y=k\mid T=h)=\begin{cases} q_{x+h}^{(SD)}/2 = {}_{0/1}q_{x+h}^{(SD)}/2 & ,\ \text{se}\ \ k=0 \\ {}_{k/1}q_{x+h}^{(SD)} & ,\ \text{se}\ \ k=1,\dots,n-h-1 \\ {}_{n-h}p_{x+h}^{(S)} & ,\ \text{se}\ \ k>n-h-1 \end{cases}$$

Owing to (2.2) and (2.5) the distribution function of (2.3) issues from

$$(2.6)\quad \mathrm{Prob}\ (T=h\cap Y=k)=\begin{cases} {}_{h/1}q_x^{(HS)}\ {}_{k/1}q_{x+h}^{(SD)} & ,\ \text{se}\ \ 1\le k\le n-h-1 \\ {}_{h/1}q_x^{(HS)}\ {}_{n-h}p_{x+h}^{(S)} & ,\ \text{se}\ \ k>n-h-1 \end{cases}$$

and the single net premium for the benefit indicated sub *a)* is

$$(2.7)\qquad E(B_{AGG}+B_{ESO1})=C\sum_{h=0}^{n-1} v^{h+0.5}\ {}_{h/1}q_x^{(HS)}+$$

$$+Q\sum_{h=0}^{n-2} v^h\ {}_{h/1}q_x^{(HS)}\left\{\sum_{k=1}^{n-h-1} a_{\overline{k}|i}\ {}_{k/1}q_{x+h}^{(SD)}+a_{\overline{n-h-1}|i}\ {}_{n-h}p_{x+h}^{(S)}\right\}$$

b) Early form of endowment coverage payable in case of death, with capital and upgraded annual premiums in advance, with exemption of remaining annual premiums after DD rise.
The r.p.v. of this insurance form is the sum of the two components' r.p.v.$_{\cdot s}$, the early value and the exemptive one.
Using again the symbol in *a)*, let

- C_0 be the capital insured on contract subscription for the endowment ;
- β_h , (h = 1,...,n-1), be the upgrading factor of C_0 to obtain the capital insured in the year (h, h+1); $\beta_0 = 1$;
- η_h , (h = 0,...,n-1), be the insured capital's early rate in case of DD rise in the year (h, h+1); $0 < \eta_h \le 1$;
- Q_{x+h+1} be the annual upgraded premium of endowment payable at h ;
- ${}_{/m}A_x^{(S)} = \sum_{k=0}^{m-1} {}_{k/1}q_x^{(SD)} v^{k+1}$ be the net single premium for a term insurance on DD policy-holder, with m years term and annual payments in arrears.

It follows:

$$(2.8)\qquad B_{ANT}=\begin{cases} C_0\,\beta_T\,\eta_T v^{T+0.5}\left[1-v^Y\right] & ,\ \text{se}\ \ Y<n-T \\ C_0\,\beta_T\,\eta_T v^{T+0.5}\left[1-v^{n-T-0.5}\right] & ,\ \text{se}\ \ Y\ge n-T \end{cases}$$

where, if $T \ge n$, is $B_{ANT} = 0$.
By analogy with (2.3) and in force of (2.4), the r.p.v.'s latter component is

$$(2.9)\qquad B_{ESO2}=\sum_{r=1}^{M} Q_{T+r+1} v^{T+r}\quad ,\quad T<n-1$$

Obviously $B_{ESO2} = 0$ if $Y = 0$ or if $T \ge n-1$.
Because of (2.2) and (2.5), tenable also as to form sub *b)* , the single net premium is

$$(2.10)\quad E(B_{ANT}+B_{ESO2})=$$

$$= C_0 \sum_{h=0}^{n-1} \beta_h \eta_h v^{h+0.5} \left\{ 1 - (1+i)_{/n-h} A_{x+h}^{(S)} - {}_{n-h}p_{x+h}^{(S)} v^{n-h-0.5} \right\} {}_{h/1}q_x^{(HS)} +$$

$$+ \sum_{h=0}^{n-2} {}_{h/1}q_x^{(HS)} \left\{ \sum_{k=1}^{n-h-1} {}_{k/1}q_{x+h}^{(SD)} \sum_{r=1}^{k} Q_{h+r+1} v^{h+r} + {}_{n-h}p_{x+h}^{(S)} \sum_{r=1}^{n-h-1} Q_{h+r+1} v^{h+r} \right\}$$

