RUIN PROBABILITY APPROXIMATION FOR A CLASS OF RENEWAL PROCESSES WITH HEAVY TAILS

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ABSTRACT

This paper examines an integro-differential equation of the survival probability $\delta(u)$ for a class of risk processes in which claims occur as an ordinary renewal process. Specifically, according to the model proposed by Dickson, claims are assumed to occur as an Erlang process. We determine $\delta(u)$ by using an exponential claim size distribution, thus comparing the results to those obtained from the classical Cramér-Lundberg model. After having transformed the integro-differential equation into an integral equation, we find an approximate solution $\hat{\delta}(u)$ for $\delta(u)$. We then test it in exponential cases. Furthermore, we consider the case of large claims characterized by heavy-tailed claim size distributions. The approximate solution found for $\delta(u)$ also applies to certain subexponential distributions. Finally, we obtain an expression for the asymptotic behaviour of $\hat{\delta}(u)$ when the claim size distribution has a regularly varying tail.

KEYWORDS

Ruin probability. Renewal process. Large claims. Subexponential distributions.

1.

Most of the results in the field of ruin theory concern the classical risk model, that is Cramér-Lundberg model, in which claims occur as a Poisson process.
In 1957, Sparre Andersen introduced a mathematical model for the risk theory based on the assumption that the claim arrival process is an ordinary renewal process. In this model:
- the interclaim times $\{T_i\}_{i=1}^{+\infty}$ form a sequence of independent and identically distributed random variables with common distribution function $K(t)$;
- the claim sizes $\{X_i\}_{i=1}^{+\infty}$ are also independent and identically distributed random variables with distribution function $F(x)$ with mean value m and density function $f(x)$;
- the processes $\{T_i\}_{i=1}^{+\infty}$ and $\{X_i\}_{i=1}^{+\infty}$ are supposed to be independent.

The probability of ultimate ruin with an initial reserve u is

$$\psi(u) = \Pr\left\{u + \sum_{i=1}^{n} \left(\bar{c}T_i - X_i\right) < 0, \text{for some } n \in N\right\},$$

where

$$\bar{c}=\frac{1}{\mu}m(1+\eta)$$

is the constant net premium income rate, and η is the relative security loading $(0<\eta<1)$. Dickson (1998) examined the case when $K(t)$ is an Erlang (2) distribution with density function

$$k(t)=\beta^2 te^{-\beta t} \qquad (t>0)$$

with mean value $\mu=\frac{2}{\beta}$. In this case, $\bar{c}=\frac{\beta}{2}m(1+\eta)$.

He obtained the following equation

(1.1) $$\bar{c}^2\delta''(u)-2\beta\bar{c}\delta'(u)+\beta^2\delta(u)=\beta^2\int_0^u f(u-x)\delta(x)dx,$$

where $\delta(u)=1-\psi(u)$ is the survival probability. The following conditions must occur:

(1.2a) $$\begin{cases}\delta(0)=\delta_0\\ \delta'(0)=\delta_0'\end{cases}$$

(1.2b) $$\bar{c}^2\delta''(0)-2\beta\bar{c}\delta'(0)+\beta^2\delta(0)=0$$

(1.2c) $$\lim_{u\to+\infty}\delta(u)=1$$

(1.2d) $$\lim_{u\to+\infty}\delta'(u)=0.$$

Since the asymptotic relations do not depend on the parameter β, let $c=\frac{\bar{c}}{\beta}$ so that equation (1.1) becomes

(1.3) $$c^2\delta''(u)-2c\delta'(u)+\delta(u)=\int_0^u f(u-x)\delta(x)dx.$$

Our purpose in this paper is to find an approximation for the survival probability $\delta(u)$ and to examine the asymptotic behaviour for the ruin probability $\psi(u)$. In section 2, we find the exact solution for $\delta(u)$ in the case of exponential claim size distribution, then making a comparison with the Poisson case. In section 3, we transform the integro-differential equation (1.3) into an integral equation, giving an analytical method to solve it. In section 4, we obtain an approximation for the survival probability. Finally, in section 5, we examine the large claims case and give asymptotic results when the claim size distribution has a regularly varying tail.

2.