3. APPLICATIONS OF RISK THEORY CONCERNING POLICY GAIN

In order to make estimates under correlation between gains of policies' couples, it is necessary first to calculate their means and variances. Such gain issues the gap between *prudential* (or 1st order) technical bases, by which net annual premium is computed, and *experience* ones (or 2nd order), by which effective benefits are estimated.
To that purpose, as to DD coverage premiums, it is necessary to estimate a random annuity-due, starting from subscription and expiring either on H insured person's death or on DD rise. Denoting by U the starting time of an H insured person's year of death ($U = \infty$ if DD arises), because of above stated hypotheses such annuity includes L+1 payments, where $L = \min(T,U)$.
The result of the preceding patterns is

$$\text{(3.1)} \qquad \text{Prob } (L = h) = \begin{cases} {}_{h/1}q_x^{(HS)} + {}_{h/1}q_x^{(HD)} & , \text{ se } h < n \\ {}_n p_x^{(H)} & , \text{ se } h \ge n \end{cases}$$

That stated, with reference to *form* sub *a)* and using already defined symbols, the random gain discounted in 0 on the n term policy j, having capital ${}^jC$ and annual premium ${}^jQ$, is

$$\text{(3.2)} \qquad {}^jX_{AGG} = {}^jP_{AGG}\, \ddot{a}_{\overline{L+1}|i} - {}^jB_{AGG} - {}^jB_{ESO1}$$

where ${}^jP_{AGG}\, \ddot{a}_{\overline{L+1}|i}$ represents the annuity-due value of the corresponding net annual premiums agreed up by contract and payable up to DD occurrence or, at the later, for n years and ${}^jB_{AGG}$, ${}^jB_{ESO1}$ are benefits defined in (2.1) and (2.3). The 1st order moment $E({}^jX_{AGG})$ is obtained by subtracting (2.7) from

$$\text{(3.3)} \quad E({}^jP_{AGG}\, \ddot{a}_{\overline{L+1}|i}) = {}^jP_{AGG} \left[\sum_{h=0}^{n-1} \ddot{a}_{\overline{h+1}|i} \left({}_{h/1}q_x^{(HS)} + {}_{h/1}q_x^{(HD)} \right) + \ddot{a}_{\overline{n}|i}\, {}_n p_x^{(H)} \right]$$

and, for the well known formula, the variance of ${}^jX_{AGG}$ is obtained in function the 1st and 2nd order moments.
In particular $E({}^jX_{AGG}^2)$ is calculated - by (2.2), (2.5), (3.1) and (3.2) - summing up the following expressions:

- $E\left[\left({}^jP_{AGG}\, \ddot{a}_{\overline{L+1}|i} \right)^2 \right] = {}^jP_{AGG}^2 \left[\sum_{h=0}^{n-1} \left(\ddot{a}_{\overline{h+1}|i} \right)^2 \left({}_{h/1}q_x^{(HS)} + {}_{h/1}q_x^{(HD)} \right) + \left(\ddot{a}_{\overline{n}|i} \right)^2 {}_n p_x^{(H)} \right]$
- $E\left[\left({}^jB_{AGG} \right)^2 \right] = {}^jC^2 \sum_{h=0}^{n-1} v^{2h+1}\, {}_{h/1}q_x^{(HS)}$
- $E\left[\left({}^jB_{ESO1} \right)^2 \right] = {}^jQ^2 \sum_{h=0}^{n-2} v^{2h}\, {}_{h/1}q_x^{(HS)} \left(\sum_{k=1}^{n-h-1} \left(a_{\overline{k}|i} \right)^2 {}_{k/1}q_{x+h}^{(SD)} + \left(a_{\overline{n-h-1}|i} \right)^2 {}_{n-h}p_{x+h}^{(S)} \right)$
- $-2\, E({}^jP_{AGG}\, \ddot{a}_{\overline{L+1}|i}\, {}^jB_{AGG}) = -2\, {}^jP_{AGG}\, {}^jC \sum_{h=0}^{n-1} \ddot{a}_{\overline{h+1}|i}\, v^{h+0.5}\, {}_{h/1}q_x^{(HS)}$

- $-2\,E({}^jP_{AGG}\,\ddot{a}_{\overline{L+1}|i}\;{}^jB_{ESO1}) =$

$$= -2\;{}^jP_{AGG}\;{}^jQ\sum_{h=0}^{n-2}\ddot{a}_{\overline{h+1}|i}\,v^h\,{}_{h/1}q_x^{(HS)}\left(\sum_{k=1}^{n-h-1}a_{\overline{k}|i}\,{}_{k/1}q_{x+h}^{(SD)}+a_{\overline{n-h-1}|i}\;{}_{n-h}p_{x+h}^{(S)}\right)$$

- $2\,E({}^jB_{AGG}\;{}^jB_{ESO1}) =$

$$= 2\;{}^jC\;{}^jQ\sum_{h=0}^{n-2}v^{2h+0.5}\;{}_{h/1}q_x^{(HS)}\left(\sum_{k=1}^{n-h-1}a_{\overline{k}|i}\;{}_{k/1}q_{x+h}^{(SD)}+a_{\overline{n-h-1}|i}\;{}_{n-h}p_{x+h}^{(S)}\right)$$

Similarly, in the *form* sub *b)*, the corresponding policy random gain is given by

$$(3.4)\qquad {}^jX_{ANT} = {}^jP_{ANT}\,\ddot{a}_{\overline{L+1}|i} - {}^jB_{ANT} - {}^jB_{ESO2}$$

which uses values defined in (2.8) and (2.9) and annual net premium ${}^jP_{ANT}$ as agreed upon. The first moment is obtained by subtracting (2.10) from $E({}^jP_{ANT}\,\ddot{a}_{\overline{L+1}|i})$, similar to (3.3) by replacing ${}^jP_{ANT}$ to ${}^jP_{AGG}$.

The variance of ${}^jX_{ANT}$ is obtained in function of $E({}^jX_{ANT})$ and $E\left[\left({}^jX_{ANT}\right)^2\right]$. Such a 2nd moment is the sum of the following ones:

- $E({}^jP_{ANT}\,\ddot{a}_{\overline{L+1}|i})^2 = {}^jP_{ANT}^2\left[\sum_{h=0}^{n-1}\left(\ddot{a}_{\overline{h+1}|i}\right)^2\left({}_{h/1}q_x^{(HS)}+{}_{h/1}q_x^{(HD)}\right)+\left(\ddot{a}_{\overline{n}|i}\right)^2\,{}_np_x^{(H)}\right]$
- $E({}^jB_{ANT})^2 =$

$$= {}^jC_j^2\sum_{h=0}^{n-1}\beta_h^2\,\eta_h^2\;{}_{h/1}q_x^{(HS)}\left\{\sum_{k=0}^{n-h-1}\left(v^{h+0.5}-v^{h+k+0.5}\right)^2{}_{k/1}q_{x+h}^{(SD)}+\left(v^{h+0.5}-v^n\right)^2{}_{n-h}p_{x+h}^{(S)}\right\}$$

- $E({}^jB_{ESO2})^2 =$

$$= \sum_{h=0}^{n-2}{}_{h/1}q_x^{(HS)}\left\{\sum_{k=1}^{n-h-1}\left[\sum_{r=1}^{k}{}^jQ_{h+r+1}v^{h+r}\right]^2{}_{k/1}q_{x+h}^{(SD)}+\left[\sum_{r=1}^{n-h-1}{}^jQ_{h+r+1}v^{h+r}\right]^2{}_{n-h}p_{x+h}^{(S)}\right\}$$

- $2\,E({}^jP_{ANT}\,\ddot{a}_{\overline{L+1}|i}\;{}^jB_{ANT}) = -2\;{}^jP_{ANT}\,{}^jC_0\sum_{h=0}^{n-1}\beta_h\,\eta_h\,\ddot{a}_{\overline{h+1}|i}\,{}_{h/1}q_x^{(HS)}\;\cdot$

$$\cdot\left\{\sum_{k=0}^{n-h-1}\left(v^{h+0.5}-v^{h+k+0.5}\right){}_{k/1}q_{x+h}^{(SD)}+\left(v^{h+0.5}-v^n\right){}_{n-h}p_{x+h}^{(S)}\right\}$$