When the claim size distribution is exponential with density function

$$f(x) = e^{-x},$$

we find the exact solution for equation (1.3).
By observing that $f'(x) + f(x) = 0$, we can use a well-known analytical method to solve the (1.3) (i.e. differentiating the (1.3) and adding to it the derivative), therefore obtaining the differential equation

$$c^2 \delta'''(u) + (c^2 - 2c)\delta''(u) + (1 - 2c)\delta'(u) = 0.$$

The solution of this differential equation using the conditions (1.2) is

$$\delta(u) = 1 + Ae^{\alpha u}$$

with

$$\alpha = \frac{3 - \eta - \sqrt{(3-\eta)^2 + 16\eta}}{2(1+\eta)}$$

$$A = -\frac{8}{(1+\eta)\left[\eta + 5 + \sqrt{(3-\eta)^2 + 16\eta}\right]}.$$

In this specific case, it is possible to find the initial value δ_0 for $\delta(u)$:[1]

$$\delta_0 = 1 + A = \frac{\eta}{1+\eta} \frac{\eta^2 + 6\eta - 3 + (1+\eta)\sqrt{(3-\eta)^2 + 16\eta}}{\eta\left[\eta + 5 + \sqrt{(3-\eta)^2 + 16\eta}\right]}$$

Through algebric calculations, it results that

$$\frac{\eta^2 + 6\eta - 3 + (1+\eta)\sqrt{(3-\eta)^2 + 16\eta}}{\eta\left[\eta + 5 + \sqrt{(3-\eta)^2 + 16\eta}\right]} > 1.$$

[1] The same result for δ_0 can be obtained following a method used by Dickson and Hipp based on the consideration of the Laplace transform of (1.3).

Subsequently $\delta_0 >_P \delta_0$, where $_P\delta_0 = \dfrac{\eta}{1+\eta}$ is the survival probability with an initial reserve u=0 in the Poisson model.
Furthermore, Lundberg's inequality still holds

$$\psi(u) \leq e^{-Ru}$$

and R is the unique positive number such that $E\left[e^{RX_i}\right]E\left[e^{-cRT_i}\right]=1$. In our case, solving

$$\int_0^{+\infty} e^{-(1-R)x}dx \int_0^{+\infty} te^{-(cR+1)t}dt = 1$$

we get

$$R = \frac{\eta - 3 + \sqrt{(\eta-3)^2 + 16\eta}}{2(1+\eta)} =$$

$$= \frac{\eta}{1+\eta} \frac{\eta - 3 + \sqrt{(\eta-3)^2 + 16\eta}}{2\eta}.$$

As we had expected, $R = |\alpha|$. It is also interesting to notice that $R >_P R$, where $_PR = \dfrac{\eta}{1+\eta}$ is the value for the adjustement coefficient if the claim number process is Poisson and the claims are exponentially distributed. In fact, it results that

$$\frac{\eta - 3 + \sqrt{(\eta-3)^2 + 16\eta}}{2\eta} > 1.$$

So, from Lundberg's inequality, it derives that the survival probability in the renewal model is higher than the survival pobability in the classical model, when the claim size is exponentially distributed.
This result can be tested using the following expressions

$$_P\delta(u) = 1 - \frac{1}{1+\eta} e^{-\frac{\eta}{1+\eta}u}$$

and

$$\delta(u) = 1 - \frac{1}{(1+\eta)} \frac{8}{\left[\eta + 5 + \sqrt{(3-\eta)^2 + 16\eta}\right]} e^{-\frac{\eta}{1+\eta}\frac{\eta-3+\sqrt{(3-\eta)^2+16\eta}}{2\eta}u}, \tag{2.1}$$

where $_P\delta(u)$ and $\delta(u)$ denote respectively the survival probability in the Poisson and in the renewal models.

In the following table, there is a numerical example for an initial reserve u=100.

η	δ_0	${}_p\delta_0$	$\delta(u)$	${}_p\delta(u)$
0.0025	0.003324	0.002494	0.285191	0.222657
0.0050	0.006630	0.004975	0.488104	0.394984
0.0075	0.009917	0.007444	0.632745	0.528525
0.0100	0.013187	0.009901	0.736037	0.632139
0.0250	0.032431	0.024390	0.962224	0.914882
0.0500	0.063149	0.047619	0.998305	0.991858
0.0750	0.092279	0.069767	0.999911	0.999132
0.1000	0.119936	0.090909	0.999995	0.999898
0.1250	0.146223	0.111111	1.000000	0.999987
0.1500	0.171235	0.130435	1.000000	0.999998
0.1750	0.195058	0.148936	1.000000	1.000000
0.2000	0.217771	0.166667	1.000000	1.000000

3.