- $-2\,E({}^jP_{ANT}\,\ddot{a}_{\overline{L+1}|i}\;{}^jB_{ESO2}) =$

$$= -2\,{}^jP_{ANT}\sum_{h=0}^{n-2}\ddot{a}_{\overline{h+1}|i}v^h\;{}_{h/1}q_x^{(HS)}\left\{\sum_{k=1}^{n-h-1}{}_{k/1}q_{x+h}^{(SD)}\sum_{r=1}^{k}{}^jQ_{h+r+1}v^{h+r}+{}_{n-h}p_{x+h}^{(S)}\left(\sum_{r=1}^{n-h-1}{}^jQ_{h+r+1}v^{h+r}\right)\right\}$$

$2\,E({}^jB_{ANT}\;{}^jB_{ESO2}) = 2\,{}^jC_0\sum_{h=0}^{n-2}\beta_h\,\eta_h\,{}_{h/1}q_x^{(HS)}\;\cdot$

$$\cdot\left\{\sum_{k=0}^{n-h-1}\left(v^{h+0.5}-v^{h+k+0.5}\right){}_{k/1}q_{x+h}^{(SD)}\sum_{r=1}^{k}{}^jQ_{h+r+1}v^{h+r}+\left(v^{h+0.5}-v^n\right){}_{n-h}p_{x+h}^{(S)}\sum_{r=1}^{n-h-1}{}^jQ_{h+r+1}v^{h+r}\right\}$$

By knowing means and variances of gains, it is possible to estimate the risk concerning portfolio management under assumptions more general than those leading to independence among policy gain r.p.v.$_{s}$ and which give way to customary formulations.

It is possible to get some results concerning the problems examined also in the assumption of a two-by-two correlation of mentioned r.p.v.,s, adopting for all couples, in order to simplify, the same Bravais correlation coefficient *r*. We shall consider only cases qualified by a weak correlation between risks consequent to the effects of the same cause (like age, environmental issues, etc.) on risk themselves, which might, in the extreme, be supposed independent conditionally to each of possible influencing factors.
We denote by m = E(X) the mean of the portfolio's random present gain X (more exactly X_{AGG} or X_{ANT}, depending on choice), obtained by summing up the single policies' ones, i.e. the values $E(^jX)$, where ^jX is $^jX_{AGG}$ or $^jX_{ANT}$. Besides, let be:

- $^j\sigma^2 \quad = \sigma^2(^jX)$
- $^j\sigma \quad = \sqrt{^j\sigma^2}$ = standard deviation of ^jX
- $\sigma \quad$ = standard deviation of X.

Denoting by G the solvency fund's amount relative to the portfolio in question, the number

$$\lambda = \frac{G+m}{\sigma} > 0 \tag{3.5}$$

represents its "safety index". It is well known the probability ruin upper bound, due to Cantelli, in function of G through (3.5), given by

$$\text{Prob}\,(X+G<0) \leq \frac{1}{1+\lambda^2} \tag{3.6}$$

where from $\lambda > 0$ follows $1/(1+\lambda^2) < 1$. (3.6) holds true, no matter the X distribution, also in case of positive correlation between risks. In this case, it will be sufficient to put in (3.5)

$$m = \sum_j E(^jX) \quad ; \quad \sigma \quad +\left(\sum_j {^j\sigma^2} + r\sum_i \sum_{j\neq i} {^i\sigma}\,{^j\sigma}\right)^{0.5}$$

Clearly, by the growing of *r* along positive numbers, σ too grows and with it the bound $1/(1+\lambda^2)$, wherefore the link weakens and probability may increase (without however ever reaching 1).
If risks are independent it is well known that, under wider assumptions (basically if each single risk's variance is negligible as to the whole portfolio's) the distribution of X converges "in law" to Gauss distribution with increase of risks' number. Therefore, being

$$\text{Prob}\,(X+G<0) = \text{Prob}\left(\frac{X-m}{\sigma} < \frac{-G-m}{\sigma} = -\lambda\right) \tag{3.7}$$

the distribution of the normalized variable correspondent to X in a portfolio with several homogeneous policies is approximately N(0,1) and we can, therefore, write:

$$\text{Prob}\,(X+G<0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\lambda} e^{-x^2/2}\,dx \quad < \quad \frac{1}{2} \tag{3.7'}$$

Anyway, in the assumption of correlation as mentioned above, the aforesaid limit theorem does not apply and with it the entire processing in general must be dropped.
However, under very specific hypothesis regarding correlation, an estimate based on (3.7') may be accepted as approximate pattern. If we follow de Finetti's theory, that is possible if correlation exists between risks of the same nature only owing to a concurrence of external circumstances acting on them simultaneously causing their probability to vary according to approximately

Gaussian distribution[1]. Indeed the situation previously assumed about correlated risks corresponds substantially to that described by de Finetti and we can therefore apply his conclusions.
If therefore we consider this framework acceptable we can take also in this case the right hand side of (3.7') as an approximate estimation of ruin probability, provided we put

$$(3.8) \qquad \lambda = \frac{G+\sum_j E({}^jX)}{\sigma}$$

where

$$(3.9) \quad \sigma = +\left[\sum_j {}^j\sigma^2 + \sum_i\sum_{j\neq i} r\; {}^i\sigma\, {}^j\sigma\right]^{0,5} = +\left[(1-r)\sum_j {}^j\sigma^2 + r\left(\sum_j {}^j\sigma\right)^2\right]^{0.5}$$

wherefrom σ^2 is linear function of r, ranging from $\sum_j {}^j\sigma^2$ to $\left(\sum_j {}^j\sigma\right)^2$ with the growing of r from 0 to +1. Clearly λ decreases and (3.7') grows with r, even if keeping below 0.5.
To counteract increase of ruin probability due to correlation, two known ways offer:
- to make recourse to a reinsurance strategy;
- to increase the solvency fund and/or loading.

The next part of this § evidences a further aspect: the correlation contained in Bravais parameter ${}^j\rho_{h,k}$ implicitly existing between cost and revenue components of a policy gain.
Let us state, for shortness sake, with reference to j policy

$${}^jA_1 = {}^jP_{AGG}\, \ddot{a}_{\overline{L+1}|i}\;;\; {}^jA_2 = {}^jB_{AGG}\;;\; {}^jA_3 = {}^jB_{ESO1}$$

$${}^jB_1 = {}^jP_{ANT}\, \ddot{a}_{\overline{L+1}|i}\;;\; {}^jB_2 = {}^jB_{ANT}\;;\; {}^jB_3 = {}^jB_{ESO2}$$

Bearing in mind the moments' formulas, the following quantities regarding specifically additional form of policies can be calculated:

$$(3.10) \qquad {}^j_A\sigma^2_h = \mathrm{var}({}^jA_h) = E({}^jA^2_h) - [E({}^jA_h)]^2\;; \qquad h = 1, 2, 3$$

$$(3.11) \qquad {}^j_A\sigma_{hk} = \mathrm{cov}({}^jA_h, {}^jA_k) = E({}^jA_h\, {}^jA_k) - E({}^jA_h)E({}^jA_k)\;; h, k = 1, 2, 3; h \neq k$$

$$(3.12) \qquad {}^j_A\rho_{hk} = \frac{{}^j_A\sigma_{hk}}{{}^j_A\sigma_h\, {}^j_A\sigma_k} \quad ; h, k = 1, 2, 3\;; \quad h \neq k$$

as well as those ${}^j_B\sigma^2_h$, ${}^j_B\sigma_{hk}$, ${}^j_B\rho_{hk}$, defined in a parallel way concerning early forms.

[1] In a fundamental paper (see "Il problema dei pieni", G.I.I.A.,1940,p. 20) de Finetti reports as follows (here translated): "...It is not difficult to extend to such a case" i.e. of correlation "the analysis till now worked out" in non-correlation hypothesis, "because it will suffice to take into account rectangular terms in the σ formula. It is necessary however to go back to the preliminary observations....... since in the new case the reasons above indicated can no longer be applied .... If we instead assume the correlation to take place mainly among risks of the same nature, owing to circumstances having the same influence, either increasing or decreasing corresponding probabilities, and if we furtherly assume probability distribution of such factors' influence on probability to follow Gauss' distribution (or to have anyway a similar form), the conclusion reached above is still tenable. Let's suppose, to refer to the simplest case, that probability distribution *f(x)dx* that a given set of circumstances may take place, conditionally to which *n* events insured be considered independent and having the some probability x. We shall indicate by $\underline{x} = \int x f(x)dx$ before getting to known the circumstances in question ...... Assuming to know them and value x therewith, probability of total deviation will (approximately and under conditions specified in n. 2) be Gaussian.........". The Author then examines further details, which we must here leave out for space reasons.