In this section, we transform equation (1.3) into a Volterra integral equation of the second kind.
Integrating the (1.3), we obtain

$$c^2\delta'(u) - 2c\delta(u) = \left(c^2\delta_0' - 2c\delta_0\right) - \int_0^u \overline{F}(u-x)\delta(x)dx,$$

where $\overline{F}(x) = 1 - F(x)$ is the tail of the claim size distribution.
Letting $c^2\delta_0' - 2c\delta_0 = a$, we note that considering conditions (1.2c) and (1.2d) from equation (1.3) as $u \to +\infty$, it follows

(3.1) $$a = m - 2c = -m\eta.$$

This same result is found by Dickson examining the Laplace transform of (1.3).
Integrating again, the (1.3) becomes the following integral equation

(3.2) $$\delta(u) = \delta_0 - \frac{m\eta}{c^2}u + \frac{1}{c^2}\int_0^u (2c - G(x))\delta(u-x)dx,$$

where $G(x) = \int_0^x \overline{F}(t)dt$.
Before proceeding to solve equation (3.2), we remark the link between a and the expected value of risk reserve variation.
Denoting with τ_i the time of i-th claim, we have

$$\tau_{i+1} = \tau_i + T_{i+1} \qquad (\tau_1 = T_1).$$

We can observe that the risk reserve variation

$$Q_k = \sum_{i=1}^{k}(cT_i - X_i) - \sum_{i=1}^{k-1}(cT_i - X_i) = cT_k - X_k$$

in the time interval $]\tau_{k-1}, \tau_k]$ has expected value

$$E[Q_k] = cE[T_k] - E[X_k] = m\eta = -a\,.$$

Finally, we give a method to solve equation (3.2). Defining

$$N(u) = \delta_0 - \frac{m\eta}{c^2}u$$
$$H(x) = 2c - G(x)\,,$$

equation (3.2) can be written as

$$\delta(u) = N(u) + \frac{1}{c^2}\int_0^u H(x)\delta(u-x)dx$$

which is Neumann's series

$$\delta(u) = N(u) + \frac{1}{c^2}N_1(u) + \frac{1}{c^4}N_2(u) + \cdots + \frac{1}{c^{2n}}N_n(u) + \cdots \tag{3.3}$$

where

$$N_1(u) = \int_0^u H(x)N(u-x)dx$$

and

$$N_n(u) = \int_0^u H(x)N_{n-1}(u-x)dx \qquad\qquad (n \in N).$$

Being

$$|N(u)| < l_1$$
$$\int_0^{\bar{u}} |H(x)|dx < l_2$$

expression (3.3) holds for $u \in [0, \bar{u}]$ and $\forall \bar{u} \in \Re^+$. As $u \to +\infty$, (3.3) cannot describe the asymptotic behaviour of $\delta(u)$, because the kernel and $N(u)$ are not upper bounded.

4.

The aim of this section is to obtain an approximation for $\delta(u)$.
Observing that $\delta(u)>0$ and $H(x)=2c-G(x)>0$, we can write

$$\int_0^u H(u-x)\delta(x)dx=\delta(\xi)\int_0^u H(x)dx \qquad (0\le\xi\le u).$$

Approximating $\delta(u)\cong\delta(\xi)$, from (3.2) we have

$$\delta(u)\cong\delta_0-\frac{m\eta}{c^2}u+\frac{1}{c^2}\delta(u)\int_0^u[2c-G(x)]dx$$

that becomes

$$\delta(u)\cong\frac{\delta_0-\frac{m\eta}{c^2}u}{1-\frac{1}{c^2}\int_0^u[2c-G(x)]dx}.$$

Denoting the right-hand member with $\hat{\delta}(u)$, we obtain the following approximate formula for the survival probability $\delta(u)$:

(4.1) $$\hat{\delta}(u)=\frac{\delta_0-\frac{m\eta}{c^2}u}{1-\frac{1}{c^2}\int_0^u[2c-G(x)]dx}.$$

Since $\hat{\delta}(0)=\delta_0$ and $\lim_{u\to+\infty}\hat{\delta}(u)=1$, $\hat{\delta}(u)$ is an approximation for $\delta(u)$ for $u\in\Re_0^+$, excluding a neighbourhood of each u_1 and u_2, where u_1 is the positive zero for the numerator and u_2 for the denominator.
Comparing the value $u_1=\frac{\delta_0c^2}{m\eta}$ to the one denoted s_0 by Dickson and Hipp (1998), we note that

$$u_1=\frac{1}{s_0},$$

where s_0 is the positive zero for the Laplace transform of $\delta(u)$ obtained from (1.1).
Expression (4.1) can be written as

(4.2) $$\hat{\delta}(u)=\frac{c^2\delta_0-m\eta u}{c^2-m\eta u-m\int_0^u \overline{F}_I(x)dx},$$

being $\overline{F}_I(x)=\frac{1}{m}\int_x^{+\infty}\overline{F}(t)dt$ the integrated tail distribution function.

As an example, we compare – for an initial reserve u=100 – the approximate solution (4.2) to the exact solution (2.1) in the exponentially distributed claim sizes case.