4. NUMERICAL EXAMPLES

It is not within the scope of this paper to analyse possible management strategies in order to limit the increase of ruin probability due to correlation[2], while we intend to examine by means of a numerical application the variations of safety index of upper bound ruin probability and of estimation (3.7') alongside correlation growth, having fixed G on the solvency margin imposed by E. C. regulations on life insurance.
On the other hand, we analyse the G variations needed to leave the ruin probability unchanged as correlation growths.
In terms of application we refer to insurance forms illustrated in § 2 and to the expressions of moments formulated in § 3. As to technical demographic bases, we have used those provided by ANIA[3], concerning 10-year-term contracts having as initial benefit 1000 currency units and an insured male population aged 35, 40, 45 and 50, whereas the technical discount rate is fixed in i=0,025.
In Tables 1 and 2, the mean and the standard deviation of X_{AGG} and X_{ANT} are indicated as well as the values of G and λ, the latter referred to the assumption of safety load implicit in the 10%, 15% and 20% technical demographic bases. The analysis of values indicated for each contract shows a very low safety index as a result of reduced margins for the Company due to a high variability benefits. The ratios $\sigma(X)/E(X)$ range from a minimum of 9,06, corresponding to the highest safety load premium, to a maximum of 78,11.

Table 1 ADDITIONAL FORM

Age	$E(X_{AGG})$			$\sigma(X_{AGG})$			S.M.	λ		
	10%	15%	20%	10%	15%	20%		10%	15%	20%
35	2,415	3,626	4,839	129,590	126,082	122,462	3,120	0,0427	0,0535	0,0650
40	4,511	6,777	9,050	167,490	163,143	158,622	3,212	0,0461	0,0612	0,0773
45	7,334	11,027	14,739	196,720	191,912	186,891	3,320	0,0542	0,0748	0,0966
50	10,939	16,468	22,036	208,990	204,443	199,643	3,451	0,0689	0,0974	0,1277

Table 2 EARLY FORM

Age	$E(X_{ANT})$			$\sigma(X_{ANT})$			S.M.	λ		
	10%	15%	20%	10%	15%	20%		10%	15%	20%
35	0,740	1,111	1,482	57,803	56,203	54,553	3,020	0,0651	0,0735	0,0825
40	1,372	2,060	2,750	76,839	74,743	72,579	3,037	0,0574	0,0682	0,0797
45	2,292	3,443	4,597	98,058	95,437	92,725	3,062	0,0546	0,0682	0,0826
50	3,228	4,852	6,483	112,327	109,391	106,347	3,086	0,0562	0,0726	0,0900

As evidenced in Tables 3 and 4 (each referred to the two insurance forms analysed), a significant improvement of the safety index is achieved in case of a portfolio holding at least 10000 contracts, assuming an uniform contracts' distribution in the four age groups considered; the ruin probability, calculated with reference both to (3.6) and (3.7') shows untenable levels in portfolios qualified by N<= 2500 policies considering that estimate is made in case of non-correlation.

[2] We refer specifically to the execution of reinsurance strategy or, alternatively, to the introduction of an increase on solvency fund and/or on loads, the latter choice being, however, market conditioned.
[3] See ANIA (1992).

Table 3 NON CORRELATED CONTRACTS - ADDITIONAL FORM

N	λ			prob ($X_{AGG} + G < 0$)					
	10%	15%	20%	Cantelli's upper bound			Normal distribution		
				10%	15%	20%	10%	15%	20%
10	0,0862	0,1176	0,1510	0,99263	0,98635	0,97770	0,46567	0,45318	0,43998
100	0,2725	0,3719	0,4776	0,93088	0,87847	0,81429	0,39262	0,35497	0,31648
400	0,5450	0,7439	0,9551	0,77101	0,64376	0,52295	0,29289	0,22847	0,16976
1000	0,8617	1,1762	1,5102	0,57389	0,41956	0,30483	0,19443	0,11976	0,06550
2500	1,3624	1,8597	2,3878	0,35011	0,22429	0,14922	0,08653	0,03146	0,00848
5000	1,9268	2,6301	3,3768	0,21220	0,12631	0,08063	0,02700	0,00427	0,00037
8000	2,4372	3,3268	4,2714	0,14409	0,08287	0,05196	0,00740	0,00044	0,00001
10000	2,7249	3,7195	4,7755	0,11870	0,06741	0,04201	0,00322	0,00010	0,00000

Table 4 NON CORRELATED CONTRACTS - EARLY FORM

N	λ			prob ($X_{ANT} + G < 0$)					
	10%	15%	20%	Cantelli's upper bound			Normal distribution		
				10%	15%	20%	10%	15%	20%
10	0,0909	0,1115	0,1334	0,99180	0,98773	0,98252	0,46378	0,45562	0,44695
100	0,2875	0,3525	0,4218	0,92366	0,88948	0,84897	0,38687	0,36224	0,33660
400	0,5750	0,7050	0,8435	0,75155	0,66801	0,58426	0,28266	0,24041	0,19946
1000	0,9091	1,1147	1,3338	0,54751	0,44593	0,35985	0,18165	0,13250	0,09114
2500	1,4374	1,7624	2,1089	0,32614	0,24353	0,18358	0,07530	0,03900	0,01748
5000	2,0328	2,4925	2,9824	0,19485	0,13865	0,10107	0,02104	0,00634	0,00143
8000	2,5713	3,1528	3,7724	0,13138	0,09141	0,06565	0,00507	0,00081	0,00008
10000	2,8748	3,5249	4,2177	0,10794	0,07449	0,05322	0,00202	0,00021	0,00001

Dropping out the hypothesis of independence and considering a weak positive correlation between contracts in the portfolio, safety conditions rapidly get worse, with a significant increase in ruin probability, also corresponding to low correlation level (r=0,001); specifically Tables 5 and 6 (with respect to the two insurance forms and to a 10000 contract portfolio) give λ values as well as ruin probabilities according to definitions given in § 3, with the correlation factor varying from a minimum of 0,001 to a maximum of 0,25.

Table 5 N=10000 CORRELATED CONTRACTS - ADDITIONAL FORM

r	λ			prob ($X_{AGG} + G < 0$)					
	10%	15%	20%	Cantelli's upper bound			Normal distribution		
				10%	15%	20%	10%	15%	20%
0,000	2,7249	3,7195	4,7755	0,11870	0,06741	0,04201	0,00322	0,00010	0,00000
0,001	1,4567	1,9884	2,5530	0,32031	0,20187	0,13302	0,07260	0,02338	0,00534
0,005	0,7418	1,0125	1,3000	0,64508	0,49379	0,37176	0,22912	0,15565	0,09680
0,010	0,5345	0,7296	0,9367	0,77780	0,65262	0,53263	0,29650	0,23282	0,17445
0,015	0,4392	0,5996	0,7698	0,83827	0,73558	0,62791	0,33025	0,27440	0,22071
0,020	0,3816	0,5209	0,6688	0,87287	0,78655	0,69092	0,35137	0,30121	0,25180
0,030	0,3126	0,4267	0,5479	0,91097	0,84595	0,76912	0,37728	0,33479	0,29188
0,050	0,2428	0,3314	0,4255	0,94433	0,90103	0,84669	0,40408	0,37016	0,33523
0,100	0,1720	0,2348	0,3015	0,97126	0,94774	0,91668	0,43171	0,40718	0,38152
0,150	0,1406	0,1919	0,2463	0,98063	0,96450	0,94279	0,44411	0,42393	0,40271
0,200	0,1218	0,1662	0,2134	0,98539	0,97312	0,95644	0,45154	0,43400	0,41551
0,250	0,1089	0,1487	0,1909	0,98827	0,97837	0,96484	0,45663	0,44090	0,42430