η	$\delta(u)$	$\hat{\delta}(u)$	$\delta(u)-\hat{\delta}(u)$
0.0025	0.285191	0.249477	0.035714
0.0050	0.488104	0.399462	0.088643
0.0075	0.632745	0.499576	0.133169
0.0100	0.736037	0.571147	0.164890
0.0250	0.962224	0.769607	0.192618
0.0500	0.998305	0.870417	0.127888
0.0750	0.999911	0.910152	0.089759
0.1000	0.999995	0.931406	0.068588
0.1250	1.000000	0.944639	0.055361
0.1500	1.000000	0.953668	0.046332
0.1750	1.000000	0.960222	0.039778
0.2000	1.000000	0.965194	0.034806

5.

This section examines the asymptotic behaviour of the ruin probability in the case of extreme values among claims, i.e. when very big claims do occur with high probability. We will give two asymptotic expressions for the ruin probability.

For convenience, the following notation is introduced.

For two functions $h_1(x)$ and $h_2(x)$, the notation $h_1(x)$~$h_2(x)$ as $x\to\infty$ denotes that $\lim_{x\to\infty}\frac{h_1(x)}{h_2(x)}=1$.

Applying l'Hôspital's rule to (4.2), we find

$$\hat{\delta}(u)\sim\frac{\eta}{\eta+\overline{F}_I(u)} \qquad (u\to+\infty)$$

and, since $\hat{\delta}(u)\sim\delta(u)$, it follows

$$\delta(u)\sim\frac{\eta}{\eta+\overline{F}_I(u)} \qquad (u\to+\infty),$$

so the ruin probability results

$$(5.1) \qquad \psi(u) \sim \frac{\overline{F}_I(u)}{\eta + \overline{F}_I(u)} \qquad (u \to +\infty).$$

We can observe that, since $u \to +\infty$, it is possible to disregard the $\overline{F}_I(u)$ at the denominator, so that (5.1) becomes

$$(5.2) \qquad \psi(u) \sim \frac{1}{m\eta} \int_u^{+\infty} \overline{F}(x) dx \qquad (u \to +\infty).$$

The above is the well-known asymptotic expression used in both classical and Sparre Andersen models.
This formula is valid under any hypothesis, although it is particularly suited when one wants to emphasize large claims. On the other hand, Lundberg's asymptotic result is not practicable under these circumstances (Embrechts and Veraverbeke, 1982).
In the Cramér-Lundberg model, a slightly modified formula (5.2) still holds in presence of reserve capitalization, as proved by Klüppelberg and Stadtmüller (1998).
Before stating our second result, we recall some definitions of classes of heavy-tailed distributions.

Def. 1 - Subexponential distributed function.
A distribution function with support $(0,+\infty)$ is subexponential, if for all $n \geq 2$,

$$\lim_{x \to \infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n$$

where $\overline{F^{n*}}(x)$ denotes the n-th convolution of $\overline{F}(x) = 1 - F(x)$. Moreover, it can be shown that for independent and identically distributed random variables $X_1, \ldots, X_n$ with subexponential distribution it happens that

$$\lim_{x \to \infty} \frac{\mathrm{P}(x_1, \ldots, x_n > x)}{\mathrm{P}(\max(x_1, \ldots, x_n) > x)} = 1.$$

This means that the sum $X_1 + \ldots + X_n$ becomes large if and only if one of the summands becomes large, thus making subexponential functions useful for modelling extremal events.

Def. 2 - Regular variation in Karamata's sense.
A positive, measurable function g on $(0,+\infty)$ is regularly varying at ∞ of index $\vartheta \in \Re$ if

$$\lim_{x \to \infty} \frac{g(tx)}{g(x)} = t^{\vartheta} \qquad (t>0)$$

and we write $g \in \Re_{\vartheta}$.
When $\vartheta = 0$, the function g is called slowly varying at ∞.
For $g \in \Re_{\vartheta}$, it happens that $g(x) = x^{\vartheta} l(x)$ with some slowly varying function $l(x) \in \Re_0$.

Consequently, if the claim size distribution function F is regularly varying, its representation is

$$\overline{F}(x) = x^{-\vartheta} l(x) \qquad (\vartheta > 1)$$

Examples of such functions are the Pareto distribution, the loggamma distribution, and certain Benktander and stable claim distributions.
According to a Karamata's theorem, it is possible to write

$$\overline{F}_I(x) \sim \frac{x^{-(\vartheta-1)} l(x)}{m(\vartheta-1)}. \tag{5.3}$$

Finally, in order to obtain the second asymptotic formula for $\psi(u)$, we apply result (5.3) to (5.1):

$$\psi(u) \sim \frac{l(u)}{l(u) + m\eta(\vartheta-1)u^{(\vartheta-1)}} \qquad \begin{pmatrix} \vartheta > 1 \\ u \to +\infty \end{pmatrix}.$$

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