Table 6 N=10000 CORRELATED CONTRACTS - EARLY FORM

r	λ			prob (X_{ANT} + G < 0)					
	10%	15%	20%	Cantelli's upper bound			Normal distribution		
				10%	15%	20%	10%	15%	20%
0,000	2,8748	3,5249	4,2177	0,10794	0,07449	0,05322	0,00202	0,00021	0,00001
0,001	1,5369	1,8844	2,2548	0,29744	0,21973	0,16436	0,06216	0,02976	0,01207
0,005	0,7826	0,9595	1,1481	0,62019	0,52064	0,43137	0,21694	0,16865	0,12546
0,010	0,5639	0,6914	0,8273	0,75873	0,67656	0,59366	0,28641	0,24465	0,20403
0,015	0,4634	0,5682	0,6799	0,82322	0,75594	0,68388	0,32154	0,28495	0,24829
0,020	0,4026	0,4937	0,5907	0,86050	0,80404	0,74132	0,34361	0,31077	0,27736
0,030	0,3298	0,4044	0,4839	0,90189	0,85944	0,81027	0,37076	0,34295	0,31423
0,050	0,2562	0,3141	0,3758	0,93842	0,91021	0,87624	0,39891	0,37673	0,35353
0,100	0,1815	0,2225	0,2663	0,96811	0,95282	0,93379	0,42799	0,41195	0,39501
0,150	0,1483	0,1818	0,2176	0,97848	0,96800	0,95481	0,44106	0,42786	0,41389
0,200	0,1285	0,1575	0,1885	0,98377	0,97579	0,96570	0,44889	0,43742	0,42525
0,250	0,1149	0,1409	0,1686	0,98696	0,98053	0,97236	0,45425	0,44397	0,43305

Therefore correlation, though weak, between risks insured produces a significant growth of ruin probability and a larger portfolio allows a better safety index only when corresponding to low correlation levels (r<0,05). In the end results evidence the shortcoming of the present solvency margin calculation. As a matter of fact Tables 7 and 8 worked out (with respect to the two insurance forms on portfolios qualified by N=100, 5000, 10000 and in connection with a presumptive 15% safety load) prove the inadequacy of the solvency margin when the risk correlation increases. The aim of stabilizing the ruin probability at the level r=0 requires an increase in the solvency margin against the one imposed from a minimum of 2,3% for r=0,001 to a maximum of 9349,2% for r=0,25 within the two insurance forms examined.

Table 7 S.M. SHORT COMING – ADDITIONAL FORM

r	N=100	S.M.=327,5	N=5000	S.M.=16375,0	N=10000	S.M.=32750,0
	G	G/S.M.	G	G/S.M.	G	G/S.M.
0,000	327,5	1,000	16375,0	1,000	32750,0	1,000
0,001	342,7	1,046	48227,1	2,945	143746,5	4,389
0,005	401,8	1,227	124213,9	7,586	373625,3	11,408
0,010	472,3	1,442	186764,8	11,405	555245,0	16,954
0,015	539,4	1,647	235821,1	14,401	696205,9	21,258
0,020	603,6	1,843	277553,7	16,950	815595,5	24,904
0,030	724,6	2,213	348018,2	21,253	1016539,4	31,039
0,050	943,6	2,881	460414,4	28,117	1336135,7	40,798
0,100	1403,5	4,285	667912,0	40,789	1924806,2	58,773
0,150	1787,0	5,457	827508,3	50,535	2377046,9	72,582
0,200	2123,1	6,483	962182,3	58,759	2758485,4	84,229
0,250	2425,8	7,407	1080894,1	66,009	3094626,8	94,492

Table 8 S.M. SHORT COMING – EARLY FORM

r	N=100	S.M.=305,1	N=5000	S.M.=15256,4	N=10000	S.M.=30512,8
	G	G/S.M.	G	G/S.M.	G	G/S.M.
0,000	305,1	1,000	15256,4	1,000	30512,8	1,000
0,001	312,2	1,023	30041,0	1,969	82030,0	2,688
0,005	339,6	1,113	65310,8	4,281	188726,9	6,185
0,010	372,3	1,220	94344,2	6,184	273024,7	8,948
0,015	403,49	1,322	117114,0	7,676	338450,9	11,092
0,020	433,3	1,420	136484,5	8,946	393864,8	12,908
0,030	489,5	1,604	169191,1	11,090	487131,8	15,965
0,050	591,1	1,937	221360,5	14,509	635470,6	20,826
0,100	804,6	2,637	317671,9	20,822	908698,6	29,781
0,150	982,6	3,220	391749,6	25,678	1118603,5	36,660
0,200	1138,6	3,731	454259,5	29,775	1295646,0	42,462
0,250	1279,1	4,192	509360,4	33,387	1451664,1	47,576

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